Fundamentals of Business Mathematics

FUNDAMENTALS OF BUSINESS MATHEMATICS

OER LAB

CONTENTS

ACKNOWLEDGEMENTS

OER Equal Love

Thank You!

Thank you to the first students employed by the OER Lab for working hard to make this book a reality. Congratulations on your achievement!

Editors: Ayman Abdulkadir, Pranjal Saloni, Rebecca Maynard.

Reviewers: Dr. Ana Duff

Project Manager: Sarah Stokes

Suggested Attribution for This Work: *OER Lab at Ontario Tech University, 2020, licensed under a CC BY NC SA 4.0 International License, unless otherwise noted.*

About the OER Lab

Ontario Tech University is proud to host the OER Lab – a student-run, staff-managed group that brings content and technological expertise to the timely **creation of high quality OER that will be used directly in an Ontario Tech undergraduate course by Ontario Tech students**.

If you adopt this book, you will be using a work created by students as an experiential learning and employment opportunity. Please let us know if you are using this work by emailing oer@ontariotechu.ca.

ATTRIBUTIONS

This text has been adapted from the following resources:

Business Math: A Step-by-Step Handbook (2018, Revision A) by J. Olivier, under a CC BY-NC-SA License.

Business Precalculus by D. Lippman, under a CC BY-SA 3.0 United States License.

1.

REVIEW OF THE BASICS

Where can you go in life and not be exposed to numbers and mathematics? Whether you are figuring out the price of a product (including shipping) on eBay, or balancing your bank accounts, you use your elementary mathematical skills from both your primary and secondary education.

Think about the math you perform every day:

- At the grocery store, you compare products to calculate the best value. One brand of potato chips retails for $3.99 for 300 g, while the equally satisfying brand beside it is priced at $3.49 for 250 g. Which offers the better value?
- As the host for a large gathering, you are preparing a homemade lasagna and need to triple the original recipe, which calls for 1 ⅔ cups of tomato sauce. In the expanded recipe, how many cups of tomato sauce do you need?
- If you are a sports fan, you know plenty of statistics about your favourite players and teams. Many come in percentage form, such as three-point throws for NBA stars or save percentages for NHL goaltenders. What exactly do those percentages mean?
- Many employers pay out bonuses. Perhaps in your company managers get twice as large a bonus as employees. Your company has five managers and 25 employees. If it announces a $35,000 total bonus, what is your share as an employee?

Mathematics and numbers surround you in the business world, where you must read many numerical reports, interpret how the numbers fit together, and create your own reports showing such metrics as sales and profit projections.

Away from work, you must manage your income and pay your bills. This is a mathematical problem you likely solve on a daily basis, ensuring that the money flowing out of your bank account does not exceed the money flowing in. To purchase groceries, vacations, or entertainment, you need numbers.

This chapter gives you a refresher on your basic mathematical skills, which are essential for success in later chapters. Some instructors will review this chapter with you, while others will leave it up to you to complete this chapter independently. Either way, this chapter is important and should be used to test your basic abilities. For example, it would be unfortunate if you got a leasing calculation wrong because you made an error by breaking an order of operation rule even though you understood basic leasing concepts.

Approach this chapter with confidence, and if you encounter any difficulties ensure that you master the concepts before moving on to future chapters.

1.1 Algebraic Expressions

If you are like most Canadians, your employer pays you biweekly. Assume you earn $12.00 per hour. How do you calculate your pay cheque every pay period? Your earnings are calculated as follows:

$$\$12.00 \times (\text{Hours worked during the biweekly pay period})$$

The hours worked during the biweekly pay period is the unknown variable. Notice that the expression appears lengthy when you write out the explanation for the variable. Algebra is a way of making such expressions more convenient to manipulate. To shorten the expression, making it easier to read, algebra assigns a letter or group of letters to represent the variable. In this case, you might choose h to represent "hours worked during the biweekly pay period." This rewrites the above expression as follows:

$$\$12.00 \times h \ \text{ or } \ \$12h$$

Unfortunately, the word algebra makes many people's eyes glaze over. But remember that algebra is just a way of solving a numerical problem. It demonstrates how the pieces of a puzzle fit together to arrive at a solution.

For example, you have used your algebraic skills if you have ever programmed a formula into Microsoft Excel. You told Excel there was a relationship between cells in your spreadsheet. Perhaps your calculation required cell A3 to be divided by cell B6 and then multiplied by cell F2. This is an algebraic equation. Excel then took your

algebraic equation and calculated a solution by automatically substituting in the appropriate values from the referenced cells (your variables).

As illustrated in the figure to the right, algebra involves integrating many interrelated concepts. This figure shows only the concepts that are important to business mathematics, which this textbook will present piece by piece. Your understanding of algebra will become more complete as more concepts are covered over the course of this book.

This section reviews the language of algebra, exponent rules, basic operation rules, and substitution. In Section 2.5 you will put these concepts to work in solving one linear equation for one unknown variable along with two linear equations with two unknown variables. Finally, in Section 2.6 you will explore the concepts of logarithms and natural logarithms.
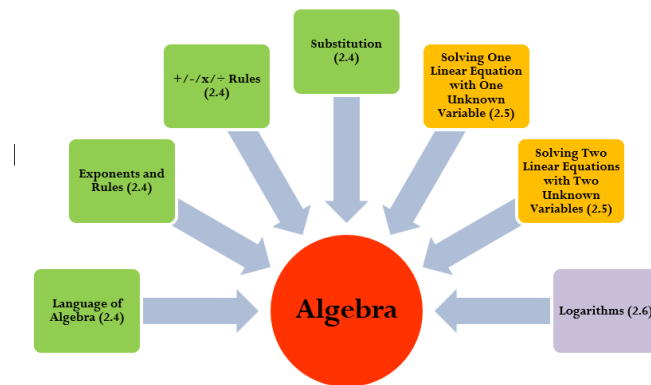
The Language of Algebra

Understanding the rules of algebra requires familiarity with four key definitions.

Algebraic Expression

A mathematical algebraic expression indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables. For example, the expression $12h says that you must take the hourly wage of $12 and multiply it by the hours worked. Note that the expression does not include an equal sign, or "=". It only tells you what to do and requires that you substitute a value in for the unknown variable(s) to solve. There is no one definable solution to the expression.

Algebraic Equation

A mathematical **algebraic equation** takes two algebraic expressions and equates them. This equation can be

solved to find a solution for the unknown variables. Examine the following illustration to see how algebraic expressions and algebraic equations are interrelated.

Algebraic Equation

6x+3y is the First Algebraic Expression: This expression on the left-hand side of the equation expresses a relationship between the variables x and y. By itself, any value can be substituted for x and y and any solution can be calculated.

4x+3 is the Second Algebraic Expression: This expression on the right-hand side of the equation expresses a relationship between x and a number. By itself, any value can be substituted for x and any solution can be generated.

$$6x+3y = 4x +3$$

= is The Equality: By having the equal sign placed in between the two algebraic expressions, they now become an algebraic equation. Now there would be a particular value for x and a particular value for y that makes both expressions equal to each other (try x = −0.75 and y = 1.125).

Term

In any algebraic expression, **terms** are the components that are separated by addition and subtraction. In looking at the example above, the expression $6x + 3y$ is composed of two terms. These terms are "$6x$" and "$3y$." A **nomial** refers to how many terms appear in an algebraic expression. If an algebraic expression contains only one term, like "$12.00h$," it is called a **monomial**. If the expression contains two terms or more, such as "$6x + 3y$," it is called a **polynomial**.

Factor

Terms may consist of one or more **factors** that are separated by multiplication or division signs. Using the $6x$ from above, it consists of two factors. These factors are "6" and "x"; they are joined by multiplication.

- If the factor is numerical, it is called the **numerical coefficient**.
- If the factor is one or more variables, it is called the **literal coefficient**

Factors

First Algebraic Expression: This first polynomial expression is composed of the two terms $\frac{7x}{3}$ and $4xy^2$.

Equal Sign: The equal sign joins the two algebraic expressions, turning this into an algebraic equation.

Second Algebraic Expression: This second polynomial expression consists of the two terms x^3 and $-2y$.

First Term: This term consists of two factors, where the numerical coefficient is $\frac{7}{3}$ and the literal coefficient is x.

$$\frac{7x}{3} + 4xy^2 = x^3 - 2y$$

Second Term: This term consists of three factors where the numerical coefficient is 4 and the literal coefficients are x and y^2.

First Term: This term consists of two factors, which are the literal coefficient x^3 and a numerical coefficient of 1 (recall that in math you do not write the 1 when writing $1x^3$ since it doesn't change anything).

Second Term: This term consists of two factors, where the numerical coefficient is -2 and the literal coefficient is y.

Exponents

Exponents are widely used in business mathematics and are integral to financial mathematics. When applying compounding interest rates to any investment or loan, you must use exponents.

Exponents are a mathematical shorthand notation that indicates how many times a quantity is multiplied by itself. The format of an exponent is illustrated below.

Exponents

Base is the quantity: This is the quantity that is to be multiplied by itself.

Power is the product: This is the result of performing the multiplication indicated, or the answer.

$$\text{Base}^{\text{Exponent}} = \text{Power}$$

Exponent is the multiplication factor: This is the number that indicates how many times the base is to be multiplied by itself.
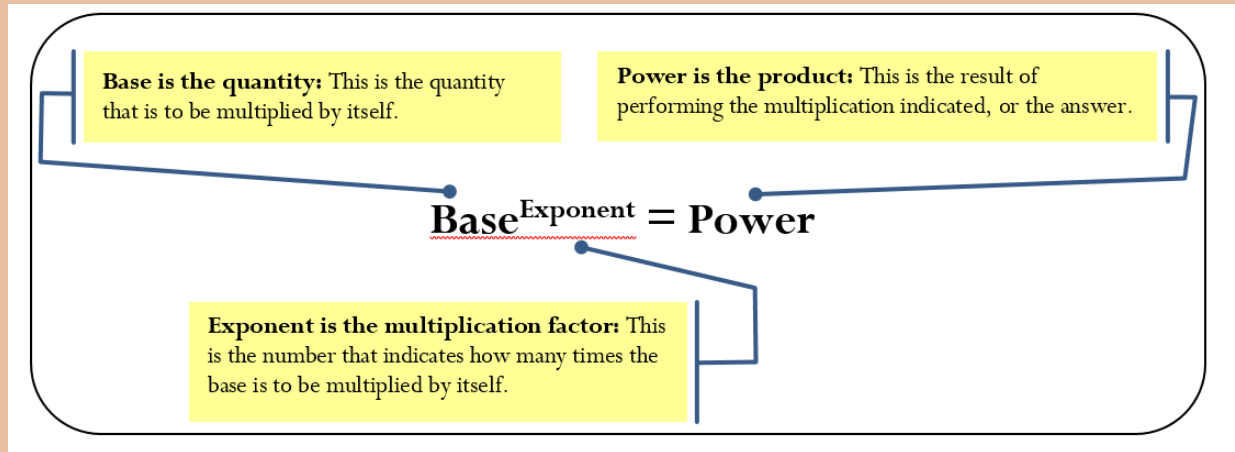
Assume you have 23 = 8. The exponent of 3 says to take the base of 2 multiplied by itself three times, or 2 × 2 × 2. The power is 8. The proper way to state this expression is "2 to the exponent of 3 results in a power of 8."

How It Works

Many rules apply to the simplification of exponents, as shown in the table below.

Rule	Illustration	Explanation
1. Multiplication	$y^a \times y^b = y^{a+b}$	If the bases are identical, add the exponents and retain the unchanged base.
2. Division	$\dfrac{y^a}{y^b} = y^{a-b}$	If the bases are identical, subtract the exponents and retain the unchanged base.
3. Raising powers to exponents	$(y^b z^c)^a = y^{b \times a} z^{c \times a}$ or $\left(\dfrac{y^b}{z^c}\right)^a = \dfrac{y^{b \times a}}{z^{c \times a}}$	If a *single* term is raised to an exponent, each factor must retain its base and you must multiply the exponents for each by the raised exponent. Note that if the expression inside the brackets has more than one term, such as $(y^b + z^c)a$, which has two terms, you cannot multiply the exponent a inside the brackets.
4. Zero exponents	$y^0 = 1$	Any base to a zero exponent will always produce a power of 1. This is explained later in this section when you review the concept of algebraic division.
5. Negative exponents	$y{-}a = \dfrac{1}{y^a}$	The negative sign indicates that the power has been shifted between the numerator and denominator. It is commonly used in this textbook to simplify appearances. Note that on the BAII Plus calculator, keying in a negative exponent requires you to key in the value of y, press y^x, enter the value of a, press ±, and then press = to calculate.
6. Fractional exponents	$y^{\frac{a}{b}} = \sqrt[b]{y^a}$	A fractional exponent is a different way of writing a radical sign. Note that the base is first taken to the exponent of a, then the root of b is found to get the power. For example, This is the same as . To key in a fractional exponent on the BAII Plus calculator, input the value of y, press y^x, open a set of brackets, key in a ÷ b, close the brackets, and press = to calculate.

Important Notes

Recall that mathematicians do not normally write the number 1 when it is multiplied by another factor because it doesn't change the result. The same applies to exponents. If the exponent is a 1, it is generally not written because any number multiplied by itself only once is the same number. For example, the number 2 could be written as 2^1, but the power is still 2. Or take the case of $(yz)^a$. This could be written as $(y^1 z^1)a$, which when simplified becomes $(y^1 \times a z^1)^a$ or $y^a z^a$. Thus, even if you don't see an exponent written, you know that the value is 1.

Example 1.1A Exponents in Algebra

[table id=2 /]

Addition and Subtraction

Simplification of unnecessarily long or complex algebraic expressions is always preferable to increase understanding and reduce the chances of error. For example, assume you are a production manager looking to order bolts for a product that you make. Your company makes three products, all in equal quantity. Product A requires seven bolts, Product B requires four bolts, and Product C requires fourteen bolts. If q represents the quantity of products required, you need to order $7q + 4q + 14q$ bolts. This expression requires four calculations to solve every time (each term needs to be multiplied by q and you then need to add everything together). With the algebra rules that follow, you can simplify this expression to $25q$. This requires only one calculation to solve. So what are the rules?

How It Works

In math, terms with the same literal coefficients are called **like terms**. Only terms with the identical literal coefficients may be added or subtracted through the following procedure:

Step 1: Simplify any numerical coefficients by performing any needed mathematical operation or converting fractions to decimals. For example, terms such as $\frac{1}{2}y$ should become $0.5y$.

Step 2: Add or subtract the numerical coefficients of like terms as indicated by the operation while obeying the rules of BEDMAS.

Step 3: Retain and do not change the common literal coefficients. Write the new numerical coefficient in front of the retained literal coefficients.

From the previous example, you require $7q + 4q + 14q$ bolts. Notice that there are three terms, each of which has the same literal coefficient. Therefore, you can perform the required addition.

Step 1: All numerical coefficients are simplified already. Skip to step 2.

Step 2: Take the numerical coefficients and add the numbers: 7 + 4 + 14 equals 25.

Step 3: Retain the literal coefficient of q. Put the new numerical coefficient and literal coefficient together. Thus, $25q$. Therefore $7q + 4q + 14q$ is the same as $25q$.

Things To Watch Out For

A common mistake in addition and subtraction is combining terms that do not have the same literal coefficient. You need to remember that the literal coefficient *must* be identical. For example, $7q$ and $4q$ have the identical literal coefficient of q. However, $7q$ and $4q^2$ have different literal coefficients, q and q^2, and cannot be added or subtracted.

Paths To Success

Remember that if you come across a literal coefficient with no number in front of it, that number is assumed to be a 1. For example, x has no written numerical coefficient, but it is the same as $1x$. Another example would be $\frac{x}{4}$ is the same as $\frac{1x}{4}$ or $\frac{1}{4}x$.

On a similar note, mathematicians also don't write out literal coefficients that have an exponent of zero. For example, $7x^0$ is just $7(1)$ or 7. Thus, the literal coefficient is always there; however, it has an exponent of zero. Remembering this will help you later when you multiply and divide in algebra.

Give It Some Thought

Examine the following algebraic expressions and indicate how many terms can be combined through addition and subtraction. No calculations are necessary. Do not attempt to simplify.

1. $\frac{3}{2}x + 4x^2 - 10x - 2y + \frac{x}{3}$
2. $23g^2 - \frac{17g^2}{5} + g^4 + g^2 - \frac{2}{3}g^2 - 0.15g + g^3$

Solutions:

1. Three terms (all of the x)
2. Four terms (all of the $g2$)

Example 1.1B: Addition and Subtraction in Algebra

[table id=3 /]

Multiplication

How It Works

Follow these steps to multiple algebraic expressions:

Step 1: Check to see if there is any way to simplify the algebraic expression first. Are there any like terms you can combine? For example, you can simplify $(3x + 2 + 1)(x + x + 4)$ to $(3x + 3)(2x + 4)$ before attempting the multiplication.

Step 2: Take *every* term in the first algebraic expression and multiply it by *every* term in the second algebraic expression. This means that numerical coefficients in both terms are multiplied by each other, and the literal coefficients in both terms are multiplied by each other. It is best to work methodically from left to right so that you do not miss anything. Working with the example, in $(3x + 3)(2x + 4)$ take the first term of the first expression, $3x$, and multiply it by $2x$ and then by 4. Then move to the second term of the first expression, 3, and multiply it by $2x$ and then by 4 (see the figure).

$$(3x + 3)(2x + 4)$$

$$(3x)(2x) + (3x)(4) + 3(2x) + 3(4)$$
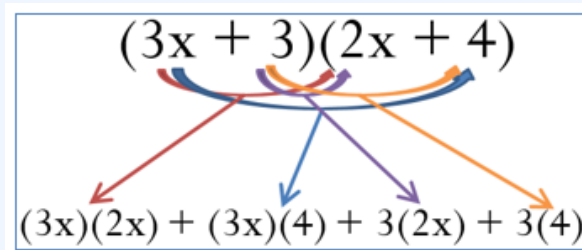
It becomes: $6x2 + 12x + 6x + 12$

Step 3: Perform any final steps of simplification by adding or subtracting like terms as needed. In the example, two terms contain the literal coefficient x, so you simplify the expression to $6x2 + 18x + 12$.

Important Notes

If the multiplication involves more than two expressions being multiplied by each other, it is easiest to work with only one pair of expressions at a time starting with the leftmost pair. For example, if you are multiplying $(4x + 3)(3x)(9y + 5x)$, resolve $(4x + 3)(3x)$ first. Then take the solution, keeping it in brackets since you have not completed the math operation, and multiply it by $(9y + 5x)$. This means you are required to repeat step 2 in the multiplication procedure until you have resolved all the multiplications.

Things To Watch Out For

The negative sign causes no end of grief for a lot of people when working with multiplication. First, if a numerical coefficient is not written explicitly, it is assumed to be 1. For example, look at $2(4a + 6b) - (2a - 3b)$. This is the same as $2(4a + 6b) + (-1)(2a - 3b)$.

When you multiply a negative through an expression, all signs in the brackets will change. Continuing with the second term in the above example, $-(2a - 3b)$ becomes $-2a + 3b$. The expression then looks like $2(4a + 6b) - 2a + 3b$.

Paths To Success

The order in which you write the terms of an algebraic expression does not matter so long as you follow all the rules of BEDMAS. For example, whether you write 3×4 or 4×3, the answer is the same because you can do multiplication in any order. The same applies to $4 + 3 - 1$ or $3 - 1 + 4$. Now let's get more complex. Whether you write $3x2 + 5x - 4$ or $5x - 4 + 3x2$, the answer is the same as you have not violated any rules of BEDMAS. You still multiply first and add last from left to right.

Although the descending exponential format for writing expressions is generally preferred, for example, $3x2 + 5x + 4$, which lists literal coefficients with higher exponents first, it does not matter if you do this or not. When checking your solutions against those given in this textbook, you only need to ensure that each of your terms matches the terms in the solution provided.

Example 1.1C: Algebraic Multiplication

[table id=4 /]

Division

You are often required to divide a monomial into either a monomial or a polynomial. In instances where the denominator consists of a polynomial, it is either not possible or extremely difficult to simplify the expression algebraically. Only division where denominators are monomials are discussed here.

How It Works

To simplify an expression when its denominator is a monomial, apply these rules:

Step 1: Just as in multiplication, determine if there is any way to combine like terms before completing the division. For example, with $\frac{3ab + 3ab - 3a^2b + 9ab^2}{3ab}$ you can simplify the numerator to $\frac{6ab - 3a^2b + 9ab^2}{3ab}$

Step 2: Take *every* term in the numerator and divide it by the term in the denominator. This means that you must divide both the numerical and the literal coefficients. As with multiplication, it is usually best to work methodically from left to right so that you do not miss anything. So in our example we get:

$\frac{6ab}{3ab} - \frac{3a^2b}{3ab} + \frac{9ab^2}{3ab}$

(2)(1)(1) − (1)(*a*)(1) + (3)(1)(*b*)

2 − *a* + 3*b*

Step 3: Perform any final simplification by adding or subtracting the like terms as needed. As there are no more like terms, the final expression remains 2 − *a* + 3*b*.

You may have heard of an outcome called "cancelling each other out." For example, in resolving the division $\frac{4a}{4a}$ many people would say that the terms cancel each other out. Many people will also mistakenly interpret this to mean that the quotient is zero and say that $\frac{4a}{4a} = 0\%$. In fact, when terms cancel each other out the quotient is one, not zero. The numerical coefficient is $\frac{4}{4} = 1$. The literal coefficient is $\frac{a}{a} = 1$ Thus, $\frac{4a}{4a} = (1)(1) = 1$. This also explains why a zero exponent equals one: $\frac{a^1}{a^1} = a^{1-1} = a^0 = 1..$

Paths To Success

A lot of people dislike fractions and find them hard to work with. Remember that when you are simplifying any algebraic expression you can transform any fraction into a decimal. For example, if your expression is $\frac{2x}{5} + \frac{3x}{4}$, you can convert the fraction into decimals: $0.4x + 0.75x$. In this format, it is easier to solve.Substitution

The ultimate goal of algebra is to represent a relationship between various variables. Although it is beneficial to simplify these relationships where possible and shorten the algebraic expressions, in the end you want to calculate a solution. **Substitution** involves replacing the literal coefficients of an algebraic expression with known numerical values. Once the substitution has taken place, you solve the expression for a final value.

Example 1.1 D

[table id=5 /]

How It Works

Follow these steps to perform algebraic substitution:

Step 1: Identify the value of your variables. Assume the algebraic equation is $PV = \frac{FV}{1 + rt}$. You need to calculate the value of PV. It is known that FV = \$5,443.84, r = 0.12, and t = $\frac{270}{365}$.

Step 2: Take the known values and insert them into the equation where their respective variables are located, resulting in .

Step 3: Resolve the equation to solve for the variable. Calculate $PV = 5{,}443.84\frac{1}{1.088767} = 5{,}000.00$

Things To Watch Out For

It is common in algebra to represent a variable with more than just one letter. As you can see from the example above, FV is a variable, and it represents *future value*. This should not be interpreted to be two variables, F and V. Similarly, PMT represents *annuity payment*. When you learn new formulas and variables, take careful note of how a variable is represented.

As well, some literal coefficients have subscripts. For example, you could see d_1 and d_2 in the same formula. What sometimes happens is that there is more than one value for the same variable. As you will learn in Chapter 6 about merchandising, when you buy an item you may receive more than one discount rate (what d stands for). Therefore, the first discount gets a subscript of 1, or d_1, and the second discount gets a subscript of 2, or d_2. This allows you to distinguish between the two values in the equation and substitute the correct value in the correct place.

Paths To Success

If you are unsure whether you have simplified an expression appropriately, remember that you can make up your own values for any literal coefficient and substitute those values into both the original and the simplified expressions. If you have obeyed all the rules and simplified appropriately, both expressions will produce the same answer. For example, assume you simplified $2x + 5x$ into $7x$, but you are not sure if you are right. You decide to let $x = 2$. Substituting into $2x + 5x$, you get $2(2) + 5(2) = 14$. Substituting into your simplified expression you get $7(2) = 14$. Since both expressions produced the same answer, you have direct confirmation that you have simplified correctly.

Example 1.1 E

[table id=6 /]

Exercises

Mechanics

For questions 1–4, simplify the algebraic expressions.

1. $2a - 3a + 4 + 6a - 3$

2. $5b(4b + 2)$

3. $\dfrac{6x^3 + 12x^2 + 13.5x}{3x}$

4. $(1 + i)^3 \times (1 + i)^{14}$

5. Evaluate the power $8^{\frac{2}{3}}$

6. Substitute the known variables and solve for the unknown variable:

I = Prt where P = $2,500, r = 0.06, and t = $\frac{135}{365}$

Applications

For questions 7–11, simplify the algebraic expressions.

7. $\left(6r^2 + 10 - 6r + 4r^2 - 3\right) - \left(-12r - 5r^2 + 2 + 3r\right)$

8. $\left[\frac{5x^9 + 3x^9}{2x}\right]^5$

9. $\frac{t}{2} + 0.75t - t^3 + \frac{5t^4}{t} - \frac{2\left(t + t^3\right)}{4}$

10. $\frac{14(1 + i) + 21(1 + i)^4 - 35(1 + i)^7}{7(1 + i)}$

11. $\frac{R}{1 + 0.08 \times \frac{183}{365}} + 3R\left(1 + 0.08 \times \frac{52}{365}\right)$

12. Evaluate the power: $\left[\left(\frac{2}{5}\right)^2\right]^2$

For questions 13 and 14, substitute the known variables and solve for the unknown variable.

13. $PV = \frac{FV}{(1+i)^N}$ where FV = $3,417.24, i = 0.05, and N = 6

14. $PMT = \frac{FV}{\left[\frac{(1+i)^N - 1}{i}\right]}$ where FV = $10,000, N = 17, and i = 0.10

Challenge, Critical Thinking, & Other Applications

For questions 15–17, simplify the algebraic expressions.

15. $\left[\frac{10a^2b^3c^4}{5b^3c^4}\right]^2 + 6\left(a^8\right)1/2 - \left(3a^2 + 6\right)\left(3a^2 - 3\right)$

16. $\frac{-(5x + 4y + 3)(2x - 5y) - (10x - 2y)(2y + 3)}{5}$

17. $\frac{(-3z)^3\left(3z^2\right)^2}{\left(2z^3\right)^{-4}}$

18. Substitute the known variables and solve for the unknown variable.

$FV_{ORD} =$
where PMT = $500, i = 0.05, Δ% = 0.02, CY = 2, PY = 4, and N = 20

For questions 19-20, evaluate the expression.

19. $50,000 \times \left(1 + \dfrac{0.10}{12}\right)^{-27}$

20. $995 \left[\dfrac{1 - (1 + 0.02)^{13}\left(1 + \frac{0.09}{4}\right)^{-13}}{\frac{0.09}{4} - 0.02} \right]$

1.2 Linear Equations: Manipulating and Solving

You are shopping at Old Navy for seven new outfits. The price points are $10 and $30. You really like the $30 outfits; however, your total budget can't exceed $110. How do you spend $110 to acquire all the needed outfits without exceeding your budget while getting as many $30 items as possible?

This is a problem of linear equations, and it illustrates how you can use them to make an optimal decision. Let L represent the quantity of clothing at the low price point of $10, and H represent the quantity of clothing at the high price point of $30. This results in the following algebraic equations:

$L + H = 7$ (the total number of outfits you need)

$\$10L + \$30H = \$110$ (your total budget)

By simultaneously solving these equations you can determine how many outfits at each price point you can purchase.

You will encounter many situations like this in your business career, for example, in making the best use of a manufacturer's production capacity. Assume your company makes two products on the same production line and sells all its output. Each product contributes differently to your profitability, and each product takes a different amount of time to manufacture. What combination of each of these products should you make such that you operate your production line at capacity while also maximizing the profits earned? This section explores how to solve linear equations for unknown variables.

Understanding Equations

To manipulate algebraic equations and solve for unknown variables, you must first become familiar with some important language, including linear versus nonlinear equations and sides of the equation.

Parts of an Equation

4x+3 is the Left Side: Every equation has two sides. Everything to the left side of the equal sign is known as the **left side of the equation**.

−2x − 3 is the Right Side: Everything to the right side of the equal sign is known as the **right side of the equation**.

$$4x + 3 = -2x - 3$$

x is the Variable x: Observe two important features of this variable:

1. Note that the variable has an exponent of 1. So if you were to plot each algebraic expression onto a Cartesian coordinate system, the expressions would form straight lines (see the left-hand figure next page). Such equations are called **linear equations**. Because each equation has only one variable, there is only one solution that makes the equation true. In business mathematics, this is the most common situation you will encounter.

2. If the exponent is anything but a one, the equation is a **nonlinear equation** since the graph of the expression does not form a straight line. The right-hand figure (next page) illustrates two algebraic expressions where the exponent is other than a 1.

Nonlinear Equations
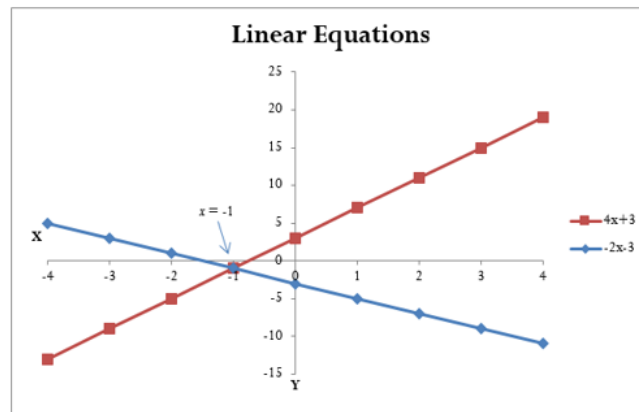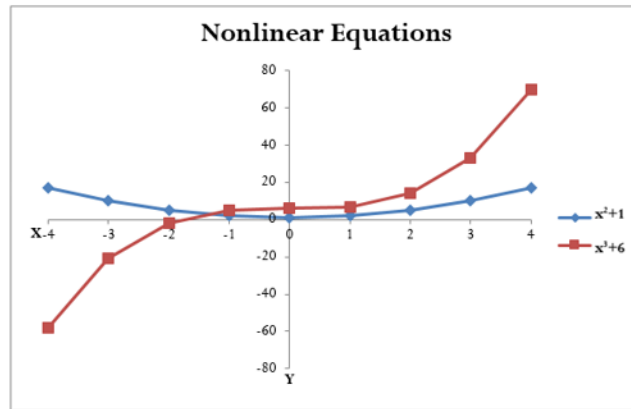
The goal in manipulating and solving a linear equation is to find a value for the unknown variable that makes the equation true. If you substitute a value of $x = -1$ into the above example, the left-hand side of the equation equals the right-hand side of the equation (see Figure 2.17). The value of $x = -1$ is known as the **root**, or solution, to the linear equation.

Solving One Linear Equation with One Unknown Variable

In your study of solving linear equations, you need to start by manipulating a single equation to solve for a single unknown variable. Later in this section you will extend from this foundation to the solution of two linear equations with two unknowns.

How It Works

To determine the root of a linear equation with only one unknown variable, apply the following steps:

Step 1: Your first goal is to separate the terms containing the literal coefficient from the terms that only have numerical coefficients. Collect all of the terms with literal coefficients on only one side of the equation and collect all of the terms with only numerical coefficients on the other side of the equation. It does not matter which terms go on which side of the equation, so long as you separate them.

To move a term from one side of an equation to another, take the mathematical opposite of the term being moved and add it to both sides. For example, if you want to move the +3 in $4x + 3 = -2x - 3$ from the left-hand side to the right-hand side, the mathematical opposite of +3 is –3. When you move a term, remember the cardinal rule: What you do to one side of an equation

you must also do to the other side of the equation. Breaking this rule breaks the equality in the equation.

Step 2: Combine all like terms on each side and simplify the equation according to the rules of algebra.

Step 3: In the term containing the literal coefficient, reduce the numerical coefficient to a 1 by dividing both sides of the equation by the numerical coefficient.

Important Notes

When you are unsure whether your calculated root is accurate, an easy way to verify your answer is to take the original equation and substitute your root in place of the variable. If you have the correct root, the left-hand side of the equation equals the right-hand side of the equation. If you have an incorrect root, the two sides will be unequal. The inequality typically results from one of the three most common errors in algebraic manipulation:

1. The rules of BEDMAS have been broken.
2. The rules of algebra have been violated.
3. What was done to one side of the equation was not done to the other side of the equation.

Things To Watch Out For

When you move a term from one side of the equation to another using multiplication or division, remember that this affects *each and every term on both sides of the equation*. To remove the x from the denominator in the following equation, multiply both sides of the equation by x:

$\frac{5}{x} + \frac{1}{x} = \frac{2}{x} + 2$ becomes $x\left(\frac{5}{x} + \frac{1}{x}\right) = \left(\frac{2}{x} + 2\right)x$ which then becomes $5 + 1 = 2 + 2x$

Multiplying every term on both sides by x maintains the equality.

Paths To Success

Negative numbers can cause some people a lot of grief. In moving terms from a particular side of the equation, many people prefer to avoid negative numerical coefficients in front of literal coefficients. Revisiting $4x + 3 = -2x - 3$, you could move the $4x$ from the left side to the right side by subtracting $4x$ from both sides. However,

on the right side this results in $-6x$. The negative is easily overlooked or accidentally dropped in future steps. Instead, move the variable to the left side of the equation, yielding a positive coefficient of $6x$.

Example 1.2 A: How to Solve the Opening Example

[table id=7 /]

Example 1.2 B: Solving a Linear Equation with One Unknown Variable

The Root of the Equation

$m = 12$

0.75m + 2m

4m − 15
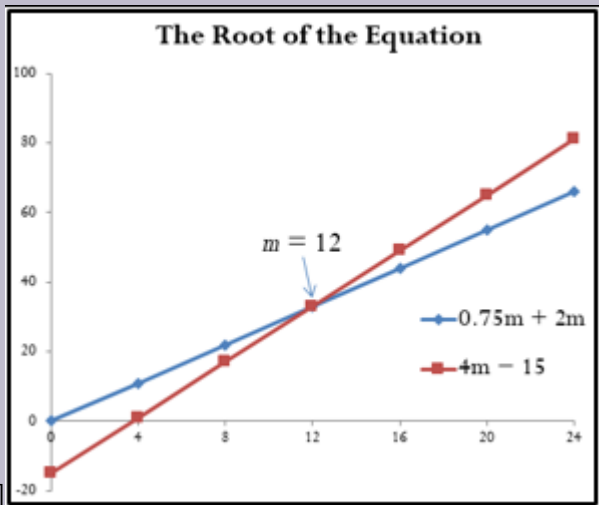
[table id=9 /]

Example 1.2 C: Solving a Linear Equation with One Unknown Variable Containing Fractions

The Root of the Equation

$$0.625b + 0.4$$

$$0.85 - 0.25b$$

$$b = 0.5143$$

[table id=10 /]

Solving Two Linear Equations with Two Unknown Variables

The manipulation process you have just practiced works well for solving one linear equation with one variable. But what happens if you need to solve two linear equations with two variables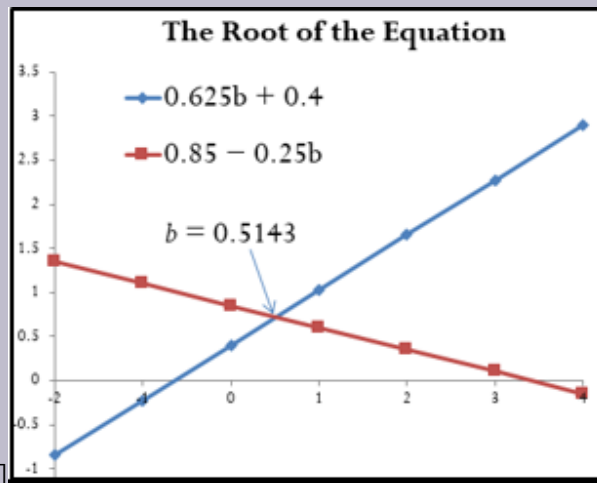 simultaneously? Remember when you were at Old Navy purchasing seven outfits earlier in this chapter (equation 1)? You needed to stay within a pricing budget (equation 2). Each equation had two unknown variables representing the number of lower-priced and higher-priced outfits.

The goal is to reduce two equations with two unknowns into a single linear equation with one unknown. Once this transformation is complete, you then identify the unknown variable by applying the three-step procedure for solving one linear equation, as just discussed.

When you work with two linear equations with two unknowns, the rules of algebra permit the following two manipulations:

1. What you do to one side of the equation must be done to the other side of the equation to maintain the equality. Therefore, you can multiply or divide any equation by any number without changing the root of the equation. For example, if you multiply all terms of $x + y = 2$ by 2 on both sides, resulting in $2x + 2y = 4$, the equality of the equation remains unchanged and the same roots exist.
2. Terms that are on the same side of an equation can be added and subtracted between equations by combining like terms. Each of the two equations has a left side and right side. This rule permits taking the left side of the first equation and either adding or subtracting like terms on the left side of the second equation. When you perform this action, remember the first rule above. If you add the left sides of the

equations together, you then must add the right side of both equations together to maintain equality.

How It Works

Follow these steps to solve two linear equations with two unknown variables:

Step 1: Write the two equations one above the other, vertically lining up terms that have the same literal coefficients and terms that have only the numerical coefficient. If necessary, the equations may need to be manipulated such that all of the literal coefficients are on one side with the numerical coefficients on the other side.

Step 2: Examine your two equations. Through multiplication or division, make the numerical coefficient on one of the terms containing a literal coefficient exactly equal to its counterpart in the other equation.

Step 3: Add or subtract the two equations as needed so as to eliminate the identical term from both equations.

Step 4: In the new equation, solve for the last literal coefficient.

Step 5: Substitute the root of the known literal coefficient into either of the two original equations. If one of the equations takes on a simpler structure, pick that equation.

Step 6: Solve your chosen equation for the other literal coefficient.

Paths To Success

Sometimes it is unclear exactly how you need to multiply or divide the equations to make two of the terms identical. For example, assume the following two equations:

$4.9x + 1.5y = 38.3$

$2.7x - 8.6y = 17.8$

If the goal is to make the terms containing the literal coefficient x identical, there are two alternative solutions:

1. Take the larger numerical coefficient for x and divide it by the smaller numerical coefficient. The resulting number is the factor for multiplying the equation containing the smaller numerical coefficient. In this case, $4.9 \div 2.7 = 1.\overline{814}$. Multiply all terms in the second equation by $1.\overline{814}$ to make the numerical coefficients for x equal to each other, resulting in this pair of equations:

$4.9x + 1.5y = 38.3$

$4.9x − 15.6\overline{074}\, y = 32.3\overline{037}$ (every term multiplied by $1.\overline{814}$)

2. Take the first equation and multiply it by the numerical coefficient in the second equation. Then take the second equation and multiply it by the numerical coefficient in the first equation. In this case, multiply all terms in the first equation by 2.7. Then multiply all terms in the second equation by 4.9.

$13.23x + 4.05y = 103.41$ (every term multiplied by 2.7)

$13.23x − 42.14y = 87.22$ (every term multiplied by 4.9)

Note that both approaches successfully result in both equations having the same numerical coefficient in front of the literal coefficient x.

Ultimately, every pairing of linear equations with two unknowns can be converted into a single equation through substitution. To make the conversion, do the following:

1. Solve either equation for one of the unknown variables.
2. Take the resulting algebraic expression and substitute it into the other equation. This new equation is solvable for one of the unknown variables.
3. Substitute your newfound variable into one of the original equations to determine the value for the other unknown variable.

Take the following two equations:

$a + b = 4$ $\qquad\qquad$ $2a + b = 6$

1. Solving the first equation for a results in $a = 4 − b$.
2. Substituting the expression for a into the second equation and solving for b results in $2(4 − b) + b = 6$, which solves as $b = 2$.
3. Finally, substituting the root of b into the first equation to calculate a gives a + 2 = 4 resulting in $a = 2$. Therefore, the roots of these two equations are $a = 2$ and $b = 2$.

Example 1.2D: Buying Those Outfits

Meeting Clothing and Budget Constraints
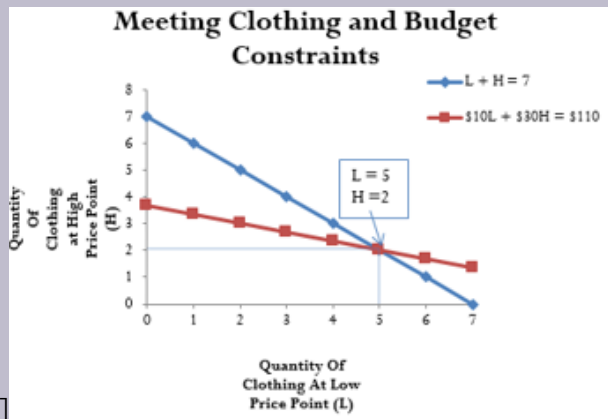
[table id=11 /]

Paths To Success

One of the most difficult areas of mathematics involves translating words into mathematical symbols and operations. To assist in this translation, the table below lists some common language and the mathematical symbol that is typically associated with the word or phrase.

Language			Math Symbol	Language			Math Symbol
Sum Addition	In addition to In excess	Increased by Plus	$+$	Subtract Decreased by Diminished by	Less Minus	Difference Reduced by	$-$
Divide Division	Divisible Quotient	Per	$\div$	Multiplied by Times	Percentage of	Product of Of	$\times$
Becomes Is/Was/Were	Will be	Results in Totals	$=$	More than Greater than			$>$
Less than	Lower than		$<$	Greater than or equal to			$\geq$
Less than or equal to			$\leq$	Not equal to			$\neq$

Example 1.2E: Solving Two Linear Equations with Two Unknowns
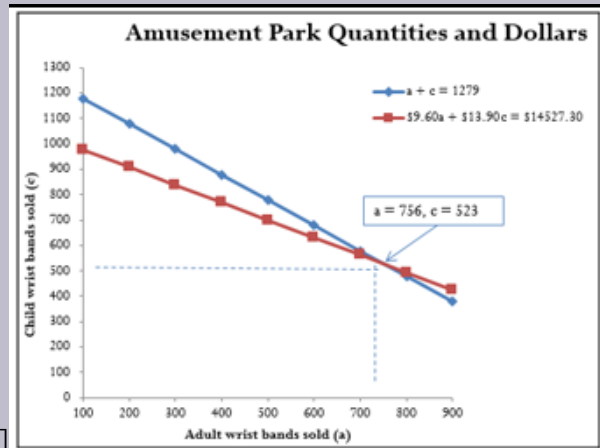
[table id=12 /]

Exercises

Mechanics

Solve the following equations for the unknown variable.

1. $3(x - 5) = 15$

2. $12b - 3 = 4 + 5b$

3. $0.75(4m + 12) + 15 - 3(2m + 6) = 5(-3m + 1) + 25$

Solve each of the following pairs of equations for both unknown variables.

4. $x + y = 6$

$3x - 2y = 8$

5. $4h - 7q = 13$

$6h + 3q = 33$

6. $0.25a + \frac{5b}{2} = 3.5$

$\frac{3a}{4} - \frac{b}{0.2} = 3$

Applications

In questions 7 and 8, solve the equation for the unknown variable.

7. $\frac{4y}{1.025^4} + y - 2y(1.05)^2 = 1,500$
8. $2,500(1 + 0.06t) + \$1,000(1 + 0.04t) = \$3,553.62$

For exercises 9–14, read each question carefully and solve for the unknown variable(s).

9. Pamela is cooking a roast for a 5:30 p.m. dinner tonight. She needs to set a delay timer on her oven. The roast takes 1 hour and 40 minutes to cook. The time right now is 2:20 p.m. How long of a delay must she set the oven for (before it automatically turns on and starts to cook the roast)?

10. In 2010, 266 million North Americans were using the Internet, which represented a 146.3% increase in Internet users over the year 2000. How many North American Internet users were there in 2000?

11. A human resource manager is trying to estimate the cost of a workforce accident. These costs usually consist of direct costs (such as medical bills, equipment damage, and legal expenses) and indirect costs (such as decreased output, production delays, and fines). From past experience, she knows that indirect costs average six times as much as direct costs. If she estimates the cost of an accident to be $21,000, what is the direct cost of the accident?

12. In 2011, Canadian federal tax rates were 0% on the first $10,527 of gross income earned, 15% on the next $31,017, 22% on the next $41,544, 26% on the next $45,712, and 29% on anything more. If a taxpayer paid $28,925.35 in federal tax, what was her gross annual income for 2011?

13. St. Boniface Hospital raises funds for research through its Mega Lottery program. In this program, 16,000 tickets are available for purchase at a price of one for $100 or three for $250. This year, the lottery sold out with sales of $1,506,050. To better plan next year's lottery, the marketing manager wants to know how many tickets were purchased under each option this year.

14. An accountant is trying to allocate production costs from two different products to their appropriate ledgers. Unfortunately, the production log sheet for last week has gone missing. However, from other documents he was able to figure out that 1,250 units in total were produced last week. The production machinery was run for 2,562.5 minutes, and he knows that Product A takes 1.5 minutes to manufacture while Product B takes 2.75 minutes to manufacture. How many units of each product were produced last week?

Challenge, Critical Thinking, & Other Applications

15. Jacob owns 15,000 shares in a corporation, which represents 2% of all issued shares for the company. He sold of his shares to another investor for $7,800. What is the total value for all of the shares issued by the company?

16. Two cellphone companies are offering different rate plans. Rogers is offering $19.99 per month, which includes a maximum of 200 weekday minutes plus $0.35 for every minute above the maximum. TELUS is offering $39.99 for a maximum 300 weekday minutes, but it charges $0.10 for every minute above the maximum. Above how many minutes would TELUS be the better choice?

17. Marianne, William, Hendrick, and Charlotte have all decided to go into business together. They need $175,000 in initial capital funding. William was able to contribute 20% less than Marianne, Hendrick contributed 62.5% more than William, and Charlotte contributed $5,000 less than half as much as Marianne. How much did each partner contribute to the initial funds?

18. A mall is being constructed and needs to meet the legal requirements for parking availability. Parking laws require one parking stall for every 100 square feet of retail space. The mall is designed to have 1,200,000 square feet of retail space. Of the total parking stalls available, 2% need to be handicap accessible, there need to be three times as many small car spaces as handicap spaces, RV spaces need to be one-quarter of the number of small car spaces, and the rest of the spaces are for regular stalls. How many of each type of parking space does the mall require?

19. Simplify the following equation into the format of #z = # and find the root. Verify the solution through substitution.

$$z \left(1 + 0.073 \times \frac{280}{365}\right) - \frac{z}{1+0.073 \times \frac{74}{365}} + 1{,}000 = 2{,}764.60$$

20. Find the roots for the following pairs of equations. Verify the solution through substitution into both equations.

$3\frac{4}{5}q + 0.18r = 12.2398$

$-5.13q - \frac{13r}{5} = -38.4674$

1.3 Natural Logarithms

Your investment is earning compound interest of 6% per year. You wonder how long it will take to double in value. Answering this question requires you to solve an equation in which the unknown variable is located in the exponent instead of the base position. To find the answer, you need logarithms, which are reviewed in this section and will be applied in Chapters 9 and 11.

Logarithms

A **logarithm** is defined as the exponent to which a base must be raised to produce a particular power. Although the base can assume any value, two values for the base are most often used.

1. **A Base Value of 10**. This is referred to as a **common logarithm**. Thus, if you have $10^2 = 100$, then 2 is the common logarithm of 100 and this is written as $\log_{10}(100) = 2$, or just $\log(100) = 2$ (you can assume the 10 if the base is not written). The format for a common logarithm can then be summarized as

$$\log(\text{power}) = \text{exponent}$$

If you have $10x = y$, then $\log(y) = x$.

2. **A Base Value of e**. This is referred to as a **natural logarithm**. In mathematics, there is a known constant e, which is a nonterminating decimal and has an approximate value of $e = 2.71828182845$. If you have $e^3 = 20.085537$, then 3 is the natural logarithm of 20.085537 and you write this as $\ln(20.085537) = 3$. The format for a natural logarithm is then summarized as:

$$\ln(\text{power}) = \text{exponent}$$

If you have $e^4 = 54.59815$, then $\ln(54.59815) = 4$

Common logarithms have been used in the past when calculators were not equipped with power functions. However, with the advent of computers and advanced calculators that have power functions, the natural logarithm is now the most commonly used format. From here on, this textbook focuses on natural logarithms only.

Properties of Natural Logarithms

Natural logarithms possess six properties:

1. **The natural logarithm of 1 is zero**. For example, if 1 is the power and 0 is the exponent, then you have $e^0 = 1$. This obeys the laws of exponents discussed in Section 2.4 of this chapter.

2. **The natural logarithm of any number greater than 1 is a positive number**. For example, the natural logarithm of 2 is 0.693147, or $e^{0.693147} = 2$.

3. **The natural logarithm of any number less than 1 is a negative number**. For example, the natural logarithm of 0.5 is −0.693147, or $e^{-0.693147} = 0.5$. Similar to property 1, this obeys the laws of exponents discussed in Section 2.4, where $e^{-0.693147} = \frac{1}{e^{0.693147}}$, always producing a proper fraction.

4. **A natural logarithm cannot be less than or equal to zero**. Since e is a positive number with an exponent, there is no value of the exponent that can produce a power of zero. As well, it is impossible to produce a negative number when the base is positive.

5. **The natural logarithm of the quotient of two positive numbers is** $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$. For example, $\ln(1.\bar{3}) = 9.903487 - 9.615805$ or $0.287682 = 0.287682$

6. **The natural logarithm of a power of a positive base is** . This property allows you to relocate the exponent down into the base. . Demonstrating this principle,

$$\ln(x^y) = y(\ln x)\%$$

$$\ln(1.05^6) = 6 \times \ln(1.05)$$

$$\ln(1.340095) = 6 \times 0.048790$$

$$0.292741 = 0.292741$$

Important Notes

Applying natural logarithms on your TI BAII Plus calculator requires two steps.

1. Input the power.
2. With the power still on the display, press the LN key located in the left-hand column of your keypad. The solution on the display is the value of the exponent. If you key in an impossible value for the natural logarithm, an "Error 2" message is displayed on the screen.

If you know the exponent and want to find out the power, remember that e^x = power. This is called the anti-log function. Thus, if you know the exponent is 2, then $e^2 = 7.389056$. On your calculator, the anti-log can be located on the second shelf directly above the LN button. To access this function, key in the exponent first and then press 2nd e^x.

Paths To Success

You do not have to memorize the mathematical constant value of *e*. If you need to recall this value, use an exponent of 1 and access the ex function. Hence, e^1 = 2.71828182845.

Give It Some Thought

For each of the following powers, determine if the natural logarithm is positive, negative, zero, or impossible.

a. 2.3   b. 1   c. 0.45   d. 0.97   e. −2   f. 4.83   g. 0

Solutions:

1. a. positive (property 2) b. zero (Property 1)   c. negative (property 3)
2. negative (property 3) e. impossible (Property 4)   f. positive (property 2)
3. impossible (property 4)

Example 1.3 A: Applying Natural Logarithms and Properties

[table id=13 /]

Exercises

Mechanics

Using your financial calculator, evaluate each of the following. Write your answer using the *ex* = power format. Round your answers to six decimals.

2. ln(0.7445)

3. ln(1.83921)

4. ln(13.2746)

5. ln(0.128555)

Applications

For each of the following, demonstrate the fifth property of natural logarithms, where $\ln\left(x^y\right) = y(\ln x)$.

6. ln[(1.02)23]

7. ln[(1.01275)41]

8. ln[(1.046)34]

Using $\dfrac{\ln\left(\frac{FV}{PV}\right)}{\ln(1 + i)}$, substitute the known values and evaluate the expression.

	FV	PV	i
12.	$78,230	$25,422	0.0225
13.	$233,120	$91,450	0.0425
14.	$18,974	$8,495	0.02175

Challenge, Critical Thinking, & Other Applications

Using FV = PV(1 + i)^N, substitute in the known values and solve for N.

	FV	PV	i
15.	$18,302.77	$14,000.00	0.015
16.	$58,870.20	$36,880	0.01

Key Takeaways

Algebraic Expressions

- Learning about the language of algebra
- The rules for manipulating exponents
- The rules of algebra for addition and subtraction
- The rules of algebra for multiplication
- The rules of algebra for division
- What is substitution and how it is performed?

Linear Equations: Manipulating and Solving

- A review of key concepts about equations
- The procedures required to solve one linear equation for one unknown variable
- The procedures required to solve two linear equations for two unknown variables

Natural Logarithms

- A review of the logarithm

The Language of Business Mathematics

algebraic equation An equation that takes two algebraic expressions and makes them equal to each other.

algebraic expression Indicates the relationship between and mathematical operations that must be conducted on a series of numbers or variables.

base The entire amount or quantity of concern.

BEDMAS An order of operations acronym standing for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.

common logarithm A logarithm with a base value of 10.

complex fraction A fraction that has fractions within fractions, combining elements of compound, proper, and/or improper fractions together.

compound fraction A fraction that combines an integer with either a proper or improper fraction.

denominator Any term by which some other term is divided; commonly the number on the bottom of a fraction.

divisor line The line that separates the numerator and the denominator in a fraction.

equivalent fractions Two or more fractions of any type that have the same numerical value upon completion of the division.

exponent A mathematical shorthand notation that indicates how many times a quantity is multiplied by itself

factor Components of terms that are separated from by multiplication or division signs.

fraction A part of a whole.

improper fraction A fraction in which the numerator is larger than the denominator.

left side of the equation The part of an equation that is to the left of the equal sign.

concept
- The rules and properties of natural logarithms

like terms Terms that have the same literal coefficient.

linear equation An algebraic expression in which the variable has an exponent of 1; when plotted, it will form a straight line.

literal coefficient A factor that is a variable.

logarithm The exponent to which a base must be raised to produce a particular power.

monomial An algebraic expression with only one term.

natural logarithm A logarithm with a base value of the mathematical constant e.

nomial The number of terms that appear in an algebraic expression.

nonlinear equation An algebraic expression in which the variable has an exponent other than 1; when plotted, it will not form a straight line.

numerator Any term into which some other term is being divided; commonly the number on the top in a fraction.

numerical coefficient A factor that is numerical.

percentage A part of a whole expressed in hundredths.

polynomial An algebraic expression with two or more terms.

portion Represents part of a whole or base.

proper fraction A fraction in which the numerator is smaller than the denominator.

rate Expresses a relationship between a portion and a base.

right side of the equation The part of an equation that is to the right of the equal sign.

root The value of the unknown variable that will make a linear equation true.

substitution Replacing the literal coefficients of an algebraic expression with their numerical values.

term In any algebraic expression, the components that are separated by addition and subtraction.

triangle technique A memorization technique that displays simple multiplication formulae in the form of a triangle. Anything on the same line is multiplied, and items above or below each other are divided to arrive at various solutions.

Chapter 1: Review of the Basics (Answers to textbook questions)

- ## 1.1 Algebraic Expressions

1. $5a+1$
2. $20b^2 +10b$
3. $2x^2+4x+4.5$
4. $(1+i)^{17}$
5. 4
6. $I = \$55.48$
7. $15r^2 +3r+5$
8. $1{,}024x^{40}$
9. $0.75t+3.5t^3$
10. $2 + 3(1+i)^3 -5(1+i)^6$
11. $3.995628R$
12. 0.0256
13. $PV=\$2{,}550$
14. $PMT= \$246.64$
15. $a^4 -9a^2 +18$
16. $-2x^2 -0.6xy+4.8y^2 -7.2x+4.2y$

17. $-3{,}888z^{19}$
18. $15,223.10
19. $39,963.05
20. $12,466.44

- ## 1.2 Linear Equation

1. x=10
2. b=1
3. m=2
4. x=4, y=2
5. h=5, q=1
6. a=8, b=0.6
7. y= $620.14
8. t=6.1513‾
9. Delay = 1.5 hours
10. 10.Users in 2000 = 107,998,376
11. Direct cost = $3,000
12. Gross income = $140,000
13. Single tickets = 10,363; 3-pack Tickets = 1,879
14. A = 700 units; B = 550 units
15. Company shares = $975,000
16. 240 minutes
17. Marianne = $50,000; William = $40,000; Hendrick = $65,000; Charlotte = $20,000
18. Regular stalls = 10,860; Handicap stalls = 240; Small car stalls = 720; RV stalls = 180
19. Z = $25,000
20. q = 2.78; r = 9.31

- ## 1.3 Natural Logarithms

1. $e^{1.367188} = 3.9243$
2. $e^{-0.295042} = 0.7445$
3. $e^{0.609336} = 1.83921$
4. $e^{2.585852} = 12.2746$
5. $e^{-2.051398} = 0.128555$

6. 0.392393
7. 2.302585
8. 1.811554
9. 0.455460
10. 0.519446
11. 1.529094
12. 50.517191
13. 22.48254
14. 36.904108
15. N=87.808096
16. N=160.423164
17. 0.583198
18. 1.451535
19. 403.428793
20. 33.115452

2.

GENERAL BUSINESS MANAGEMENT APPLICATIONS

Chapter 2 Topics

Basic calculations are so much a part of your everyday life that you could not escape them if you tried. While driving to school, you may hear on the radio that today's forecasted temperature of 27°C is 3°C warmer than the historical average. In the news, consumers complain that gas prices rose 11% in three months. A commercial tells you that the Toyota Prius you are driving uses almost 50% less gas than the Honda Civic Coupe Si. After class you head to Anytime Fitness to cancel a gym membership because you are too busy to work out. You have used only four months of an annual membership for which you paid $449, and now you want to know how large a prorated refund you are eligible for.

On a daily basis you use basic calculations such as averages (the temperature), percent change (gas prices), ratios and proportions (energy efficiency), and prorating (the membership refund). To invest successfully, you must also apply these basic concepts. Or if you are an avid sports fan, you need basic calculations to understand your favourite players' statistics.

And it's not hard to see how these calculations would be used in the business world. Retail management examines historical average daily sales to predict future sales and to schedule the employees who will service those sales. Human resource managers continually calculate ratios between accounting and performance data

to assess how efficiently labour is being used. Managers proration a company's budget across its various departments.

This chapter covers universal business mathematics you will use whether your chosen business profession is marketing, accounting, production, human resources, economics, finance, or something else altogether. To be a successful manager you need to understand percent changes, averages, ratios, proportions, and prorating.

2.1 Percent Change

On your way to work, you notice that the price of gasoline is about 10% higher than it was last month. At the office, reports indicate that input costs are down 5.4% and sales are up 3.6% over last year. Your boss asks you to analyze the year-over-year change in industry sales and submit a report. During your coffee break, you look through the day's e-flyers in your inbox. Home Depot is advertising that all garden furniture is 40% off this week; Safeway's ad says that Tuesday is 10% off day; and a *Globe and Mail* story informs you that the cost of living has risen by 3% since last year. You then log in to your financial services account, where you are happy to find that the mutual funds in your RRSP are up 6.7% from last year. What are you going to do with all this information?

Understanding how data changes from one period to the next is a critical business skill. It allows for quick assessment as to whether matters are improving or getting worse. It also allows for easy comparison of changes in different types of data over time. In this section, the concept of *percent change* is explored, which allows for the calculation of change between two points in time. Then a rate of change over time is introduced, which allows you to determine the change per period when multiple points in time are involved in the calculation.

Percent Change

It can be difficult to understand a change when it is expressed in absolute terms. Can you tell at a glance how good a deal it is to buy a $359 futon for $215.40? It can also be difficult to understand a change when it is expressed as a percentage of its end result. Are you getting a good deal if that $215.40 futon is 60% of the regular price? What most people do find easier to understand is a change expressed as a percentage of its starting amount. Are you getting a good deal if that $359 futon is on sale at 40% off? A **percent change** expresses in percentage form how much any quantity changes from a starting period to an ending period.

The Formula

To calculate the percent change in a variable, you need to know the starting number and the ending number. Once you know these, Formula 3.1 (on the next page) represents how to express the change as a percentage.

Remember two critical concepts about percent change:

1. ***Do Not Include the Original Quantity in the Change.*** Percent change represents the changes in the quantities, not the values of the quantities themselves. The original quantity does not count toward the percent change. Therefore, if any quantity doubles, its percent change is 100%, not 200%. For example, if the old quantity was 25 and the new quantity is 50, note that the quantity has doubled. However, 25 of the final 50 comes from the original amount and therefore does not count toward the change. We subtract it out of the numerator through calculating 50 − 25 = 25. Therefore, the change as a percentage is $\frac{50 - 25}{25} \times 100 = 100\%$. The same applies to a tripling of quantity. If our new quantity is 75 (which is triple the old quantity of 25), then $\frac{75 - 25}{25} \times 100 = 200\%$.% The original value of 25 is once again subtracted out of the numerator. The original 100% is always removed from the formula.

2. ***Negative Changes***. A negative change must be expressed with a negative sign or equivalent wording. For example, if the old quantity was 20 and the new quantity is 15, this is a decrease of 5, or a change of 15 − 20 = −5. The percent change is $\frac{15 - 20}{20} \times 100 = -25\%$. Be careful in expressing a negative percent change. There are two correct ways to do this properly:
   1. "The change is −25%."
   2. "It has decreased by 25%."

Note in the second statement that the word "decreased" replaces the negative sign. To avoid confusion, do not combine the negative sign with the word "decreased"—recall that two negatives make a positive, so "It has decreased by −25%" would actually mean the quantity has *increased* by 25%.

Percent Change

△% is Percent Change: The change in the quantity is always expressed in percent format. The Δ symbol is a commonly used symbol that represents the word *change*. Therefore, this algebraic symbol, which matches the BAII Plus calculator key, is read as "change percent."

New is New Quantity Value: This is the value that the quantity has become, or the number that you want to compare against a starting point.

Formula 3.1 – Percent Change: $$\Delta\% = \frac{New-Old}{Old} \times 100$$

Old is Old Quantity Value: This is the value that the quantity used to be, or the number that you want all others to be compared against. Notice how the formula is structured:
1. First calculate "New – Old," the change in the quantity. This is the numerator.
2. Divide the change by "Old" to express the quantity change first as a fraction, then convert it into its decimal format by performing the division.

× 100 is Percent Conversion: Recall from Section 2.3 that you convert a decimal to a percentage by multiplying it by 100. As the language suggests, percent changes are always percentages; therefore, you must include this component in the formula.

How It Works

To solve any question about percent change, follow these steps:

Step 1: Notice that there are three variables in the formula. Identify the two known variables and the one unknown variable.

Step 2: Solve for the unknown variable using Formula 3.1.

Assume last month your sales were $10,000, and they have risen to $15,000 this month. You want to express the percent change in sales.

Step 1: The known variables are Old = $10,000 and New = $15,000. The unknown variable is percent change, or Δ%.

Step 2: Using Formula 3.1, Δ%=

```
*** QuickLaTeX cannot compile formula:
```

```
\frac{{


*** Error message:
File ended while scanning use of \frac .
Emergency stop.
```

15,000-10,000}{10,000}}\times 100=\;\frac{{5,000}{10,000}}\times 100=0.5\times 100=50\%

$Observe that the change in sales is +$5,000$ month-over-month. Relative to sales of $10,000 last month, this month's sales have risen by 50%.

Important Notes

To access the percent change function on your BAII Plus calculator, press 2nd Δ% located above the number 5 on your keypad. Always clear the memory of any previous question by pressing 2nd CLR Work once the function is open. Use the ↑ and ↓ arrows to scroll through the four lines of this function. To solve for an unknown variable, key in three of the four variables and then press Enter. With the unknown variable on your display, press CPT. Each variable in the calculator is as follows:

- OLD = The old or original quantity; the number that represents the starting point
- NEW = The new or current quantity; the number to compare against the starting point
- %CH = The percent change, or Δ% in its percent format without the % sign
- #PD = Number of consecutive periods for change. By default, it is set to 1. For now, do not touch this variable. Later in this section, when we cover rate of change over time, this variable will be explained.
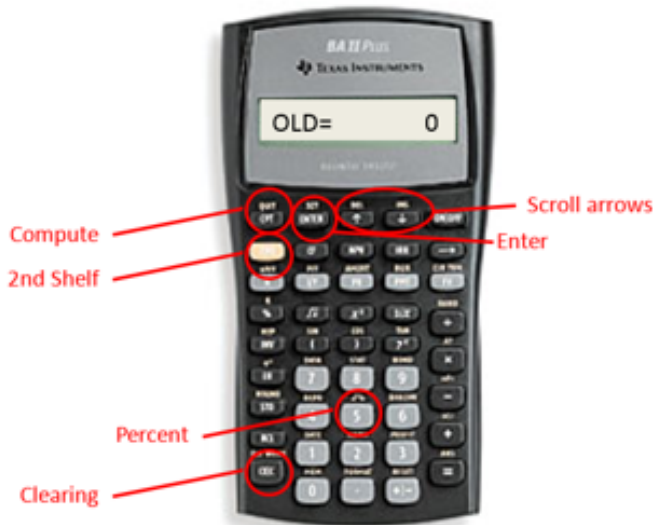
Things To Watch Out For

Watch out for two common difficulties involving percent changes.

1. **Rates versus Percent Changes**. Sometimes you may be confused about whether questions involve rates (Section 2.3) or percent changes. Recall that a *rate* expresses the relationship between a portion and a base. Look for some key identifiers, such as "is/are" (the portion) and "of" (the base). For percent change, key identifiers are "by" or "than." For example, "*x* has increased *by y%*" and "*x* is *y%* more *than* last year" both represent a percent change.

2. **Mathematical Operations**. You may imagine that you are supposed to add or subtract percent changes, but you cannot do this. Remember that percentages come from fractions. According to the rules of algebra, you can add or subtract fractions only if they share the same base (denominator). For example, if an investment increases in value in the first year by 10% and then declines in the second year by 6%, this is not an overall increase of 10% − 6% = 4%. Why? If you originally had $100, an increase of 10% (which is $100 × 10% = $10) results in $110 at the end of the first year. You must calculate the 6% decline in the second year using the $110 balance, not the original $100. This is a decline of $110 × (−6%) = −$6.60, resulting in a final balance of $103.40. Overall, the percent change is 3.4%.

Paths To Success

A percent change extends the rate, portion, and base calculations introduced in Section 2.3. The primary difference lies in the portion. Instead of the portion being a part of a whole, the portion represents the change of the whole. Putting the two formulas side by side, you can calculate the rate using either approach.

$$\text{Rate} = \frac{\text{Portion}}{\text{Base}} = \frac{\text{New}-\text{Old}}{\text{Old}}$$

Give It Some Thought

1. It has been five years since Juan went shopping for a new car. On his first visit to a car lot, he had sticker shock when he realized that new car prices had risen by about 20%. What does this situation involve?
    1. Percent change
    2. Rate, portion, base
2. Manuel had his home custom built in 2006 for $300,000. In 2014 he had it professionally appraised at $440,000. He wants to figure out how much his home has appreciated. How would he do so?
    1. The 2006 price is the "New," and the 2014 price is the "Old."
    2. The 2006 price is the "Old," and the 2014 price is the "New."

Solutions:

1. a (the question involves how car prices have changed; note the key word "by")
2. b (the 2006 price is what the house used to be worth, which is the old quantity; the 2014 price represents the new value of the home)

Example 2.1 A: Price of New Cars in Canada

[table id=14 /]

Example 2.1 B: Price Changes

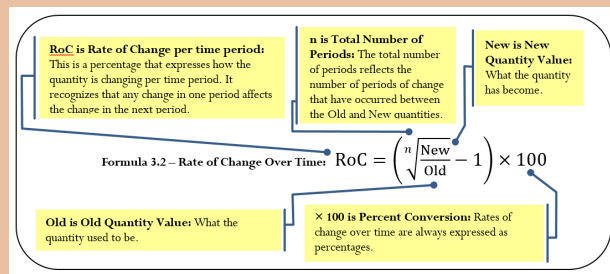[table id=15 /]

Rate of Change over Time

The percent change measures the change in a variable from start to end overall. It is based on the assumption

that only a single change occurs. What happens when the ending number results from multiple changes and you want to know the typical value of each change? For example, the population of the Toronto census metropolitan area (CMA) has grown from 4,263,759 in 1996 to 5,113,149 in 2006. What annual percentage growth in population does this reflect? Notice that we are not interested in calculating the change in population over the 10 years; instead we want to determine the percentage change in *each* of the 10 years. The **rate of change over time** measures the percent change in a variable per time period.

The Formula

Calculating the rate of change over time is not as simple as dividing the percent change by the number of time periods involved, because you must consider the change for each time period relative to a different starting quantity. For example, in the Toronto census example, the percent change from 1996 to 1997 is based on the original population size of 4,263,759. However, the percent change from 1997 to 1998 is based on the new population figure for 1997. Thus, even if the same number of people were added to the city in both years, the percent change in the second year is smaller because the population base became larger after the first year. As a result, when you need the percent change per time period, you must use Formula 3.2.

Rate of Change

RoC is Rate of Change per time period: This is a percentage that expresses how the quantity is changing per time period. It recognizes that any change in one period affects the change in the next period.

n is Total Number of Periods: The total number of periods reflects the number of periods of change that have occurred between the Old and New quantities.

New is New Quantity Value: What the quantity has become.

Formula 3.2 – Rate of Change Over Time: $\text{RoC} = \left(\sqrt[n]{\dfrac{\text{New}}{\text{Old}}} - 1 \right) \times 100$

Old is Old Quantity Value: What the quantity used to be.

× 100 is Percent Conversion: Rates of change over time are always expressed as percentages.

How It Works

When you work with any rate of change over time, follow these steps:

Step 1: Identify the three known variables and the one unknown variable.

Step 2: Solve for the unknown variable using Formula 3.2.

Important Notes

On your calculator, calculate the rate of change over time using the percent change (Δ%) function. Previously, we had ignored the #PD variable in the function and it was always assigned a value of 1. In rate of change, this variable is the same as n in our equation. Therefore, if our question involved 10 changes, such as the annual population change of the Toronto CMA from 1996 to 2006, then this variable is set to 10.

Paths To Success

You may find it difficult to choose which formula to use: percent change or rate of change over time. To distinguish between the two, consider the following:

1. If you are looking for the *overall* rate of change from beginning to end, you need to calculate the percent change.
2. If you are looking for the rate of change *per interval*, you need to calculate the rate of change over time.

Ultimately, the percent change formula is a simplified version of the rate of change over time formula where n = 1. Thus you can solve any percent change question using Formula 3.2 instead of Formula 3.1.

Give It Some Thought

For each of the following, distinguish whether you should solve the question by the percent change formula or the rate of change over time formula.

1. When Peewee started five-pin bowling with the Youth Bowling Canada (YBC) in 1997, his average was 53. In 2011, he finished his last year of the YBC with an average of 248. How did his average change from 1997 to 2011?
2. A stock was priced at $4.34 per share in 2006 and reached $7.15 per share in 2012. What annual return did a shareholder realize?
3. In 2004, total sales reached $1.2 million. By 2010, sales had climbed to $4.25 million. What is the growth in sales per year?

Solutions:

1. Percent change; looking for overall change
2. Rate of change over time; looking for change per year
3. Rate of change over time; looking for change per year

Example 2.1 D: Population Growth

[table id=16 /]

Example 2.1 E: Percent Changes and Rate of Change Together

[table id=17 /]

Exercises

Mechanics

For questions 1–3, solve for the unknown (?) using Formula 3.1 (percent change).

	Old	New	Δ%
1.	$109.95	$115.45	?
2.	?	$622.03	13.25%
3.	5.94%	?	−10%

4. If $9.99 is changed to $10.49, what is the percent change?
5. $19.99 lowered by 10% is what dollar amount?
6. What amount when increased by 40% is $3,500?
7. If 10,000 grows to 20,000 over a period of 10 years, what is the annual rate of change?

Applications

8. How much, including taxes of 12%, would you pay for an item with a retail price of $194.95?
9. From September 8, 2007 to November 7, 2007, the Canadian dollar experienced a rapid appreciation against the US dollar, going from $0.9482 to $1.1024. What was the percent increase in the Canadian dollar?
10. From 1996 to 2006, the "big three" automakers in North America (General Motors, Ford, and Chrysler) saw their market share drop from 71.5% to 52.7%. What is the overall change and the rate of change per year?
11. The average price of homes in Calgary fell by $10,000 to $357,000 from June 2009 to July 2009. The June 2009 price was 49% higher than the June 2005 price. a. What was the percent change from June 2009 to July 2009 b. What was the average price of a home in June 2005? c. What was the annual rate of change from June 2005 to June 2009?
12. On October 28, 2006, Saskatchewan lowered its provincial sales tax (PST) from 7% to 5%. What percent reduction does this represent?
13. A local Superstore sold 21,983 cases of its President's Choice cola at $2.50 per case. In the following year, it sold 19,877 cases at $2.75 per case.

    a. What is the percent change in price year-over-year?
    b. What is the percent change in quantity year-over-year?
    c. What is the percent change in total revenue year-over-year? (Hint: revenue = price × quantity)

14. A bottle of liquid laundry detergent priced at $16.99 for a 52-load bottle has been changed to $16.49 for a 48-load bottle. By what percentage has the price per load changed?

Challenge, Critical Thinking, & Other Applications

15. At a boardroom meeting, the sales manager is happy to announce that sales have risen from $850,000 to $1,750,000 at a rate of 4.931998% per year. How many years did it take for the sales to reach $1,750,000?
16. The Nova Scotia Pension Agency needs to determine the annual cost of living adjustment

(COLA) for the pension payments made to its members. To do this, it averages the consumer price index (CPI) for both the previous fiscal year and the current fiscal year. It then calculates the percent change between the two years to arrive at the COLA. If CPI information is as follows, determine the COLA that the pensioners will receive.

Previous Fiscal Year	Current Fiscal Year							
Nov.	109.2	May	112.1	Nov.	111.9	May	114.6
Dec.	109.4	June	111.9	Dec.	112.0	June	115.4
Jan.	109.4	July	112.0	Jan.	111.8	July	115.8
Feb.	110.2	Aug.	111.7	Feb.	112.2	Aug.	115.6
Mar.	111.1	Sep.	111.9	Mar.	112.6	Sep.	115.7
Apr.	111.6	Oct.	111.6	Apr.	113.5	Oct.	114.5

17. During The Bay's warehouse clearance days, it has reduced merchandise by 60%. As a bonus, today is Scratch 'n' Save day, where you can receive up to an additional 25% off the reduced price. If you scratched the maximum of 25% off, how many dollars would you save off an item that is regularly priced at $275.97? What percent savings does this represent?

18. Federal Canadian tax rates for 2010 and 2011 are listed below. For example, you pay no tax on income within the first bracket, 15% on income within the next bracket, and so on. If you earned $130,000 in each year, by what percentage did your federal tax rate change? In dollars, what was the difference?

2010 Tax Brackets	Taxed at	2011 Tax Brackets
$0–$10,382	0%	$0–$10,527
$10,383–$40,970	15%	$10,528–$41,544
$40,971–$81,941	22%	$41,545–$83,088
$81,942–$127,021	26%	$83,089–$128,800
$127,022+	29%	$128,801+

19. Melina is evaluating two colour laser printers for her small business. A Brother model is capable of printing 21 colour pages per minute and operates 162.5% faster than a similar Hewlett-Packard model. She needs to print 15,000 pages for a promotion. How much less time (stated as a percentage) will it take on the Brother model?

20. A chocolate bar has been priced at $1.25 for a 52 gram bar. Due to vending machine

restrictions, the manufacturer needs to keep the price the same. To adjust for rising costs, it lowers the weight of the bar to 48 grams.a. By what percentage has the price per gram changed? b. If this plan is implemented over two periods, what rate of change occurs in each period?

2.2 Averages

No matter where you go or what you do, averages are everywhere. Let's look at some examples:

- Three-quarters of your student loan is spent. Unfortunately, only half of the first semester has passed, so you resolve to squeeze the most value out of the money that remains. But have you noticed that many grocery products are difficult to compare in terms of value because they are packaged in different sized containers with different price points? For example, one tube of toothpaste sells in a 125 mL size for $1.99 while a comparable brand sells for $1.89 for 110 mL. Which is the better deal? A fair comparison requires you to calculate the average price per millilitre.
- Your local transit system charges $2.25 for an adult fare, $1.75 for students and seniors, and $1.25 for children. Is this enough information for you to calculate the average fare, or do you need to know how many riders of each kind there are?
- Five years ago you invested $8,000 in Roller Coasters Inc. The stock value has changed by 9%, −7%, 13%, 4%, and −2% over these years, and you wonder what the average annual change is and whether your investment kept up with inflation.
- If you participate in any sport, you have an average of some sort: bowlers have bowling averages; hockey or soccer goalies have a goals against average (GAA); and baseball pitchers have an earned run average (ERA).

Averages generally fall into three categories. This section explores simple, weighted, and geometric averages.

Simple Averages

An **average** is a single number that represents the middle of a data set. It is commonly interpreted to mean the "typical value." Calculating averages facilitates easier comprehension of and comparison between different data sets, particularly if there is a large amount of data. For example, what if you want to compare year-over-year sales? One approach would involve taking company sales for each of the 52 weeks in the current year and

comparing these with the sales of all 52 weeks from last year. This involves 104 weekly sales figures with 52 points of comparison. From this analysis, could you concisely and confidently determine whether sales are up or down? Probably not. An alternative approach involves comparing last year's average weekly sales against this year's average weekly sales. This involves the direct comparison of only two numbers, and the determination of whether sales are up or down is very clear.

In a **simple average**, all individual data share the same level of importance in determining the typical value. Each individual data point also has the same frequency, meaning that no one piece of data occurs more frequently than another. Also, the data do not represent a percent change. To calculate a simple average, you require two components:

1. The data itself—you need the value for each piece of data.
2. The quantity of data—you need to know how many pieces of data are involved (the count), or the total quantity used in the calculation.

Simple Average

SAvg is Simple Average: A simple average for a data set in which all data has the same level of importance and the same frequency.

Σ is Summation: This symbol is known as the Greek capital letter *sigma*. In mathematics it denotes that all values written after it (to the right) are summed.

Formula 3.3 – Simple Average:
$$\text{SAvg} = \frac{\sum x}{n}$$

n is Total Quantity: This is the physical total count of the number of pieces of data or the total quantity being used in the average calculation. In business, the symbol n is a common standard for representing counts.

x is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

As expressed in Formula 3.3, you calculate a simple average by adding together all of the pieces of data then taking that total and dividing it by the quantity.

How It Works

The steps required to calculate a simple average are as follows:

Step 1: Sum every piece of data.

Step 2: Determine the total quantity involved.

Step 3: Calculate the simple average using Formula 3.3.

Assume you want to calculate an average on three pieces of data: 95, 108, and 97. Note that the data are equally important and each appears only once, thus having the same frequency. You require a simple average.

Step 1: Σx = 95 + 108 + 97 = 300.

Step 2: There are three pieces of data, or n = 3.

Step 3: Apply Formula 3.3: SAvg = $\frac{300}{3} = 100$. The simple average of the data set is 100.

Important Notes

Although mentioned earlier, it is critical to stress that a simple average is calculated only when all of the following conditions are met:

1. All of the data shares the same level of importance toward the calculation.
2. All of the data appear the same number of times.
3. The data does not represent percent changes or a series of numbers intended to be multiplied with each other.

If any of these three conditions are not met, then either a weighted or geometric average is used depending on which of the above criteria failed. We discuss this later when each average is introduced.

Give It Some Thought

It is critical to recognize if you have potentially made any errors in calculating a simple average. Review the following situations and, without making any calculations, determine the best answer.

1. The simple average of 15, 30, 40, and 45 is: a. lower than 20. b. between 20 and 40, inclusive. c. higher

than 40.

2. If the simple average of three pieces of data is 20, which of the following data do not belong in the data set? Data set: 10, 20, 30, 40 a. 10 b. 20 c. 30 d. 40

Solutions:

1. b (a simple average should fall in the middle of the data set, which appears spread out between 15 and 45, so the middle would be around 30)
2. d (if the number 40 is included in any average calculation involving the other numbers, it is impossible to get a low average of 20)

Example 2.2 A: Comparing Average Sales

Weighted Averages

Have you considered how your grade point average (GPA) is calculated? Your business program requires the successful completion of many courses. Your grades in each course combine to determine your GPA; however, not every course necessarily has the same level of importance as measured by your course credits.

Perhaps your math course takes one hour daily while your communications course is only delivered in one-hour sessions three times per week. Consequently, the college assigns the math course five credit hours and the communications course three credit hours. If you want an average, these different credit hours mean that the two courses do not share the same level of importance, and therefore a simple average cannot be calculated.
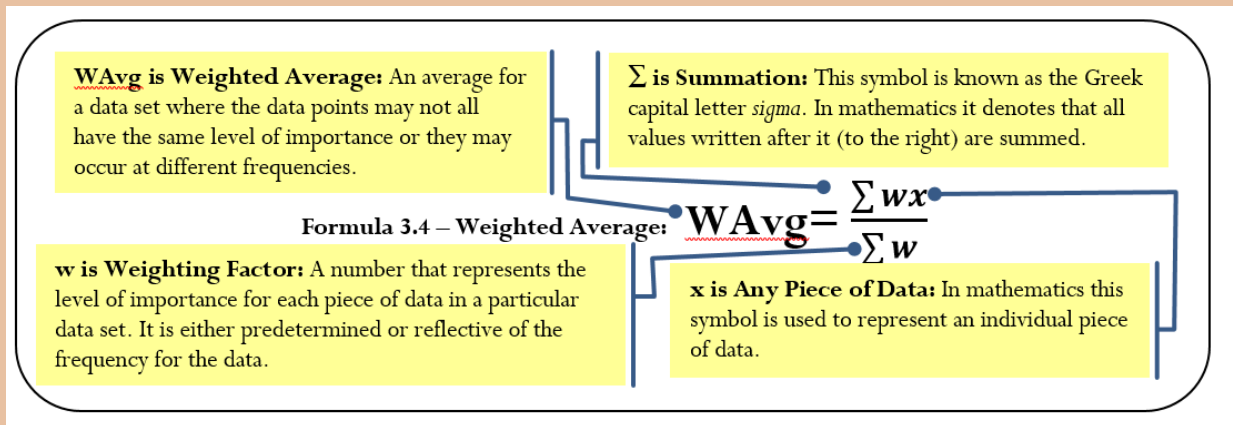
In a **weighted average**, not all pieces of data share the same level of importance or they do not occur with the same frequency. The data cannot represent a percent change or a series of numbers intended to be multiplied with each other. To calculate a weighted average, you require two components:

1. The data itself—you need the value for each piece of data.
2. The weight of the data—you need to know how important each piece of data is to the average. This is either an assigned value or a reflection of the number of times each piece of data occurs (the frequency).

The Formula

As expressed in Formula 3.4, calculate a weighted average by adding the products of the weights and data for the entire data set and then dividing this total by the total of the weights.

Weighted Average

WAvg is Weighted Average: An average for a data set where the data points may not all have the same level of importance or they may occur at different frequencies.

Σ is Summation: This symbol is known as the Greek capital letter *sigma*. In mathematics it denotes that all values written after it (to the right) are summed.

w is Weighting Factor: A number that represents the level of importance for each piece of data in a particular data set. It is either predetermined or reflective of the frequency for the data.

x is Any Piece of Data: In mathematics this symbol is used to represent an individual piece of data.

Formula 3.4 – Weighted Average:

$$WAvg = \frac{\Sigma wx}{\Sigma w}$$

How It Works

The steps required to calculate a weighted average are:

Step 1: Sum every piece of data multiplied by its associated weight.

Step 2: Sum the total weight.

Step 3: Calculate the weighted average using Formula 3.4.

Let's stay with the illustration of the math and communications courses and your GPA. Assume that these are the only two courses you are taking. You finish the math course with an A, translating into a grade point of 4.0. In the communications course, your C+ translates into a 2.5 grade point. These courses have five and three credit hours, respectively. Since they are not equally important, you use a weighted average.

Step 1: In the numerator, sum the products of each course's credit hours (the weight) and your

grade point (the data). This means (math credit hours × math grade point) + (communications credit hours × communications grade point). Numerically, this is **Σ**wx = (5 × 4) + (3 × 2.5) = 27.5.

Step 2: In the denominator, sum the weights. These are the credit hours. You have **Σ**w = 5 + 3 = 8.

Step 3: Apply Formula 3.4 to calculate your GPA.

WAvg = $\frac{27.5}{8} = 3.44$ (GPAs have two decimals).

Note that your GPA is higher than if you had just calculated a simple average of $\frac{4 + 2.5}{2} = 3.25$. This happens because your math course, in which you scored a higher grade, was more important in the calculation.

Things To Watch Out For

The most common error in weighted averages is to confuse the data with the weight. If you have the two backwards, your numerator is still correct; however, your denominator is incorrect. To distinguish the data from the weight, notice that the data forms a part of the question. In the above example, you were looking to calculate your *grade point* average; therefore, grade point is the data. The other information, the credit hours, must be the weight.

Paths To Success

The formula used for calculating a simple average is a simplification of the weighted average formula. In a simple average, every piece of data is equally important. Therefore, you assign a value of 1 to the weight for each piece of data. Since any number multiplied by 1 is the same number, the simple average formula omits the weighting in the numerator as it would have produced unnecessary calculations. In the denominator, the sum of the weights of 1 is no different from counting the total number of pieces of data. In essence, you can use a weighted average formula to solve simple averages.

Give It Some Thought

In each of the following, determine which information is the data and which is the weight.

1. Rafiki operates a lemonade stand during his garage sale today. He has sold 13 small drinks for $0.50, 29 medium drinks for $0.90, and 21 large drinks for $1.25. What is the average price of the lemonade sold?
2. Natalie received the results of a market research study. In the study, respondents identified how many times per week they purchased a bottle of Coca-Cola. Calculate the average number of purchases made

per week.

Purchases per Week	# of People
1	302
2	167
3	488
4	256

Solutions:

1. The price of the drinks is the data, and the number of drinks is the weight.
2. The purchases per week is the data, and the number of people is the weight.

Example 2.2 B: Calculating Weighted GPA

[table id=19 /]

Geometric Averages

How do you average a percent change? If sales increase 100% this year and decrease 50% next year, is the average change in sales an increase of $\frac{(100\% - 50\%)}{2} = 25\%$ per year? The answer is clearly "no." If sales last year were $100 and they increased by 100%, that results in a $100 increase. The total sales are now $200. If sales then decreased by 50%, you have a $100 decrease. The total sales are now $100 once again. In other words, you started with $100 and finished with $100. That is an average change of nothing, or 0% per year! Notice that the second percent change is, in fact, multiplied by the result of the first percent change. A **geometric average** finds the typical value for a set of numbers that are meant to be multiplied together or are exponential in nature.

The Formula

In business mathematics, you most commonly use a geometric average to average a series of percent changes. Formula 3.5 is specifically written to address this situation.

Geometric Average

GAvg is Geometric Average: The average of a series of percent changes expressed in percent format. Every percent change involved in the calculation requires an additional $(1 + \Delta\%)$ to be multiplied under the radical. The formula accommodates as many percent changes as needed.

n is Total Quantity: The physical total count of how many percent changes are involved in the calculation.

Formula 3.5 – Geometric Average: $\text{GAvg} = \left[\sqrt[n]{(1 + \Delta\%_1) \times (1 + \Delta\%_2) \times \cdots \times (1 + \Delta\%_n)} - 1\right] \times 100$

Δ% is Percent Change: The value of each percent change in the series from which the average is calculated. You need to express the percent changes in decimal format.

× 100 is Percent Conversion: Because you are averaging percent changes, convert the final result from decimal form into a percentage.

How It Works

To calculate a geometric average follow these steps:

Step 1: Identify the series of percent changes to be multiplied.

Step 2: Count the total number of percent changes involved in the calculation.

Step 3: Calculate the geometric average using Formula 3.5.

Let's use the sales data presented above, according to which sales increase 100% in the first year and decrease 50% in the second year. What is the average percent change per year?

Step 1: The changes are $\Delta\%_1$ = +100% and $\Delta\%_2$ = -50%.

Step 2: Two changes are involved, or n = 2.

Step 3: Apply Formula 3.5:

$$\text{GAvg} = \left[\sqrt[2]{(1 + 100\%) \times (1 - 50\%)} - 1\right] \times 100$$
$$= \left[\sqrt[2]{2 \times 0.5} - 1\right] \times 100 = \left[\sqrt[2]{1} - 1\right] \times 100 = 0 \times 100 = 0\%$$

> The average percent change per year is 0% because an increase of 100% and a decrease of 50% cancel each other out.

Important Notes

A critical requirement of the geometric average formula is that every $(1 + \Delta\%)$ expression must result in a number that is positive. This means that the $\Delta\%$ cannot be a value less than -100% else Formula 3.5 cannot be used.

Paths To Success

An interesting characteristic of the geometric average is that it will always produce a number that is either smaller than (closer to zero) or equal to the simple average. In the example, the simple average of +100% and -50% is 25%, and the geometric average is 0%. This characteristic can be used as an error check when you perform these types of calculations.

Give It Some Thought

For the first three questions, determine whether you should calculate a simple, weighted, or geometric average.

1. Randall bowled 213, 245, and 187 in his Thursday night bowling league and wants to know his average.
2. Cindy invested in a stock that increased in value annually by 5%, 6%, 3%, and 5%. She wants to know her average increase.
3. A retail store sold 150 bicycles at the regular price of $300 and 50 bicycles at a sale price of $200. The manager wants to know the average selling price.
4. Gonzalez has calculated a simple average of 50% and a geometric average of 60%. He believes his numbers are correct. What do you think?

Solutions:

1. Simple; each item has equal importance and frequency.
2. Geometric; these are a series of percent changes on the price of stock.
3. Weighted; each item has a different frequency.
4. At least one of the numbers is wrong since a geometric average is always smaller than or equal to the

simple average

Example 2.2 C: Changing Prices

[table id=20 /]

[1] WestJet, *WestJet Fact Sheet*, www.westjet.com/pdf/investorMedia/ investorFactSheet.pdf (accessed May 10, 2011).

Exercises

Mechanics

Calculate a simple average for questions 1 and 2.

Data

1. 8, 17, 6, 33, 15, 12, 13, 16
2. $1,500 $2,000 $1,750 $1,435 $2,210

Calculate a weighted average for questions 3 and 4.

3.	4	4	4	4	12	12	12	12	12	12	12	15	15

4.	Data	$3,600	$3,300	$3,800	$2,800	$5,800
Weight	2	5	3	6	4	

Calculate a geometric average for exercises 5 and 6. Round all percentages to four decimals.

5.	5.4%	8.7%	6.3%	
6.	10%	-4%	17%	-10%

Applications

7. If a 298 mL can of soup costs $2.39, what is the average price per millilitre?
8. Kerry participated in a fundraiser for the Children's Wish Foundation yesterday. She sold 115 pins for $3 each, 214 ribbons for $4 each, 85 coffee mugs for $7 each, and 347 baseball hats for $9 each. Calculate the average amount Kerry raised per item.
9. Stephanie's mutual funds have had yearly changes of 9.63%, -2.45%, and 8.5%. Calculate the annual average change in her investment.
10. In determining the hourly wages of its employees, a company uses a weighted system that factors in local, regional, and national competitor wages. Local wages are considered most important and have been assigned a weight of 5. Regional and national wages are not as important and have been assigned weights of 3 and 2, respectively. If the hourly wages for local, regional, and national competitors are $16.35, $15.85, and $14.75, what hourly wage does the company pay?
11. Canadian Tire is having an end-of-season sale on barbecues, and only four floor models remain, priced at $299.97, $345.49, $188.88, and $424.97. What is the average price for the barbecues?
12. Calculate the grade point average (GPA) for the following student. Round your answer to two decimals.

Course	Grade	Credit Hours	Grade	Grade Point	Grade	Grade Point
Economics 100	D	5	A+	4.5	C+	2.5
Math 100	B	3	A	4.0	C	2.0
Marketing 100	C	4	B+	3.5	D	1.0
Communications 100	A	2	B	3.0	F	0.0
Computing 100	A+	3				
Accounting 100	B+	4				

13. An accountant needs to report the annual average age (the length of time) of accounts receivable (AR) for her corporation. This requires averaging the monthly AR averages, which are listed below. Calculate the annual AR average.

Month	Monthly AR Average	Month	Monthly AR Average	Month	Monthly AR Average
January	$45,000	May	$145,000	September	$185,000
February	$70,000	June	$180,000	October	$93,000
March	$85,000	July	$260,000	November	$60,000
April	$97,000	August	$230,000	December	$50,000

14. From January 2007 to January 2011, the annual rate of inflation has been 2.194%, 1.073%, 1.858%, and 2.346%. Calculate the average rate of inflation during this period.

Challenge, Critical Thinking, & Other Applications

15. Gabrielle is famous for her trail mix recipe. By weight, the recipe calls for 50% pretzels, 30% Cheerios, and 20% peanuts. She wants to make a 2 kg container of her mix. If pretzels cost $9.99/kg, Cheerios cost $6.99/kg, and peanuts cost $4.95/kg, what is the average cost per 100 g rounded to four decimals?

16. Caruso is the marketing manager for a local John Deere franchise. He needs to compare his average farm equipment sales against his local Case IH competitor's sales. In the past three months, his franchise has sold six $375,000 combines, eighteen $210,000 tractors, and fifteen $120,000 air seeders. His sales force estimates that the Case IH dealer has sold four $320,000 combines, twenty-four $225,000 tractors, and eleven $98,000 air seeders. Express the Case IH dealer's average sales as a percentage of the John Deere dealer's average sales.

17. You are shopping for shampoo and consider two brands. Pert is sold in a bundle package of two 940 mL bottles plus a bonus bottle of 400 mL for $13.49. Head & Shoulders is sold in a bulk package of three 470 mL bottles plus a bonus bottle of 280 mL for $11.29.

    a. Which package offers the best value?
    b. If the Head & Shoulders increases its package size to match Pert at the same price per mL, how much money do you save by choosing the lowest priced package?

18. The following are annual net profits (in millions of dollars) over the past four years for three divisions of Randy's Wholesale:

   ◦ Cosmetics: $4.5, $5.5, $5.65, $5.9
   ◦ Pharmaceutical: $15.4, $17.6, $18.5, $19.9
   ◦ Grocery: $7.8, $6.7, $9.87, $10.75

Rank the three divisions from best performing to worst performing based on average annual percent change.

19. You are shopping for a Nintendo Wii gaming console and visit www.shop.com, which finds online sellers and lists their prices for comparison. Based on the following list, what is the average price for a gaming console (rounded to two decimals)?

NothingButSoftware.com	$274.99
eComElectronics	$241.79
NextDayPC	$241.00
Ecost.com	$249.99
Amazon	$169.99
eBay	$165.00
Buy.com	$199.99
HSN	$299.95
Gizmos for Life	$252.90
Toys 'R' Us	$169.99
Best Buy	$169.99
The Bay	$172.69
Walmart	$169.00

20. Juanita receives her investment statement from her financial adviser at Great-West Life. Based on the information below, what is Juanita's average rate of return on her investments?

Investment Fund	Proportion of Entire Portfolio Invested in Fund	Fund Rate of Return
Real Estate	0.176	8.5%
Equity Index	0.073	36.2%
Mid Cap Canada	0.100	−1.5%
Canadian Equity	0.169	8.3%
US Equity	0.099	−4.7%
US Mid Cap	0.091	−5.7%
North American Opportunity	0.063	2.5%
American Growth	0.075	−5.8%
Growth Equity	0.085	26.4%
International Equity	0.069	−6.7%

2.3 Ratios, Proportions, and Prorating

You and your business partner have a good problem: Consumers are snapping up packets of your new eucalyptus loganberry facial scrub as fast as you can produce them. Each packet of the scrub contains 600 mg of loganberry extract and 80 mg of eucalyptus oil, as well as water and clay and other ingredients. Ratios are invaluable in understanding the relationship among different quantities, such as how much of each ingredient you need.

You have no trouble obtaining the water and clay, but the loganberry extract and eucalyptus oil are in short supply because of poor weather. In any time period, your suppliers can provide seven times as much loganberry extract as eucalyptus oil. To figure out which ingredient is limiting your production, you need proportions.

Sometimes you need to relate a proportion to the total of the quantities. This requires prorating skills. For example, once your business has grown, you start using a production line, which follows the common practice of producing more than one product. The papaya facial scrub you have introduced recently and the eucalyptus loganberry scrub require the same amount of production time. Your equipment capacity is 1,000 units per day. How many units of each type of scrub must you produce to meet market demand in the ratio of nine to two?

Years later, you are splitting the profits of your business partnership in proportion to each partner's total investment. You invested $73,000, while your partner invested $46,000. With total profits of $47,500, what is your share?

Understanding the relationships among various quantities and how the components relate to an overall total emphasizes the need for understanding ratios, proportions, and prorating.

What Is a Ratio?

A **ratio** is a fixed relationship between two or more quantities, amounts, or sizes of a similar nature. For a ratio to exist, all terms involved in the ratio must be nonzero. Examine the criteria of this definition more closely:

1. *There Must Be Two or More Quantities*. A ratio does not exist if only one quantity is involved. For example, the fuel tank on your Mustang takes 60 litres of gasoline. This is not a ratio, as there is no relationship to any other quantity, amount, or size. On the other hand, if you compare your fuel tank to the fuel tank of your friend's Hummer, you now have two quantities involved and could say that her fuel tank has twice the capacity of yours.
2. *All Terms Must Be of a Similar Nature*. In all of the examples provided note that all quantities, amounts, or sizes are based on the same unit. In the cosmetics formulation it was milligrams to milligrams; in the production line, it was units to units; in the investment scenario, it was dollars to dollars. For a ratio to have meaning and to be properly interpreted, all terms of the ratio must be expressed in a similar nature. When you place different units such as kilometres and metres into the same ratio, the result is confusing and will lead to misinterpretation of the relationship.
3. *All Terms Must Be Nonzero*. The numbers that appear in a ratio are called the **terms of the ratio**. If we have a recipe with four cups of flour to one cup of sugar, there are two terms: four and one. If any term is zero, then the quantity, amount, or size does not exist. For example, if the recipe called for four cups of flour to zero cups of sugar, there is no sugar! Therefore, every term must have some value other than zero.

Let's continue using the example of four cups of flour to one cup of sugar. Business ratios are expressed in five common formats, as illustrated in the table below.

Format	Ratio Example	Interpretation
To	4 to 1	Four cups of flour to one cup of sugar
: (colon)	4 : 1	Four cups of flour to one cup of sugar
Fraction	$\frac{4}{1}$	Four cups of flour per one cup of sugar
Decimal	4	Four times as much flour than sugar
Percent	400%	Flour is 400% of sugar (Hint: think rate, portion, base)

All of these formats work well when there are only two terms in the ratio. If there are three or more terms,

ratios are best expressed in the colon format. For example, if the recipe called for four parts of flour to one part of sugar to two parts of chocolate chips, the ratio is 4 : 1 : 2. The fraction, decimal, and percent forms do not work with three or more terms.

Simplification and Reduction of Ratios

When a ratio is used to express a relationship between different variables, it must be easy to understand and interpret. Sometimes when you set up a ratio initially, the terms are difficult to comprehend. For example, what if the recipe called for 62½ parts flour to 25 parts sugar? That is not very clear. Expressing the same ratio another way, you can say the recipe requires 5 parts flour to 2 parts sugar. Note how the relationship is clearer in the latter expression. Either way, though, both ratios mean the same thing; in decimal format this ratio is expressed as a value of 2.5. Recall that Section 2.2 discussed how fractions are expressed in higher and lower terms. We now apply the same knowledge to ratios to make the relationship as clear as possible.

When you reduce ratios to lower terms, remember two important characteristics involving the cardinal rule and integers:

1. ***The Cardinal Rule.*** Recall from Section 2.5 that the cardinal rule of algebra states, "What you do to one you must do to the other." In other words, whatever mathematical operation is performed on a term in a ratio must be equally performed on every other term in the ratio. If this rule is violated then the relationship between the terms is broken.
2. ***Maintaining Integers.*** Integers are easier to understand than decimals and fractions. In reducing a ratio to lower terms, aim to maintain every term as an integer (much as in Section 2.2).

How It Works

The steps involved in reducing ratios to lower terms are listed below. You may not need some steps, so skip them if the characteristic is not evident in the ratio.

Step 1: Clear any fractions that, when divided, produce a nonterminating decimal. Apply the rules of algebra and multiply each term by the denominator being cleared from the ratio. For example, if the ratio is , the first term when divided produces a nonterminating decimal. Clear the fraction by multiplying every term by the denominator of 3, resulting in a ratio of 1 : 6.

Step 2: Perform division on all fractions that produce a terminating decimal. For example, if the ratio is , both terms convert to terminating decimals, resulting in a ratio of 0.4 : 0.3.

Step 3: Eliminate all decimals from the ratio through multiplication. In other words, express the ratio in higher terms by multiplying every term by a power of 10. The power of 10 you choose must be large enough to eliminate all decimals. For example, if the ratio is 0.2 : 0.25 : 0.125, notice that the third term has the most decimal positions. A power of 1,000 (10^3) is required to move the decimal three positions to the right. Multiply every term by a power of 1,000, resulting in a ratio of 200 : 250 : 125.

Step 4: Find a common factor that divides evenly into every term, thus producing integers. If you find no such factors, then the ratio is in its lowest terms. For example, if the ratio is 10 : 4 : 6 it can be factored by dividing every term by 2, resulting in a ratio of 5 : 2 : 3. There is no common factor that reduces this ratio further; therefore, it is in its lowest terms.

Important Notes

To factor any term, recall your multiplication tables. Assume the ratio is 36 : 24. It is always best to pick the lowest term in the ratio when factoring. Looking at the 24 and recall what multiplies together to arrive at 24. In this case, your factors are 1 × 24, 2 × 12, 3 × 8, and 4 × 6. The goal is to find the largest value among these factors that also divides evenly into the other term of 36. The largest factor that makes this true is 12. Therefore, perform step 4 in our reduction process by dividing every term by a factor of 12, resulting in a ratio of 3 : 2.

Things To Watch Out For

Always ensure that before you apply any of the reduction steps your relationship meets the requirements of being a ratio in the first place. For example, the expression of 10 km : 500 m is not a 1 : 50 ratio since it violates the "similar nature" characteristic of the ratio definition. You need to convert the metres into kilometres, producing 10 km : 0.5 km. Now that you have a ratio, the reduction to lowest terms results in a 20 : 1 ratio. This is very different than 1 : 50! The lesson learned is to make sure you are working with a proper ratio before you manipulate it.

Paths To Success

In the fourth step of the procedure, do not feel compelled to find what is called the "magic factor." This is the single factor that reduces the ratio to its lowest terms in a single calculation. Although this is nice if it happens, you may as well find any factor that will make the ratio smaller and easier to work with.

- For example, assume a ratio of 144 : 72 : 96. It may not be very apparent what factor goes into each of these terms on first glance. However, they are all even numbers, meaning they can all be divided by 2.

This produces 72 : 36 : 48.

- Once again, the "magic factor" may not be clear, but every term is still even. Divide by 2 again, producing 36 : 18 : 24.
- If no magic factor is apparent, again note that all numbers are even. Divide by 2 yet again, producing 18 : 9 : 12.
- The common factor for these terms is 3. Divided into every term you have 6 : 3 : 4. There is no common factor for these terms, and the ratio is now in its lowest terms.

In the above example it took four steps to arrive at the answer—and that is all right. Did you notice the "magic factor" that could have solved this in one step? You can find it if you multiply all of the factors you have: 2 × 2 × 2 × 3 = 24. Dividing every term in the original ratio by 24 produces the solution in one step. If you did not notice this "magic factor," there is nothing wrong with taking four steps (or more!) to get the answer. In the end, both methods produce the same solution.

Example 2.3 A: Reducing Ratios to Lowest Terms

[table id=21 /]

Reducing a Ratio to the Smallest Term of One

Your goal in reducing a ratio is to make it easier to understand. Sometimes, even after you have applied the techniques for reducing the ratio, the end result is still hard to grasp. Look at part (d) of Example 3.3A. You arrived at a final solution of 41 : 55, with no further reduction possible. Think of this as 41 cups of flour to 55 cups of sugar. The relationship is not very clear.

In these circumstances, although integers are preferable in general, you must reduce the ratio further, reintroducing decimals to make the relationship more comprehensible. This means you apply a technique called "reducing the ratio to the smallest term of one." In this technique, the smallest term in the ratio will have a value of 1 once you perform the ratio reduction and simplification.

How It Works

Follow these steps to reduce a ratio to the smallest term of one:

Step 1: Locate the smallest term in the ratio. (Do not just pick the first term.)

Step 2: Divide every term in the ratio by the selected smallest term. Every other term becomes a decimal number, for which either a clear rounding instruction is provided or you must obey the rounding rules used in this textbook. The smallest term by nature of the division equals one.

Let's continue with part (d) of Example 3.3A, in which the reduced ratio is 41 : 55.

Step 1: Locate the smallest term in the ratio. It is the first term and has a value of 41.

Step 2: Take every term and divide it by 41 to arrive at 1 : $1.\overline{34146}$. The decimal number allows you to roughly interpret the ratio as "one cup of flour to a touch over 1⅓ cups of sugar." Although not perfect, this is more understandable than 41 : 55.

Important Notes

You may have to make a judgment call when you decide whether to leave a reduced ratio alone or to reduce it to a ratio where the smallest term is one. There is no clear definitive rule; however, keep the following two thoughts in mind:

1. For purposes of this textbook, you are provided with clear instructions on how to handle the ratio, allowing everyone to arrive at the same solution. Your instructions will read either "Reduce the ratio to its smallest terms" or "Reduce the ratio to the smallest term of one."
2. In the real world, no such instructions exist. Therefore you should always base your decision on which format is easier for your audience to understand. If the reduced fraction leaves your audience unable to understand the relationship, then use the reduction to a ratio with the smallest term of one instead.

Example 2.3 B: Maintaining Inventory

[table id=22 /]

What Are Proportions?

Knowing the relationship between specific quantities is helpful, but what if your quantity happens not to match the specific quantity expressed in the existing ratio? Example 3.3B illustrated that you need to learn how to apply the ratio to meet current conditions. The ratio in the inventory system was expressed in terms of 27 Nintendo Wiis; however, the shelf displayed 38 units. How can we relate an existing ratio to a needed ratio?

A **proportion** is a statement of equality between two ratios. Just as we have both algebraic expressions and algebraic equations, there are ratios and proportions. With algebraic expressions, only simplification was possible. When the expression was incorporated into an algebraic equation, you solved for an unknown. The same is true for ratios and proportions. With ratios, only simplification is possible. Proportions allow you to solve for any unknown variable. The Wii and PlayStation example could have been set up as follows:

Proportions

Left-Hand Side Ratio: The known ratio forms the left side of the proportion. It expresses the known relationship between the units of Wiis and PlayStations.

Right-Hand Side Ratio: The unknown ratio forms the right side of the proportion. It expresses the relationship between the variables in the same order, but uses the currently known information. Any unknown quantities are algebraically assigned to an unknown variable (in this case p for PlayStations).

$$27 : 47 = 38 : p$$

Equal Sign: By placing the equal sign between the two ratios, you create a proportion.

A proportion must adhere to three characteristics, including ratio criteria, order of terms, and number of terms.

- ***Characteristic #1: Ratio Criteria Must Be Met***. By definition, a proportion is the equality between two ratios. If either the left side or the right side of the proportion fails to meet the criteria for being a ratio, then a proportion cannot exist.
- ***Characteristic #2: Same Order of Terms***. The order of the terms on the left side of the proportion must be in the exact same order of terms on the right side of the proportion. For example, if your ratio is the number of MP3s to CDs to DVDs, then your proportion is set up as follows:

MP3 : CD : DVD = MP3 : CD : DVD

- ***Characteristic #3: Same Number of Terms***. The ratios on each side must have the same number of terms such that every term on the left side has a corresponding term on the right side. A proportion of MP3 : CD : DVD = MP3 : CD is not valid since the DVD term on the left side does not have a corresponding term on the right side.

When you work with proportions, the mathematical goal is to solve for an unknown quantity or quantities. In order to solve any proportion, always obey the following four rules:

1. ***Rule #1****: At Least One Value for Any Term Is Known*. At least one of the left side or right side values for each term must be known. For example, $x : 5 = y : 10$ is not a solvable proportion since the corresponding first terms on both sides are unknown. However, $15 : 5 = y : 10$ is solvable since at least one of the first corresponding terms (the 15 and the y) is known.
2. ***Rule #2****: One Pair of Corresponding Terms Must Be Known*. At least one pair of corresponding terms on the left side and right side must have both quantities known. For example, $3 : x : 6 = y : 4 : z$ is not a solvable proportion since there is no pair of first terms (3 and y), second terms (x and 4), or third terms (6 and z) that produces a pair of known values. However, $3 : x : 6 = 9 : 4 : z$ is a solvable proportion since the first terms on both sides (the 3 and 9) are known.
3. ***Rule #3****: Obey BEDMAS and Perform Proper Algebraic Manipulation*. To manipulate a proportion, you must satisfy the rules of BEDMAS (Section 2.1) and all of the rules of algebra (Sections 2.4 and 2.5). Violating any of these rules breaks the equality of the ratios and produces an incorrect proportion.
4. ***Rule #4****: Use the Fractional Format*. The fractional format for ratios is recommended for solving a proportion. The other four formats generally make solving the proportion much more difficult, and the mathematical operations required become unclear.

How It Works

Follow these steps in solving any proportion for an unknown variable or variables:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Work with only two terms at a time, and express the two terms in fractional format. This is not a problem if the proportion has only two terms on each side of the equation, for example, 27 : 47 = 38 : *p*. This is expressed as $\frac{27}{47} = \frac{38}{p}$. If the proportion has three or more terms on each side, you can pick any two terms from each side of the proportion so long as you pick the same two terms on each side. In making your selection, aim to have a pair of terms on one side of the equation where both values are known while the other side of the equation is made up of one known term and one unknown term. For example, assume the proportion 6 : 5 : 4 = 18 : 15 : *y*. The selection of the first and third terms only on each side of the equation produces 6 : 4 = 18 : *y*. Notice that this is now a proportion with only two terms on each side, which you can express as $\frac{6}{4} = \frac{18}{y}$

Step 3: Solve for the unknown variable. Obey the rules of BEDMAS and perform proper algebraic manipulation.

Step 4: If the proportion contained more than one unknown variable, go back to step 2 and select another pairing that isolates one of the unknown variables. Although one of the unknown variables is now known as a result of step 3, do not use this known value in making your selection. The danger in using a solved unknown variable is that if an error has occurred, the error will cascade through all other calculations. For example, assume the original proportion was 3 : 7 : 6 : 8 = *x* : *y* : *z* : 28. From step 2 you may have selected the pairing of 3 : 8 = *x* : 28, and in step 3 you may have erroneously calculated *x* = 11.5 (the correct answer is *x* = 10.5). In returning to step 2 to solve for another unknown variable, do not involve *x* in your next pairing. To isolate *y*, use 7 : 8 = *y* : 28 and not 3 : 7 = 11.5 : *y*, which will at least ensure that your calculation of *y* is not automatically wrong based on your previous error.

Things To Watch Out For

You must always pick terms from the same positions on both sides of the proportion. Otherwise, you will violate the equality of the proportion, since the terms are no longer in the same order on both sides and

Characteristic #2 is not satisfied. For example, in working with the proportion $6 : 5 : 4 = 18 : 15 : y$ from above, you cannot select the first and third terms on the left side and also select the second and third terms on the right side. In other words, $6 : 4 \neq 15 : y$ since 1st term : 3rd term ≠ 2nd term : 3rd term.

Paths To Success

It is always easiest to solve a proportion when the unknown variable is in the numerator. This characteristic requires minimal algebra and calculations to isolate the variable. If you find yourself with an unknown variable in the denominator, you can mathematically invert the fraction on both sides, since this obeys the cardinal rule of "what you do to one, you must do to the other." When you invert, the numerator becomes the denominator and vice versa. For example, if the proportion is $\frac{3}{4} = \frac{12}{b}$, inversion then produces a proportion of 2.09 and later!

$$\frac{4}{3} = \frac{b}{12}$$

Notice that isolating the unknown variable in the inverted proportion requires only a multiplication of 12 on both sides. This is a lot less work!

Give It Some Thought

1. Some of the following proportions violate the characteristics or rules of proportions. Examine each and determine if all the rules and characteristics are met. If not, identify the problem.

---

a. $4 : 7 = 6 : y$ b. $5 : 3 = 6 : a : b$ c. $6 \text{ km} : 3 \text{ m} = 2 \text{ m} : 4 \text{ km}$ d. $6 : k = 18 : 12$

e. $4 : 0 = 8 : z$ f. $9 = p$ g. $4 : 7 : 10 = d : e : f$ h. $y : 10 : 15 = x : 30 : z$

---

2. In the following solved proportions, which person properly executed step 2 of the proportion steps?

$$6 : 5 : 4 : 3 = x : y : z : 9$$

Person A: $6 : 5 = x : y$ Person B: $4 : 3 = y : 9$

Solutions:

1. a. OK
2. Not the same number of terms on each side (Characteristic #3)
3. Terms are not in same units; terms are not in the same order (Characteristics #1 and #2)
4. OK

5. Does not meet ratio criteria; there is a term of zero (Characteristic #1)
6. Does not meet ratio criteria; must have at least two terms on each side (Characteristic #1)
7. There is no corresponding known term on both sides; every pair of terms contains an unknown (Rule #2)
8. The first corresponding term is not known on both sides; at least one needs to be known (Rule #1)
9. Person C did it right. Person A failed to isolate a variable, and different terms were extracted by Person B (3rd term : 4th term ≠ 2nd term : 4th term).

Example 2.3 C: Estimating Competitor Profits

[table id=23 /]

What Is Prorating?

Ratios and proportions are commonly used in various business applications. But there will be numerous situations where your business must allocate limited funds across various divisions, departments, budgets, individuals, and so on. In the opener to this section, one example discussed the splitting of profits with your business partner, where you must distribute profits in proportion to each partner's total investment. You invested $73,000 while your partner invested $46,000. How much of the total profits of $47,500 should you receive?

The process of **prorating** is the taking of a total quantity and allocating or distributing it proportionally. In the above example, you must take the total profits of $47,500 and distribute it proportionally with your business partner based on the investment of each partner. The proportion is:

your investment : your partner's investment = your profit share : your partner's profit share

This proportion has two major concerns:

1. You don't know either of the terms on the right side. As per the rules of proportions, this makes the proportion unsolvable.
2. There is a piece of information from the situation that you didn't use at all! What happened to the total profits of $47,500?

Every prorating situation involves a **hidden term**. This hidden term is usually the sum of all the other terms on the same side of the proportion and represents a total. In our case, it is the $47,500 of total profits. This quantity must be placed as an extra term on both sides of the proportion to create a proportion that can actually be solved.

How It Works

Prorating represents a complex proportion. As such, the steps involved in prorating are similar to the steps for solving any proportion:

Step 1: Set up the proportion with the known ratio on the left side. Place the ratio with any unknown variables on the right side.

Step 2: Insert the hidden term on both sides of the proportion. Usually this term represents the total of all other terms on the same side of the proportion.

Step 3: Working with only two terms at a time, express the two terms in fractional format. Ensure that only one unknown variable appears in the resulting proportion.

Step 4: Solve for the unknown variable. Ensure that the rules of BEDMAS are obeyed and proper algebraic manipulation is performed.

Step 5: If the prorating contains more than one unknown variable, go back to step 3 and select a pairing that isolates another one of the unknown variables.

To solve your profit-splitting scenario, let y represent your profit share and p represent your partner's share:

Step 1: your investment : your partner's investment = your profit share : your partner's profit share

$73,000 : $46,000 = $y : p$

Step 2: Insert the hidden total term on both sides:

your investment	:	your partner's investment	:	total investment	=	your profit share	:	your partner's profit share	:	total profits

$73,000 : $46,000 : $119,000 = $y : p$: $47,500

Step 3: Set up one proportion:

Step 4: Solve for y. Calculate y = $29,138.66.

Step 5: Set up the other proportion:

Solve for *p*. Calculate *p* = $18,361.34.

Your final proportion is:

$73,000 : $46,000 : $119,000 = $29,138.66 : $18,361.34 : $47,500.00

So you will receive $29,138.66 of the total profits and your partner will receive $18,361.34.

Important Notes

To make prorating situations easier to solve, it is always best to insert the hidden term as the *last term* on both sides of the equation. This forces the unknown variables into the numerator when you select pairs of terms. It then takes less algebra for you to isolate and solve for the unknown variable.

Things To Watch Out For

Ensure that when you insert the hidden term you put it in the same position on both sides of the proportion. A common error is to put the total on the "outsides" of the proportion:

A : B = C : D when prorated becomes

Total : A : B = C : D : Total

This violates the proportion characteristics, since the terms are not in the same order on both sides. The correct insertion of the hidden total term makes the proportion look like this:

A : B : Total = C : D : Total

Paths To Success

Another way of approaching prorating is to calculate a "per unit" basis and then multiply each term in the proportion by this base. For example, assume you are distributing $100 across three people, who have shares of three, five, and two. This is a total of 10 shares (the hidden term). Therefore, $100 divided by 10 shares is $10 per share. If the first person has three shares, then $10 × 3 = $30. The second and third people get $10 × 5 = $50 and $10 × 2 = $20, respectively. Therefore,

3 : 5 : 2 : 10 = $30 : $50 : $20 : $100.

Example 2.3 E: A Refund on Your Car Insurance Premiums

[table id=24 /]

Exercises

Mechanics

For questions 1–4, reduce the ratios to their smallest terms.

1. a. 66 : 12 b. 48 : 112 : 80 c. 24 : 36 : 84 : 108
2. a. 7.2 : 6 b. 0.03 : 0.035 : 0.02 c. 0.27 : 0.18 : 0.51 : 0.15
3. a. $5\frac{1}{4} : 6$ b. $\frac{3}{2} : \frac{9}{5} : \frac{21}{10}$ c. $2\frac{1}{8} : \frac{9}{8} : 3\frac{3}{8} : 1$
4. a. $\frac{8}{3} : \frac{5}{9}$ b. $\frac{5}{6} : \frac{1}{3} : 2$ c. $\frac{1}{7} : \frac{1}{3} : \frac{1}{4} : \frac{1}{6}$
5. Reduce the following ratios to the smallest term of one. Round to two decimals as needed. a. 48 : 53 b. $5\frac{7}{8} : 2\frac{3}{4} : 3\frac{1}{12}$ c. $\frac{2}{3} : 11\frac{1}{11} : 5\frac{5}{9} : 2.08$
6. Solve the following proportions for the unknown variable. Round to two decimals as needed. a. 7 : 12 = 109 : y b. 3.23 : 4.07 : 2.12 = x : 55.9625 : 29.15 c. $q : \frac{5}{3} = 13.75 : 25$
7. Solve the following proportions for all of the unknown variables. a. $12.15 : $38.30 : r = x : $59,184.53 : $26,998.93 b. 16 : 5 : 9 : 30 = g : h : i : $397,767

For questions 8 and 9, prorate the total as indicated by the ratio.

8. Ratio = 3 : 2, Total = $11,368.25
9. Ratio = 7 : 5 : 3, Total = $46,923.90

Applications

10. EB Games sells gaming consoles. Last month, it sold $22,500 worth of Nintendo Wii consoles, $31,500 worth of Microsoft Xbox consoles, and $18,000 worth of Sony PlayStation

consoles. Express the ratio of Wii to Xbox to PlayStation sales in its lowest terms.

11. The manufacturing cost of a deluxe candle is made up of $2.40 in paraffin, $1.20 in dye, $1.60 in overhead, and $4.40 in direct labour. Express the ratio between these costs respectively in their lowest terms.

12. If one Canadian dollar buys $1.0385 of US dollars, how much US money can you spend so that you do not exceed the maximum duty-free amount of $800 Canadian on your next US vacation?

13. For every $100 of retail spending, the average Canadian spends approximately $19.12 at motor vehicle dealerships and $18.35 at grocery stores. If the average Canadian spends $49,766 per year on retail spending, how much more does a Canadian spend annually at motor vehicle dealerships than at grocery stores?

14. Marina has a three-sevenths interest in a partnership. If she sells one-quarter of her interest for $8,250, what is the implied value of the partnership?

15. You are making a punch for an upcoming party. The recipe calls for 2½ parts ginger ale to ⅘ parts grenadine to 1⅜ parts vodka. If your punch bowl can hold 8.5 litres, how much (in litres) of each ingredient is needed? Round the answers to two decimals as needed.

Challenge, Critical Thinking, & Other Applications

16. A manufacturing facility requires one member of the board of directors for every 10 executives. There are five managers for every executive and eight workers for every manager. The average salary paid to directors is $105,000 along with $80,000 to executives, $55,000 to managers, and $30,000 to workers. If the facility has 1,383 employees, what is the total labour cost?

17. It is common for strip mall owners to allocate to their tenants general overhead costs such as snow clearing, security, and parking lot maintenance in one of two ways. The first method involves allocating these costs on the basis of the square metres each tenant operates. The second method involves allocating these costs on the basis of the number of annual transactions made by each tenant. At Charleswood Square, there are three tenants. The 7-Eleven takes up 150 square metres, Tim Hortons takes up 235 square metres, and Quiznos takes up 110 square metres. Last year, 7-Eleven had 350,400 transactions, Tim Hortons had 657,000 transactions, and Quiznos had 197,100 transactions. Total overhead costs for Charleswood Square last year were $25,980. Determine which method of allocation each tenant would prefer and how much of a savings in percentage its preferred method would represent relative to the other allocation method.

18. The average annual Canadian cellphone plan covers 780 minutes of voice time, 600 text

messages, and 8 video messages with a total annual cost of $500.63. However, some Canadians are high-usage consumers and average 439 minutes per month.

a. Assuming the same proportion of usage and that each component shares the cost equally on a per-unit basis, how many text messages and video messages do these high-usage Canadian consumers average per year? How much should they pay annually for their cellphone usage?

b. A similar typical cellphone plan in the Netherlands has a total annual cost of $131.44. How much would a high-usage plan cost annually in the Netherlands?

19. Procter & Gamble (P&G) is analyzing its Canadian sales by region: 23% of sales were on the West coast; 18% were in the Prairies, 45% were in central Canada, and 14% were in Atlantic Canada. Express and reduce the ratio of P&G's respective sales to the smallest term of one. Round to two decimals.

20. A company has 14 managers, 63 supervisors, and 281 workers. After a record profit year, the company wants to distribute an end-of-year bonus to all employees. Each worker is to get one share, each supervisor gets twice as many shares as workers, and each manager gets twice as many shares as supervisors. If the bonus to be distributed totals $1,275,000, how much does each manager, supervisor, and worker receive?

The Language of Business Mathematics

average A single number that represents the middle of a data set.

geometric average A typical value for a set of numbers that are meant to be multiplied together or are exponential in nature. This may take the form of a series of percent changes.

hidden term In prorating, this is the sum of all the other terms on the same side of the proportion and represents a total.

percent change An expression in percent form of how much

Section 2.1: Percent Change (Are We Up or Down?)

Measuring the percent change in a quantity from one value to another

Measuring the constant rate of change over time in a quantity

Section 2.2: Averages (What Is Typical?)

The calculation of simple averages when everything is equal

The calculation of weighted averages when not everything is equal

The calculation of geometric averages when everything is multiplied together

Section 2.3: Ratios, Proportions, and Prorating (It Is Only Fair)

The characteristics of a ratio

How to simplify and reduce a ratio to its lowest terms

How to simplify a ratio by reducing its smallest term to a value of one

How to equate two ratios into the form of a proportion

How to use proportions to prorate

any quantity changes from a starting period to an ending period.

proportion A statement of equality between two ratios.

prorating The process of taking a total quantity and allocating or distributing it proportionally.

rate of change over time A measure of the percent change in a variable per time period.

ratio A fixed relationship between two or more quantities, amounts, or sizes of a similar nature where all terms are nonzero.

simple average An average where each piece of data shares the same level of importance and frequency but does not represent percent changes or numbers that are intended to be multiplied with each other.

terms of the ratio The numbers appearing in a ratio.

weighted average An average where not all pieces of data share the same level of importance or they do not occur with the same frequency; the data cannot represent percent changes or numbers that are intended to be multiplied with each other.

The Formulas You Need to Know

Symbols Used

$\Delta\%$ = percent change

$\sum$ = summation symbol, which means you add everything up

GAvg = geometric average

n = a count of something, whether a total number of periods or a total quantity

New = the value that a quantity has become; the number that is being compared

Old = the value that a quantity used to be; the number to compare to

RoC = rate of change per time period

SAvg = simple average

x = any individual piece of data

w = weighting factor

WAvg = weighted average

Chapter 2: General Business Management Applications (Answers to textbook questions)

- ## 2.1 Percent Change

1. 5.0023%
2. $549.25
3. 5.346%
4. 5.0005%
5. $17.99
6. $2,500
7. 7.1773%
8. $218.34
9. 16.2624%
10. Δ% = −26.2937%; RoC = −3.0048%
11. a. −2.7248%, b. $246,308.72, c. 10.4833%
12. −28.5714%
13. a. 10%, b. −9.5801%, c. −0.5381%

14. 5.1452%
15. 15 years
16. 2.5148%
17. Amount saved = $193.18; Δ%=−70.0004%
18. −0.6155%
19. 61.9048% less time.
20. a. 8.$\overline{3}$%, b. 4.08$\overline{3}$% per period.

- ## 2.2 Averages

1. 15
2. $1,779
3. 10
4. $3,795
5. 6.7910%
6. 2.6888%
7. $0.00802/ml
8. $6.46
9. 5.0821%
10. $15.88
11. $314.83
12. 2.74
13. $125,000
14. 1.8666%
15. $0.8082
16. 99.0805%
17. a.Pert better; Pert=0.005916\overline{6}$/ml; H&S=$0.006680/ml, b. Pert saves $1.74
18. Grocery 11.2853%; Cosmetics 9.4493%; Pharmaceuticals 8.9208%
19. $213.64
20. 10.2809%

- ## 2.3 Ratios, Proportions and Prorating

1. a. 11:2; 1b. 6:13:10; 1c. 2:3:7:9
2. a. 6:5; 2b. 6:7:4
3. a. 35:4; 3b. 5:6:7; c. 17:9:27:8

4. a. 24:5; 4b. 5:2:12; 4c. 12:28:21:15
5. a. 1:1.10; 5b. 2.14:1:1.12; 5c. 1:16.64:8.33:3.12
6. a. y=1.87; b. x=44.55; c. q=0.92
7. a. r=$17.47; x=$18,775.25; b. g=212,142.40; h = $66,294.50; i=$119,330.10
8. $6,820.95:$4,547.30
9. $21,897.82:$15,641.30:$9,384.78
10. 5:7:4
11. 6:3:4:11
12. m=$778.88
13. $383.20
14. $44,916.67
15. Gingerale = 4.08L, Grenadine =2.176L; Vodka = 2.244L
16. $46,965,000
17. 7-11: transactions, 4% savings; Tim Horton's: square metres, 12.963% savings; Quizno's: transactions, 26.3636% savings
18. a. Texts = 4,052; Video = 54, Cost =$3,381.18; 18b. $887.73.
19. 1.64 : 1.29 : 3.21 : 1
20. Manager = $11,015.12; Supervisor = $5,507.56; Worker = $2,753.78

3.

HUMAN RESOURCES AND ECONOMIC APPLICATIONS

Chapter 3 Topics

Has your employer paid you all that you have earned? While this problem is difficult to track, the Canadian Payroll Association estimates the frequency of payroll errors to be 5.7%.[1] Studies in the United States have found that 33% of employers have made some form of payroll mistake, whether in the calculation of pay or in deductions.[2] This means that you must always check your paycheque to ensure that you receive the correct amount of pay. If you are overpaid, you are not entitled to keep the overage and must reimburse your employer. If you are underpaid, your employer owes you money.

We all work to earn enough income to cover our expenses, such as mortgages, car payments, and recreational activities. There are many ways to earn that income, ranging from straight salaries and hourly wages to commission rates and piecework wages. The amount of money you generate by working is called your *gross earnings*. If you are an employee, this amount is then subject to automatic deductions, including federal taxes, provincial taxes, employment insurance, Canada Pension Plan, along with any other deductions required by your employer. All of these government deductions are submitted to the government on your behalf. What is left over after all of these deductions is your net pay, which is your take-home amount.

Payroll forms one of the largest and most contentious expenses in most organizations. It is critical to get it right.

Workers often go on strike if they believe their take-home pay is unfair. And payroll errors are costly and time consuming to fix, aside from their impact on employee satisfaction.

Payroll decisions are affected by inflation. Ask your parents how much things cost when they were young. In the 1950s, my parents attended movie theatres for 25¢ admission. A chocolate bar was 5¢. Today, attending a first-run movie costs $8.99 or more, and chocolate bars are typically around one dollar. To keep up with these rising prices, incomes have constantly increased. In 1959, family income was approximately $4,500 compared to the 2009 median family income of $68,410.[3] Can you still afford the same things that people could afford back then? Business indexes are regularly used to understand the change over time in various quantities or how a quantity at one time point compares to the corresponding value at a certain reference time point.

For example, the next time you are offered a raise at work, an index can tell you whether you are truly receiving a raise or not! Assume that product prices have risen by 2% year-over-year. If you are offered a raise less than 2%, you have taken a wage cut and have less purchasing power than before. A raise of 2% allows you to at least break even, and anything above 2% increases your purchasing power.

Whether you are pursuing a career in human resources, economics, or some other field, the issues of gross earnings, income taxes, and indexes remain important both personally and professionally. This chapter examines some of the basic mathematical concepts involved in these calculations.

[1] Steven Van Alstine, Vice-President of Education, the Canadian Payroll Association.

[2] Mie-Yun Lee, "Outsource Your Payroll," *Entrepreneur*. www.entrepreneur.com/humanresources/article47340.html (accessed November 29, 2009).

[3] Statistics Canada, "Median Total Income, by Family Type, by Province and Territory," CANSIM table 111-0009, http://www40.statcan.ca/l01/cst01/famil108a-eng.htm (accessed October 19, 2010).

3.1 Gross Earnings

You work hard at your job, and you want to be compensated properly for all the hours you put in. Assume you work full time with an hourly rate of pay of $10. Last week you worked eight hours on Sunday and eight hours on Monday, which was a statutory holiday. Then you took Tuesday off, worked eight hours on each of Wednesday and Thursday, took Friday off, and worked 10 hours on Saturday. That's a total of 42 hours of work for the week. What is your gross pay? Give or take a small amount depending on provincial employment standards, it should be about $570. But if you don't understand how to calculate gross earnings, you could be underpaid without ever realizing it.

Here are some notes about the content in this chapter: About 10% of Canadian workers fall under federal employment standards, which are not discussed here. This textbook generalizes the most common provincial employment standards; however, to calculate your earnings accurately requires you to apply your own provincial employment standards legislation. Part-time employment laws are extremely complex, so this textbook assumes in all examples that the employee is full time.

This section addresses the calculation of **gross earnings**, which is the amount of money earned before any deductions from your paycheque. The four most common methods of employee remuneration include salaries, hourly wages, commissions, and piecework wages.

Salary and Hourly Wages

One ad in the employment classifieds indicates compensation of $1,270 biweekly, while a similar competing ad promotes wages of $1,400 semi-monthly. If both job ads are similar in every other way, which job has the higher annual gross earnings? To make this assessment, you must understand how salaries work. A **salary** is a fixed compensation paid to a person on a regular basis for services rendered. Most employers pay employees by salary in occupations where the employee's work schedule generally remains constant.

In contrast, an **hourly wage** is a variable compensation based on the time an employee has worked. In contrast to a salary, this form of compensation generally appears in occupations where the number of hours is unpredictable or continually varies from period to period.

Employment Contract Characteristics[1]

Salaried and hourly full-time employees are similar with regard to their gross earnings. The major earnings issues in an employment contract involve regular earnings structure, overtime, and holidays.

[1] Special thanks to Steven Van Alstine (CPM, CAE), Vice-President of Education, the Canadian Payroll Association, for assistance in summarizing Canadian payroll legislation and jurisdictions.

Regular Earnings Structure

An agreement with your employer outlines the terms of your employment, including the time frame and frequency of pay.

1. *Time Frame.* For salaried employees, the time frame that the salary covers must be clearly stated. For example, you could receive a salary of $2,000 monthly or $50,000 annually. Notice that each of these salaries is followed by the specific time frame for the compensation. For hourly employees, the time

frame requires identification of the wage earned per hour.

2. *Frequency.* How often the gross earnings are paid out to the employee must be defined.

   a. Monthly: Earnings are paid once per month. By law, employees must receive compensation from their employer at least once per month, which equals 12 times per year.

   b. Daily: Earnings are paid at the end of every day. This results in about 260 paydays per year (5 days per week multiplied by 52 weeks per year). In a leap year, there might be one additional payday.

   c. Weekly: Earnings are paid once every week. This results in 52 paydays in any given year since there are 52 weeks per year.

   d. Biweekly: Earnings are paid once every two weeks. This results in 26 paydays in any given year since there are $52 \div 2 = 26$ biweekly periods per year.

   e. Semi-monthly: Earnings are paid twice a month, usually every half month (meaning on the 15th and last day of the month). This results in 24 paydays per year.

Thus, the earnings structure specifies both the time frame and the frequency of earnings. For a salaried employee, this may appear as "$2,000 monthly paid semi-monthly" or "$50,000 annually paid biweekly." For an hourly employee, this may appear as "$10 per hour paid weekly." No matter whether you are salaried or hourly, earnings determined by your regular rate of pay are called your **regular earnings.**

Overtime

Overtime is work time in excess of your regular workday, regular workweek, or both. In most jurisdictions it is paid at 1.5 times your regular hourly rate (called *time-and-a-half*), though your company may voluntarily pay more or a union may have negotiated a more favourable rate such as two times your regular hourly rate (called *double time*). A contract with an employer will specify your regular workday and workweek, and some occupations are exempt from overtime. Due to the diversity of occupations, there is no set rule on what constitutes a regular workday or workweek. In most jurisdictions, a regular workweek is eight hours per day and 40 hours per week. Once you exceed these regular hours, you are eligible to receive **overtime or premium earnings**, which are based on your overtime rate of pay.

Holidays

A **statutory holiday** is a legislated day of rest with pay. Five statutory holidays are recognized throughout Canada, namely, New Year's Day, Good Friday (or Easter Monday in Quebec), Canada Day, Labour Day, and Christmas Day. Each province or territory has an additional four to six **public holidays** (or general holidays), which may include Family Day (known as Louis Riel Day in Manitoba and Islander Day in PEI) in February,

Victoria Day in May, the Civic Holiday in August, Thanksgiving Day in October, Remembrance Day in November, and Boxing Day in December. These public holidays may or may not be treated the same as statutory holidays, depending on provincial laws.

Statutory and public holidays generally require employees to receive the day off with pay. If the holiday falls on a nonworking day, it is usually the next working day that is given off instead. For example, if Christmas Day falls on a Saturday, typically the following Monday is given off with pay. Here's how holidays generally work (though you should always consult legislation for your specific jurisdiction):

- You should be given the day off with pay, called **holiday earnings**. The holiday earnings are in the amount of a regular day's earnings, and the hours involved count toward your weekly hourly totals for overtime purposes (preventing employers from shifting your work schedule that week).
- If you are required to work, the employer must offer another day off in lieu with pay. Your work on the statutory holiday is then paid at *regular earnings* and the hours involved contribute toward your weekly hourly totals for overtime purposes. You are paid holiday earnings on your future day off.
- If you are required to work and no day or rest is offered in lieu, this poses the most complex situation. Under these conditions:

- You are entitled to the holiday earnings you normally would have received for the day off. The hours that make up your holiday earnings contribute toward your weekly hourly totals for overtime purposes (again, consult your local jurisdiction).

In addition, for the hours you worked on the statutory holiday you are entitled to overtime earnings known as **statutory holiday worked earnings**. These hours do not contribute toward your weekly hourly totals for overtime purposes since you are already compensated at a premium rate of pay. For example, assume you work eight hours on Labour Day, your normal day is eight hours, and you won't get another day off in lieu. Your employer owes you the eight hours of holiday earnings you should have received for getting the day off *plus* the eight hours of statutory holiday worked earnings for working on Labour Day.

The Formula

The four forms of compensation consist of regular earnings, overtime earnings, holiday earnings, and statutory holiday worked earnings. Add these four elements together to determine total gross earnings. Formula 4.1 shows the relationship.

Gross Earnings

GE is Gross Earnings: Gross earnings are earning before any deductions and represent the amount owed to the employee for services rendered. This is commonly called the *gross amount* of the paycheque.

Statutory Holiday Worked Earnings: This pay shows up only if a statutory holiday is worked and the employee will not receive another paid day off in lieu. It is received in addition to the holiday pay and must be paid at a premium rate.

Formula 4.1 – Salary & Hourly Gross Earnings:
GE = Regular Earnings + Overtime Earnings + Holiday Earnings + Statutory Holiday Worked Earnings

Regular Earnings: Unless the employees have exceeded their daily or weekly thresholds or a holiday is involved, all hours worked are considered regular earnings.

Holiday Earnings: If a statutory holiday occurs during a pay period, this is holiday pay in an amount that represents a regular shift.

Overtime Earnings: Any hours worked that exceed daily or weekly thresholds fall under overtime earnings. For most individuals, this is calculated at 1.5 times their regular hourly rate.

How It Works

Salaried Employees

To calculate the total gross earnings for a salaried employee, follow these steps:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. If the employee has only regular hours of pay, skip to step 6.

Step 2: Calculate the employee's equivalent hourly rate of pay. This means converting the salary into an equivalent hourly rate:

$$\text{Equivalent Hourly Rate} = \frac{\text{Annual Salary}}{\text{Annual Hours Worked}}$$

For example, use a $2,000 monthly salary requiring 40 hours of work per week. Express the salary annually by multiplying it by 12 months, yielding $24,000. Express the 40 hours per week annually by multiplying by 52 weeks per year, yielding 2,080 hours worked. The equivalent hourly rate is $24,000 ÷ 2,080 = $11.538461.

Step 3: Calculate any holiday earnings. Take the unrounded hourly rate and multiply it by the number of hours in a regular shift, or

Holiday Earnings = Unrounded Hourly Rate × Hours in a Regular Shift

A salaried employee earning $11.538461 per hour having a daily eight-hour shift receives $11.538461 × 8 = $92.31 in holiday earnings.

Step 4: Calculate any overtime earnings.

1. Determine the overtime hourly rate of pay by multiplying the unrounded hourly rate by the minimum standard overtime factor of 1.5 (this could be higher if the company pays a better overtime rate than this):

Overtime Hourly Rate = Unrounded Hourly Rate × 1.5

11. Round the final result to two decimals. For the salaried worker, $11.538461 × 1.5 = $17.31 per overtime hour.
12. Multiply the overtime hourly rate by the overtime hours worked.

Step 5: Calculate any statutory holiday worked earnings which is similar to calculating overtime earnings:

Statutory Holiday Worked Earnings = Statutory Hourly Rate × Statutory Hours Worked

The statutory hourly rate is at minimum 1.5 times the unrounded hourly rate of pay. The salaried employee working eight hours on a statutory holiday receives $17.31 × 8 = $138.48.

Step 6: Calculate the gross earnings paid at the regular rate of pay. Take the amount of the salary and divide it by the number of pay periods involved, then subtract any holiday earnings:

$$\text{Regular Earnings} = \frac{\text{Salary}}{\text{Pay Periods}} - \text{Holiday Earnings}$$

You need to calculate the number of pay periods based on the regular earnings structure. For example, an annual $52,000 salary paid biweekly would have 26 pay periods annually. Therefore, a regular paycheque is $52,000 ÷ 26 = $2,000 per paycheque. As another example, a $2,000 monthly salary paid semi-monthly has two pay periods in a single month, resulting in regular earnings of $2,000 ÷ 2 = $1,000 per paycheque. If a holiday is involved in the pay period, you must deduct the holiday earnings from these amounts.

Step 7: Calculate the total gross earnings by applying Formula 4.1.

Hourly Employees

To calculate the total gross earnings for an hourly employee, follow steps similar to those for the salaried employee:

Step 1: Analyze the employee's work performed and assign hours as needed into each of the four categories of pay. It is usually best to set up a table similar to the one below. This table allows you to visualize the employee's week at a glance along with totals, enabling proper assessment of their hours worked.

This table separates the four types of earnings into different rows. Enter the information into the table about the employee's workweek, placing it in the correct day and on the correct row. If any daily thresholds are exceeded, place the appropriate hours into the overtime row. Once you have completed this, total the regular hours and holiday hours for the week and check to see if they exceed any regular weekly threshold. If so, starting with the last workday of the week and working backwards, convert regular hours into overtime hours until you have reduced the regular hours to the regular weekly total.

Type of Pay	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Total Hours	Rate of Pay	Earnings
Regular										
Holiday										
Overtime										
Holiday Worked										
								TOTAL EARNINGS		

Once you have completed the table, if the employee has only regular hours of pay, skip to step 5. Otherwise, proceed with the next step.

Step 2: Calculate any holiday earnings. Take the hourly rate and multiply it by the number of hours in a regular shift:

Holiday Earnings = Hourly Rate × Hours in a Regular Shift

Step 3: Calculate any overtime earnings.

1. Determine the overtime hourly rate of pay rounded to two decimals by multiplying the hourly rate by the minimum standard overtime factor of 1.5 (or higher).

Overtime Hourly Rate = Hourly Rate × 1.5

1. Multiply the overtime hourly rate by the overtime hours worked.

Step 4: Calculate any statutory holiday worked earnings. This is the same procedure as for a salaried employee.

Step 5: Calculate the gross earnings paid at the regular rate of pay. Take the number of hours worked and multiply it by the hourly rate of pay:

Regular Earnings = Hours Worked × Hourly Rate

For example, 20 hours worked at $10 per hour with no holiday earnings is 20 × $10 = $200.

Step 6: Calculate the total gross earnings by applying Formula 4.1.

Things To Watch Out For

Be careful about the language of the payment frequency. It is very common to confuse *semi* and *bi*, and sometimes businesses use the terms incorrectly. The term *semi* generally means half. Therefore, to be paid semi-monthly means to be paid every half month. The term *bi* means two. Therefore, to be paid biweekly means to be paid every two weeks. Some companies that pay semi-monthly mistakenly state that they pay bimonthly, which in fact would mean they paid every two months.

Paths To Success

In calculating the pay for a salaried employee, this textbook assumes for simplicity that a year has exactly 52 weeks. In reality, there are 52 weeks plus one day in any given year. In a leap year, there are 52 weeks plus two days. This extra day or two has no impact on semi-monthly or monthly pay, since there are always 24 semi-months and 12 months in every year. However, weekly and biweekly earners are impacted as follows:

- If employees are paid weekly, approximately once every six years there are 53 pay periods in a single year. This would "reduce" the employees' weekly paycheque in that year. For example, assume they earn $52,000 per year paid weekly. Normally, they are paid $52,000 ÷ 52 = $1,000 per week. However, since there are 53 pay periods approximately every sixth year, this results in $52,000 ÷ 53 = $981.13 per week for that year.
- If employees are paid biweekly, approximately once every 12 years there are 27 pay periods in a single year. This has the same effect as the extra pay period above. For example, if they are paid $52,000 per year biweekly they normally receive $52,000 ÷ 26 = $2,000 per biweekly cheque. Approximately every twelfth year, they are paid $52,000 ÷ 27 = $1,925.93 per biweekly cheque for that year.

Many employers ignore these technical nuances in pay structure since the extra costs incurred to modify payroll combined with the effort required to calm down employees who don't understand the smaller paycheque are not worth the savings in labour. Therefore, most employers treat every year as if it has 52 weeks (26 biweeks)

regardless of the reality. In essence, employees receive a bonus paycheque approximately once every six or twelve years!

Give It Some Thought

A salaried employee whose normal workweek is 8 hours per day and 40 hours per week works 8 hours each day from Monday to Saturday inclusive, where Monday was a statutory holiday. Which of the following statements is correct (assuming she will not get another day off in lieu of the holiday)?

1. The employee receives only her regular weekly earnings for 40 hours.
2. The employee receives 32 hours of regular earnings, 8 hours of holiday earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
3. The employee receives 40 hours of regular earnings, 8 hours of overtime earnings, and 8 hours of statutory holiday worked earnings.
4. The employee receives 40 hours of regular earnings and 8 hours of overtime earnings.

Solutions:

The correct answer is b. When working on the statutory holiday and not getting another day off in lieu, the salaried employee is eligible for eight hours of holiday earnings plus eight hours of statutory holiday worked earnings. The holiday earnings count toward the weekly total, but not the statutory holiday worked earnings. Thus the employee from Tuesday to Friday inclusive worked an additional 32 regular hours, bringing her weekly total to 40 hours. The work on Saturday exceeds her 40 hour workweek, and therefore all eight hours are paid as overtime earnings.

Example 3.1 A: Salary with Overtime and a Holiday

Tristan is compensated with an annual salary of $65,000 paid biweekly. His regular workweek consists of four 10-hour days, and he is eligible for overtime at 1.5 times pay for any work in excess of his regular requirements. Tristan worked regular hours for the first two weeks. Over the next two weeks, Tristan worked his regular hours and became eligible for 11 hours of overtime. During these two weeks, he worked his regular shift on Good Friday but his employer has agreed to give him another day off with pay in the future.

a. Determine Tristan's gross earnings for the first two-week pay period.
b. Determine Tristan's gross earnings for the second two-week pay period.

You have been asked to calculate Tristan's gross earnings, or GE, for two consecutive pay periods.

What You Already Know	How You Will Get There
Step 1: You know Tristan's compensation:	For each biweekly pay period, apply the following steps:
Annual Salary = $65,000 Pay Periods = biweekly = 26 times per year Annual Hours= 10/day × 4 days/week × 52 weeks = 2,080	Step 2: Calculate Tristan's equivalent hourly rate of pay.
	Step 3: Calculate holiday earnings using the equivalent hourly rate.
You also know his work schedule:	Step 4: Calculate overtime earnings by taking the overtime hourly pay rate multiplied by hours worked.
First Two Weeks = regular pay Overtime in First Two Weeks = 0 Second Two Weeks = regular pay Overtime in Second Two Weeks = 11 hours	Step 5: Calculate statutory holiday worked earnings at the premium rate of pay.
	Step 6: Calculate regular earnings:
There is a holiday in the second two weeks, but he will receive another day off in lieu.	Step 7: Determine total gross earnings using Formula 4.1.

Step	First Two Weeks	Second Two Weeks
2.	There are only regular earnings; skip to step 6.	Equivalent Hourly Rate $= \dfrac{\$65,000}{2,080} = \$31.25$
3.	Holiday earnings = $0	Tristan will take his holiday pay another day. Therefore, he is not eligible for this pay, and his work counts as regular hours. Holiday earnings = $0
4.	Overtime earnings = $0	Overtime hourly rate = $31.25 × 1.5 = $46.88 Overtime earnings = $46.88 × 11 = $515.68
5.	Statutory worked = $0	Since Tristan is receiving another day off in lieu, he is not eligible for this pay; Statutory worked = $0
6.	Regular earnings $= \dfrac{\$65,000}{26} - \$0 = \$2,500$	Regular earnings $= \dfrac{\$65,000}{26} - \$0 = \$2,500$
7.	GE = $2,500 + $0 + $0 + $0 = $2,500	GE = $2,500 + $515.68 + $0 + $0 = $3,015.68

For the first two-week pay period, Tristan worked only his regular hours and therefore is compensated $2,500 as per his salary. For the second two-week pay period, Tristan is eligible to receive his regular hours plus his overtime, but he receives no additional pay for the worked holiday since he will receive another day off in lieu. His total gross earnings are $3,015.68.

Example 3.1 B: A Week in the Life of an Hourly Wage Earner

Marcia receives an hourly wage of $32.16 working on an automotive production line. Her union has negotiated a regular work day of 7.25 hours for five days totalling 36.25 hours for the week. Overtime is paid at 1.5 times her regular rate for any work that exceeds the daily or weekly limits. If work is required on a statutory holiday, her company does not give a day off in lieu and pays a premium rate of 2.5 times her regular rate. Last week, Marcia worked nine hours on Monday, her regular hours on Tuesday through Friday inclusive, and three hours on Saturday. Friday was a statutory holiday. Calculate Marcia's gross earnings for the week.

You need to calculate Marcia's gross earnings, or GE, for the week.

What You Already Know

Step 1: You know Marcia's pay structure and workweek:

Regular Hourly Rate = $32.16

Overtime Hourly Pay = 1.5 times

Statutory Holiday Worked Rate = 2.5 times

Exceeding 7.25 hours daily or 36.25 hours weekly is overtime.

Sunday	0
Monday	9
Tuesday	7.25
Wednesday	7.25
Thursday	7.25
Friday (statutory holiday)	7.25
Saturday	3

How You Will Get There

Step 1 (continued): Take Marcia's hours and place them into the table. Assess whether any daily or weekly totals are considered overtime and make any necessary adjustments.

Step 2: Calculate holiday earnings using the hourly rate.

Step 3: Calculate overtime earnings by determining the overtime hourly rate of pay multiplied by hours worked.

Step 4: Calculate statutory holiday worked earnings at the premium rate of pay.

Step 5: Calculate regular earnings.

Step 6: Determine total gross earnings.

Step 1:

Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings
Regular	0	7.25 [1]	7.25	7.25	7.25		3	39.25 [3]	$32.16	
Holiday						7.25 [2]				
Overtime		1.75 [1]						1.75		
Holiday Worked						7.25 [2]		7.25		
								TOTAL EARNINGS		

[1] She worked nine hours. Therefore, the first 7.25 hours are regular pay and the last 1.75 are overtime pay.

[2] Friday was a statutory holiday, and she will not receive another day off in lieu. She must receive statutory holiday worked pay in addition to her hours worked.

[3] Note the weekly total of 36.25 has been exceeded by three hours. Move Saturday's hours into overtime.

The following table is the final layout of her workweek:

Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings
Regular	0	7.25	7.25	7.25	7.25			36.25	$32.16	
Holiday						7.25				
Overtime		1.75					3	4.75		
Holiday Worked						7.25		7.25		
								TOTAL EARNINGS		

Steps 2-6: Explanations are noted in the table below.

Type	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Total	Rate	Earnings
Regular	0	7.25	7.25	7.25	7.25			36.25	$32.16	$932.64 [4]
Holiday						7.25				$233.16 [1]
Overtime		1.75					3	4.75	$48.24 [2]	$229.14 [2]
Holiday Worked						7.25		7.25	$80.40 [3]	$582.90 [3]
								TOTAL EARNINGS		$1,977.84 [5]

[1] Holiday Earnings = $7.25 \times \$32.16 = \$233.16$

[2] Overtime Hourly Rate = $\$32.16 \times 1.5 = \$48.24$
Overtime Earnings = $4.75 \times \$48.24 = \$229.14$

[3] Statutory Holiday Worked Rate = $\$32.16 \times 2.5 = \$80.40$
Statutory Worked Earnings = $7.25 \times \$80.40 = \$582.90$

[4] Regular Earnings $= 29 \times \$32.16 = \$932.64$

[5] GE $= \$932.64 + \$229.14 + \$233.16 + \$582.90 = \$1,977.84$

Marcia will receive total gross earnings of $1,977.84 for the week.

Marcia's Weekly Gross Earnings

$1,977.84

- Regular Earnings --> $932.64
- Holiday Earnings --> $233.16
- Overtime Earnings --> $229.14
- Holiday Worked Earnings --> $582.90
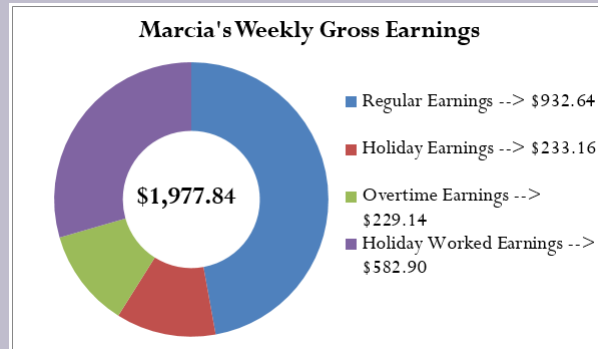
Commission

Over the last two weeks you sold $50,000 worth of machinery as a sales representative for IKON Office Solutions Canada. IKON's compensation plan involves a straight commission rate of 3.5%. What are your gross earnings? If you sold an additional $12,000 in machinery, how much more would you earn?

Particularly in the fields of marketing and customer service, many workers are paid on a commission basis. A **commission** is an amount or a fee paid to an employee for performing or completing some form of transaction. The commission typically takes the form of a percentage of the dollar amount of the transaction. Marketing and customer service industries use this form of compensation as an incentive to perform: If the representative doesn't sell anything then the representative does not get paid. Issues to be discussed about commission include what constitutes regular earnings, how to handle holidays and overtime, and the three different types of commission structures.

- *Regular Earnings.* All commissions are considered to be regular earnings. To calculate the gross earnings for an employee, take the total amount of the transactions and multiply it by the commission rate:

Gross Earnings = Total Transaction Amount × Commission Rate

This is not a new formula. It is a specific application of Formula 2.2: Rate, Portion, Base. In this case, the Base is the total amount of the transactions, the Rate is the commission rate, and the Portion is the gross earnings for the employee.

- *Holidays and Overtime.* Commission earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the provincial standards on these matters vary widely and the mathematics involved do not necessarily follow any one procedure or calculation. As such, this textbook leaves these issues to be covered in a payroll administration course.
- *Types of Commission.* Commission earnings typically follow one of the following three structures:
- *Straight Commission*. If your entire earnings are based on your dollar transactions and calculated strictly as a percentage of the total, you are on **straight commission**. An application of Formula 2.2 (Rate, Portion, Base) calculates your gross earnings under this structure.
- *Graduated Commission*. Within a **graduated commission** structure, you are offered increasing rates of commission for higher levels of performance. The theory behind this method of compensation is that the higher rewards motivate employees to perform better. An example of a graduated commission scale is found in the table below.

Transaction Level	Commission Rate
$0–$100,000.00	3%
$100,000.01–$250,000.00	4.5%
$250,000.01–$500,000.00	6%
Over $500,000.00	7.5%

Recognize that the commission rates are applied against the portion of the sales falling strictly into the particular category, not the entire balance. Thus, if the total sales equal $150,000, then the first $100,000 is paid at 3% while the next $50,000 is paid at 4.5%.

- *Salary Plus Commission*. If your earnings combine a basic salary together with commissions on your dollar transactions, you have a **salary plus commission** structure. No new mathematics are required for this commission type. You must combine salary calculations, discussed earlier in this section, with either a straight commission or graduated commission, as discussed above. Usually this form of compensation pays the lowest commission rate since a basic salary is already provided.

How It Works

Follow these steps to calculate commission earnings:

Step 1: Determine which commission structure is used to pay the employee. Identify information on commission rates, graduated scales, and any salary.

Step 2: Determine the dollar amounts that are eligible for any particular commission rate and calculate commissions.

Step 3: Sum all earnings from every eligible commission rate plus any salary.

Let's assume $380,000 of merchandise is sold. Using the previous table as our graduated commission scale, calculate commission earnings.

Step 1: Sales total $380,000 and all commission rates and scales are found in the table.

Step 2: The first $100,000 is compensated at 3%, equalling $3,000. The next $150,000 is compensated at 4.5%, equalling $6,750. The last $130,000 is compensated at 6%, equalling $7,800. There is no compensation at the 7.5% level since sales did not exceed $500,000.

Step 3: The total commission on sales of $380,000 is $3,000 + $6,750 + $7,800 = $17,550.

Example 3.1 C: Different Types of Commissions
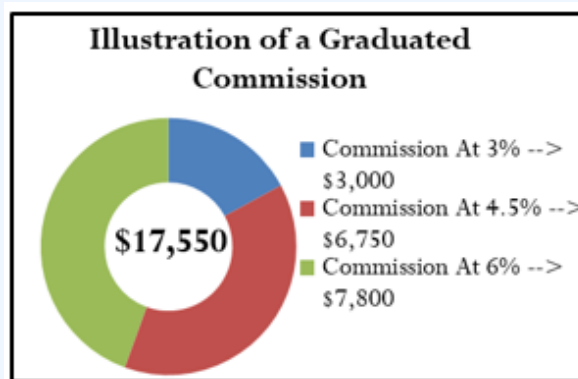
Josephine is a sales representative for Kraft Foods Canada. Over the past two weeks, she closed $325,000 in retail distribution contracts. Calculate the total gross earnings that Josephine earns if

a. She is paid a straight commission of 3.45%.
b. She is paid 2% for sales on the first $100,000, 3% on the next $100,000, and 4% on all remaining sales.
c. She is paid a base salary of $2,000 plus a commission of 3.5% on all sales above $100,000.

You need to calculate the gross earnings, or GE, for Josephine under various commission structures.

What You Already Know

Step 1: For all three option, total sales = $325,000

a. This is a straight commission where the commission rate = 3.45%.
b. This is a graduated commission, structured as follows:

Sales Level	Rate
$0-$100,000.00	2%
$100,000.01-$200,000.00	3%
$200,000.01 and above	4%

c. This is a salary plus graduated commission, with salary = $2,000 and graduated commission structure as follows:

Sales Level	Rate
$0-$100,000.00	0%
$100,000.01 and above	3.5%

How You Will Get There

Step 2: Determine the sales eligible for each commission and calculate total commissions. Note that all calculations are $Portion = Rate \times Base.$

Step 3: Sum all commissions plus any salary to calculate total gross earnings.

Steps 2 & 3:

a. $3.45\% \times \$325,000 = 0.0345 \times \$325,000 = \$11,212.50$ = Total Gross Earnings

b.

Sales Level	Rate	Eligible Sales at Each Rate	Commission Earned
$0-$100,000.00	2%	$\$100,000 - \$0 = \$100,000$	$0.2 \times \$100,000 = \$2,000$
$100,000.01-$200,000.00	3%	$\$200,000 - \$100,000 = \$100,000$	$0.03 \times \$100,000 = \$3,000$
$200,000.01 and above	4%	$\$325,000 - \$200,000 = \$125,000$	$0.04 \times \$125,000 = \$5,000$

Total Gross Earnings = $2,000 + $3,000 + $5,000 = $10,000

c. Recall Base Salary = $2,000

Sales Level	Rate	Base	Commission Earned
$0-$100,000.00	0%	$\$100,000 - \$0 = \$100,000$	$0 \times \$100,000 = 0$
$100,000.01 and above	3.5%	$\$325,000 - \$100,000 = \$225,000$	$0.035 \times \$225,000 = \$7,875$

Total Gross Earnings = $0 + $7,875 + $2,000 = $9,875

If Josephine is paid under straight commission, her total gross earnings will be $11,212.50. Under the graduated commission, she will receive $10,000 in total gross earnings. For the salary plus commission, she will receive $9,875 in total gross earnings.

Josephine's Gross Earnings under Different Commission Structures
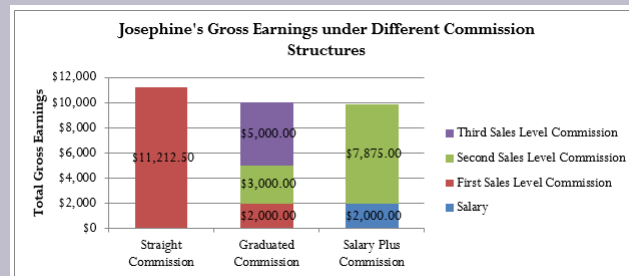
Piecework

Have you ever heard the phrase "pay-for-performance"? Although this phrase has many interpretations in different industries, for some people this phrase means that they get paid based on the quantity of work that they do. For example, many workers in clothing manufacturing are paid a flat rate for each article of clothing

they produce. As another example, employees in fruit orchards may get paid by the number of pieces of fruit that they harvest, or simply by the kilogram. As you can see, these workers are neither salaried nor paid hourly, nor are they on commission. They earn their paycheque for performing a specific task. Therefore, a **piecework** wage compensates such employees on a per-unit basis.

This section focuses on the regular earnings only for piecework wage earners. Similar to workers on commission, piecework earners are eligible to receive overtime earnings, holiday earnings, and statutory holiday worked earnings. However, the standards vary widely from province to province, and there is not necessarily any one formula to calculate these earnings. As with commissions, this textbook leaves those calculations for a payroll administration course.

The Formula

To calculate the regular gross earnings for a worker paid on a piecework wage, you require the piecework rate and how many units they are to be paid for:

$$\text{Gross Earnings} = \text{Piecework Rate} \times \text{Eligible Units}$$

This is not a new formula but another application of Formula 2.2 (Rate, Portion, Base). The Piecework Rate is the Rate, the Eligible Units are the Base, and the Gross Earnings are the Portion.

How It Works

To calculate an employee's gross earnings for piecework, follow these steps:

Step 1: Identify the piecework rate and the level of production or units.

Step 2: Perform any necessary modifications on the production or units to match how the piecework is paid.

Step 3: Calculate the commission gross earnings by multiplying the rate by the eligible units.

Assume that Juanita is a piecework earner at a blue jean manufacturer. She is paid daily and earns $1.25 for every pair of jeans that she sews. On a given day, Juanita sewed 93 pairs of jeans. Her gross earnings are calculated as follows:

Step 1: Her piecework rate = $1.25 per pair with production of 93 units.

Step 2: The rate and production are both expressed by the pair of jeans. No modification is necessary.

Step 3: Her gross piecework earnings are the product of the rate and units produced, or $1.25 × 93 = $116.25.

Things To Watch Out For

Pay careful attention to step 2 in the procedure. In some industries, the piecework rate and the units of production do not match. For example, a company could pay a piecework rate per kilogram, but a single unit may not represent a kilogram. This is typical in some canning industries, where workers are paid per kilogram for canning the products, but the cans may only be 200 grams in size. Therefore, if workers produce five cans, they are not paid for five units produced. Rather, they are paid for only one unit produced since five cans × 200 g = 1,000 g = 1 kg. Before calculating piecework earnings, ensure that both the piecework rate and the eligible units are in the same terms, whether it be metric tonnes, kilograms, or otherwise.

Example 3.1 D: A Telemarketer Earning Piecework Wages

In outbound telemarketing, some telemarketers are paid on the basis of "completed calls." This is not commission since their pay is not based on actually selling anything. Rather, a completed call is defined as simply any phone call for which the agent speaks with the customer and a decision is reached, regardless of whether the decision was to accept, reject, or request further information. If a telemarketer produces five completed calls per hour and works 7½-hour shifts five times per week, what are the total gross earnings she can earn over a biweekly pay period if her piecework wage is $3.25 per completed call?

We are looking for the total gross earnings, or GE, for the telemarketer over the biweekly pay period.

What You Already Know	How You Will Get There
Step 1: The frequency of the telemarketer's pay, along with her hours of work, piecework wage, and unit of production are known:	Step 2: You must determine the telemarketer's production level. Calculate how many completed calls she achieves per biweekly pay period:
Piecework Rate = $3.25 per completed call	Eligible Units = Units Produced per Hour × Hours per Day × Days per Week × Weeks
Hourly Units Produced = 5	Step 3: Apply Formula 2.2 (adapted for piecework wages) to get the portion owing.
Hours of Work = 7½ hours per day, five days per week	
Frequency of Pay = biweekly	

Step 2: Eligible Units = $1.5 \times 7.5 \times 5 \times 2 = 375$

Step 3: GE = $\$3.25 \times 375 = \$1,218.75$

Over a biweekly period, the telemarketer completes 375 calls. At her piecework wage, this results in total gross earnings of $1,218.75.

Exercises

Mechanics

1. Laars earns an annual salary of $60,000. Determine his gross earnings per pay period under each of the following payment frequencies:
   a. Monthly   b. Semi-monthly   c. Biweekly   d. Weekly

2. A worker earning $13.66 per hour works 47 hours in the first week and 42 hours in the second week. What are his total biweekly earnings if his regular workweek is 40 hours and all overtime is paid at 1.5 times his regular hourly rate?

3. Marley is an independent sales agent. He receives a straight commission of 15% on all sales from his suppliers. If Marley averages semi-monthly sales of $16,000, what are his total annual gross earnings?

4. Sheila is a life insurance agent. Her company pays her based on the annual premiums of the customers that purchase life insurance policies. In the last month, Sheila's new customers purchased policies worth $35,550 annually. If she receives 10% commission on the first $10,000 of premiums and 20% on the rest, what are her total gross earnings for the month?

5. Tuan is a telemarketer who earns $9.00 per hour plus 3.25% on any sales above $1,000 in any given week. If Tuan works 35 regular hours and sells $5,715, what are his gross earnings for the week?

6. Adolfo packs fruit in cans on a production line. He is paid a minimum wage of $9.10 per hour and earns $0.09 for every can packed. If Adolfo manages to average 160 cans per hour, what are his total gross earnings daily for an eight-hour shift?

Applications

7. Charles earns an annual salary of $72,100 paid biweekly based on a regular workweek of 36.25 hours. His company generously pays all overtime at twice his regular wage. If Charles worked 85.5 hours over the course of two weeks, what are his gross earnings?

8. Armin is the payroll administrator for his company. In looking over the payroll, he notices the following workweek (from Sunday to Saturday) for one of the company's employees: 0, 6, 8, 10, 9, 8, and 9 hours, respectively. Monday was a statutory holiday, and with business booming the employee will not be given another day off in lieu. Company policy pays all overtime at time-and-a-half, and all hours worked on a statutory holiday are paid at twice the regular rate. A normal workweek consists of five, eight-hour days. If the employee receives $22.20 per hour, what are her total weekly gross earnings?

9. In order to motivate a manufacturer's agent to increase his sales, a manufacturer offers

monthly commissions of 1.2% on the first $125,000, 1.6% on the next $150,000, 2.25% on the next $125,000, and 3.75% on anything above. If the agent managed to sell $732,000 in a single month, what commission is he owed?

10. Humphrey and Charlotte are both sales representatives for a pharmaceutical company. In a single month, Humphrey received $5,545 in total gross earnings while Charlotte received $6,388 in total gross earnings. In sales dollars, how much more did Charlotte sell if they both received 5% straight commission on their sales?

11. Mayabel is a cherry picker working in the Okanagan Valley. She can pick 17 kg of cherries every hour. The cherries are placed in pails that can hold 13.6 kg of cherries. If she works 40 hours in a single week, what are her total gross earnings if her piecework rate is $17.00 per pail?

12. Miranda is considering three relatively equal job offers and wants to pick the one with the highest gross earnings. The first job is offering a base salary of $1,200 semi-monthly plus 2% commission on monthly sales. The second job offer consists of a 9.75% straight commission. Her final job offer consists of monthly salary of $1,620 plus 2.25% commission on her first $10,000 in monthly sales and 6% on any monthly sales above that amount. From industry publications, she knows that a typical worker can sell $35,000 per month. Which job offer should she choose, and how much better is it than the other job offers?

13. A Canadian travel agent is paid a flat rate of $37.50 for every vacation booked through a certain airline. If the vacation is in North America, the agent also receives a commission of 2.45%. If the vacation is international, the commission is 4.68%. What are the total monthly gross earnings for the agent if she booked 29 North American vacations worth $53,125 and 17 international vacations worth $61,460?

14. Vladimir's employer has just been purchased by another organization. In the past, he has earned $17.90 per hour and had a normal workweek of 37.5 hours. However, his new company only pays its employees a salary semi-monthly. How much does Vladimir need to earn each paycheque to be in the same financial position?

Challenge, Critical Thinking, & Other Applications

15. An employee on salary just received his biweekly paycheque in the amount of $1,832.05, which included pay for five hours of overtime at time-and-a-half. If a normal workweek is 40 hours, what is the employee's annual salary?

16. A graduated commission scale pays 1.5% on the first $50,000, 2.5% on the next $75,000, and 3.5% on anything above. What level of sales would it take for an employee to receive total gross earnings of $4,130?

17. A sales organization pays a base commission on the first $75,000 in sales, base + 2% on the

next $75,000 in sales, and base + 4% on anything above. What is the base commission if an employee received total gross earnings of $7,500 on $200,000 in sales?

18. A typical sales agent for a company has annual sales of $4,560,000, equally spread throughout the year, and receives a straight commission of 2%. As the new human resource specialist, to improve employee morale you have been assigned the task of developing different pay options of equivalent value to offer to the employees. Your first option is to pay them a base salary of $2,000 per month plus commission. Your second option is to pay a base commission monthly on their first $100,000 in sales, and a base + 2.01% on anything over $200,000 per month. In order to equate all the plans, determine the required commission rates, rounded to two decimals in percent format, in both options.

19. Shaquille earns an annual salary of $28,840.50 paid biweekly. His normal workweek is 36.25 hours and overtime is paid at twice the regular rate. In addition, he is paid a commission of 3% of sales on the first $25,000 and 4% on sales above that amount. What are his total gross earnings during a pay period if he worked 86 hours and had sales of $51,750?

20. Mandy is paid $9.50 per hour and also receives a piecework wage of $0.30 per kilogram, or portion thereof. A regular workday is 7.5 hours and 37.5 hours per week. Overtime is paid at time-and-a-half, and any work on a statutory holiday is paid at twice the regular rate. There is no premium piecework wage. Mandy's work record for a two-week period is listed below. Determine her total gross earnings.

Week		Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
1	Hours worked		7.5	7.5	9	8	7.5	3
250 g items produced		1,100	1,075	1,225	1,150	1,025	450
2	Hours worked		4 — Statutory holiday, no day off in lieu	7.5	10	7.5	8
250 g items produced		575	1,060	1,415	1,115	1,180	

3.2 Personal Income Tax

You have just filled out your income tax return, where you see that your total payable for the year is more than

$5,000. You are glad you do not have to pay this lump sum all at once! Instead, almost all of it has actually been paid already: Your employer has made regular deductions from each paycheque to cover your federal and provincial taxes.

In Canada, businesses are taxed differently than individuals and there are many nuances and complexities to address. For these reasons, this textbook does not cover business taxes, which require a course in corporate taxes.

For individuals, taxes are relatively straightforward. This section of the book uses the 2013 tax year federal and provincial/territorial personal income tax brackets to calculate the amount of these taxes that are ultimately deducted from your gross earnings. You can find current-year tax brackets at www.taxtips.ca/marginaltaxrates.htm.

Federal and Provincial/Territorial Tax Brackets and Tax Rates

Personal income tax is a tax on gross earnings levied by both the federal and provincial/territorial governments. All of Canada uses a **progressive tax system** in which the tax rate increases as the amount of income increases; however, the increased tax rates apply only to income amounts above minimum thresholds. Thus, higher income earners pay higher marginal tax rates than lower income earners. The format is similar to a graduated commission structure.

The federal and provincial/territorial governments offer a **basic personal amount**, which is the amount of income for which the wage earner is granted a tax exemption. In other words, it is tax-free income. This is designed to help low-income earners in Canada. Tax brackets are adjusted annually federally and in most provinces and territories to reflect increases in the cost of living, as measured by the consumer price index (CPI).

2013 Provincial Taxes

Province/Territory	Taxable Income	Tax Rate
British Columbia	$0.00–$10,276.00*	0%
	$10,276.01–$37,568.00	5.06%
	$37,568.01–$75,138.00	7.7%
	$75,138.01–$86,268.00	10.5%
	$86,268.01–$104,754.00	12.29%
	$104,754.01 and up	14.7%
Alberta	$0.00–$17,593.00*	0%
	$17,593.01 and up	10%
Saskatchewan	$0.00–$15,241.00*	0%
	$15,241.01–$42,906.00	11%
	$42,906.01–$122,589.00	13%
	$122,589.01 and up	15%
Manitoba	$0.00–$8,884.00*	0%
	$8,884.01–$31,000.00	10.8%
	$31,000.01–$67,000.00	12.75%
	$67,000.01 and up	17.4%
Ontario**	$0.00–$9,574.00*	0%
	$9,574.01–$39,723.00	5.05%
	$39,723.01–$79,448.00	9.15%
	$79,488.01–$509,000.00	11.16%
	$509,000.01 and up	13.16%
Quebec	$0.00–$11,195.00*	0%
	$11,195.01–$41,095.00	16%
	$41,095.01–$82,190.00	20%
	$82,190.01 and up	24%
New Brunswick	$0.00–$9,388.00*	0%
	$9,388.01–$38,954.00	9.1%
	$38,954.01–$77,908.00	12.1%
	$77,908.01–$126,662.00	12.4%
	$126,662.01 and up	14.3%

Province/Territory	Taxable Income	Tax Rate
Nova Scotia	$0.00–$8,481.00*	0%
	$8,481.01–$29,590.00	8.79%
	$29,590.01–$59,180.00	14.95%
	$59,180.01–$93,000.00	16.67%
	$93,000.01–$150,000.00	17.5%
	$150,000.01 and up	21%
Prince Edward Island**	$0.00–$7,708.00*	0%
	$7,708.01–$31,984.00	9.8%
	$31,984.01–$63,969.00	13.8%
	$63,969.01 and up	16.7%
Newfoundland and Labrador	$0.00–$8,451.00*	0%
	$8,451.01–$33,748.00	7.7%
	$33,748.01–$67,496.00	12.5%
	$67,496.01 and up	13.3%
Yukon**	$0.00–$11,038.00*	0%
	$11,038.01–$43,561.00	7.04%
	$43,561.01–$87,123.00	9.68%
	$87,123.01–$135,054.00	11.44%
	$135,054.01 and up	12.76%
Northwest Territories	$0.00–$13,546.00*	0%
	$13,546.01–$39,453.00	5.9%
	$39,453.01–$78,908.00	8.6%
	$78,908.01–$128,286.00	12.2%
	$128,286.01 and up	14.05%
Nunavut	$0.00–$12,455.00*	0%
	$12,455.01–$43,535.00	4%
	$43,535.01–$83,071.00	7%
	$83,071.01–$135,054.00	9%
	$135,054.01 and up	11.5%

* These amounts represent the basic personal amount.

**There are additional income surtaxes in these provinces and Yukon. These surtaxes are not covered in this textbook.

2013 Federal Taxes

Taxable Income	Tax Rate
$0.00–$11,038.00*	0%
$11,038.01–$43,561.00	15%
$43,561.01–$87,123.00	22%
$87,123.01–$135,054.00	26%
$135,054.01 and up	29%

*This is the basic personal amount.

The Formula

To calculate the income taxes for any individual, you must total all taxes from all eligible gross earnings in every tax bracket. Formula 4.2 shows the calculation.

Annual Income Taxes

Income Tax: The total of all tax portions in all tax brackets represents the amount of taxes that must be deducted from annual gross earnings and remitted to the appropriate government(s). You must complete these calculations both federally and provincially to determine the total annual income taxes. The formula is an adaptation of Formula 2.2: Rate, Portion, and Base, where:

- Each eligible income in any tax bracket is the base.
- Each associated tax bracket rate is the rate.

Multiplying each base by each rate produces the portion of eligible gross earnings owed in taxes.

Formula 4.2 – Annual Income Tax:

$$\text{Income Tax} = \sum (\text{Eligible Income in Tax Bracket} \times \text{Tax Bracket Rate})$$

Eligible Income in Tax Bracket: All earnings fall into tax brackets, as listed in the two tables above. Therefore, you must separate gross earnings into the different tax brackets. The amount of gross earnings in any tax bracket is then taxed at the associated tax rate.

Tax Bracket Rate: Every tax bracket has an associated tax rate as listed in the two tables above. This rate increases as the gross earnings increase. Multiply the gross earnings in any tax bracket by this rate to calculate the portion of the earnings that are owed in taxes.

How It Works

To calculate the total annual income tax owing in any tax year, follow these steps:

Step 1: Determine the total amount of gross earnings that are taxable for the individual.

Step 2: Calculate the federal income taxes owing. Using the federal tax table, split the gross earnings into each of the listed tax brackets, then calculate the taxes owing in each tax bracket by multiplying the income in the bracket and the tax rate. Round any tax amounts to two decimals.

Step 3: Calculate the provincial/territorial income taxes owing. Repeat the same process as in step 2, but use the appropriate tax brackets from the provincial tax table instead. Round any tax amounts to two decimals.

Step 4: Calculate the total annual income taxes owing by applying Formula 4.2. This means adding all calculated tax amounts from both steps 2 and 3 together.

Assume you live in British Columbia and your taxable gross income is $44,000. Here is how you would calculate your annual total federal and provincial taxes:

Step 1: The gross earnings are $44,000.

Step 2: Federally, your income falls into the first three brackets. Your first $11,038 is taxed at 0%; therefore, there are no income taxes on this amount. The next $32,523 (from $11,038.01 to $43,561.00) is taxed at 15%, thus $32,523 × 15% = $4,878.45. The last $439 (from $43,561.01 to $44,000.00) is taxed at 22%, thus $439 × 22% = $96.58.

Step 3: Provincially, your income falls into the first three brackets for British Columbia. Your first $10,276 is taxed at 0%; therefore, there are no income taxes on this amount. The next $27,292 (from $10,276.01 to $37,568.00) is taxed at 5.06%, thus $27,292 × 5.06% = $1,380.98. The last $6,432 (from $37,568.01 to $44,000) is taxed at 7.7%, thus $6,432 × 7.7% = $495.26.

Step 4: Your total federal income tax is $0.00 + $4,878.45 + $96.58 = $4,975.03. Your total provincial income tax is $0.00 + $1,380.98 + $495.26 = $1,876.24. Applying Formula 4.1, income tax = $4,975.03 + $1,876.24 = $6,851.27, which is the amount that will be deducted from total gross earnings.

Things To Watch Out For

Have you ever heard someone say, "I earn more income and moved to a higher tax bracket, so now I am earning less money and my paycheque is lower"? The progressive tax system used in Canada makes this statement untrue.

Remember that tax rates apply only to the portion of the gross earnings in the tax bracket and are not retroactive to lower levels of income. For example, if your taxable gross income increased from $80,000 to $88,000, your highest tax bracket is now 26% instead of 22%. However, your federal income tax is not calculated at 26% for your entire income. Rather, the first $11,038 is tax free, the next $32,523 is taxed at 15%, the next $43,562 is taxed at 22%, and the final $877 is taxed at 26%.

Paths To Success

It is sometimes beneficial to pre-calculate the total income taxes in any tax bracket under the assumption that the individual's income comprises the entire tax bracket. This technique is particularly useful when repetitive calculations are required, such as determining the federal income taxes for each employee in an entire company. For example, the second federal tax bracket extends from $11,038.01 to $43,561, representing $32,523 of employee income. Since this bracket is taxed at 15%, someone who earns a higher income always owes the full

amount of tax in this category, which is $32,523 × 15% = $4,878.45. The next tax bracket covers $43,561.01 to $87,123, representing $43,562 of income. Someone earning a higher income always owes the full amount of tax in this category, which is $43,562 × 22% = $9,583.64.

Assume taxable gross earnings of $88,000. With your pre-calculated income taxes in each bracket, you know the income taxes in the first three tax brackets are $0, $4,878.45, and $9,583.64, or $14,462.09 in total. You only need to calculate the tax on the portion of income in the final tax bracket of $877 × 26% = $228.02. The grand total is $14,462.09 + $228.02 = $14,690.11.

Example 3.2 A: Federal and Provincial Income Taxes

A Canadian wage earner has taxable gross income of $97,250. Calculate the federal and provincial annual income taxes individually and then sum the amounts to calculate the total annual income taxes if the wage earner lives in

a. Ontario
b. New Brunswick
c. Alberta

For each of the provinces, you need to calculate the individual's total federal income tax and total provincial income tax. Then sum these amounts to arrive at the annual income tax.

What You Already Know	How You Will Get There
Step 1: The employee's gross taxable earnings are $97,250. From the tax tables, you also know the federal and provincial income tax brackets and corresponding tax rates.	Step 2: Calculate the federal income tax. This amount is the same in all three provinces and needs to be calculated only once. Looking at the federal tax table, you see that this wage earner falls into the first four tax brackets, which requires the sum of the income tax calculations for each bracket.
	Step 3: Calculate the provincial income tax. Using the provincial tax table, the income spans all four of Ontario's tax brackets. In New Brunswick, the individual falls into the first four tax brackets. In Alberta, the individual spans both of Alberta's tax brackets. In all three provinces, the total provincial income is the sum of the income tax calculations for each bracket.
	Step 4: Calculate total annual income taxes using Formula 4.2.

Step 2:

$$(\$11,038 - \$0) \times 0 = \$0.00$$
$$(\$43,561 - \$11,038) \times 0.15 = \$4,878.45$$
$$(\$87,123 - \$43,561) \times 0.22 = \$9,583.64$$
$$(\$97,250 - \$87,123) \times 0.26 = \$2,633.02$$

Total federal income tax = $0.00 + $4,878.45 + $9,583.64 + $2,633.02 = $17,095.11

Step 3:

Ontario	New Brunswick	Alberta
($9,574 − $0) × 0 = $0.00 ($39,723 − $9,574) × 0.0505 = $1,522.52 ($79,448 − $39,723) × 0.0915 = $3,634.84 ($97,250 − $79,448) × 0.1116 = $1,986.70	($9,388 − $0) × 0 = $0.00 ($38,954 − $9,388) × 0.091 = $2,690.51 ($77,908 − $38,954) × 0.121 = $4,713.43 ($97,250 − $77,908) × 0.124 = $2,398.41	($17,593 − $0) × 0 = $0.00 ($97,250 − $17,593) × 0.10 = $7,965.70
Total Ontario income tax = $0.00 + $1,522.52 + $3,634.84 + $1,986.70 = $7,144.06	Total New Brunswick income tax = $0.00 + $2,690.51 + $4,713.43 + $2,398.41 = $9,802.35	Total Alberta income tax = $0.00 + $7,965.70 = $7,965.70

Step 4:

Total income tax if living in Ontario = $17,095.11 + $7,144.06 = $24,239.17

Total income tax if living in New Brunswick = $17,095.11 + $9,802.35 = $26,897.46

Total income tax if living in Alberta = $17,095.11 + $7,965.70 = $25,060.81

For an individual earning $97,250 in taxable gross income, the person will pay $24,239.17 in total income tax if living in Ontario, $26,897.46 total income tax if living in New Brunswick, and $25,060.81 if living in Alberta.

Exercises

Use the provincial and federal tax tables introduced earlier in this chapter when you determine income taxes in these exercises.

Mechanics

For questions 1–3, determine only the federal income taxes on the following gross taxable incomes.

1. $22,375
2. $158,914
3. $102,100

For questions 4–6, determine only the provincial income taxes on the following gross taxable incomes.

4. $61,000 in Newfoundland and Labrador
5. $83,345 in Saskatchewan
6. $78,775 in British Columbia

For questions 7 and 8, determine the total federal and provincial/territorial taxes on the following gross taxable incomes.

7. $48,910 in the Northwest Territories
8. $65,525 in Prince Edward Island

Applications

9. Nadia lives in Manitoba and has three part-time jobs. Her gross annual taxable income from each job was $5,300, $21,450, and $25,390. How much federal and provincial income tax does Nadia owe?

10. If Delaney's gross taxable income increases from $71,000 to $79,000, by what dollar amount will her after-tax pay change once federal and provincial income taxes are deducted in the province of Quebec?

11. Helen just moved from Saskatchewan to her new job in Alberta at the same rate of pay of $51,225. By what dollar amount will her provincial income taxes change?

12. Jane has received two job offers that she thinks are relatively equal. The only difference lies in the salary. The first job offer is for a position in Nova Scotia earning gross taxable income of $63,375. The second job offer is for a position in British Columbia earning gross taxable income of $61,990. If Jane will select the job that has the highest after-tax income, which one should she choose? How much better is this option in dollars?

13. Suppose an employee earns $111,300 in gross taxable income. Calculate the federal and provincial income taxes that need to be deducted if he lives in either Saskatchewan, Ontario, or the Northwest Territories. In which province or territory will he earn the most income after income tax deductions? The least? What is the dollar amount difference among the three alternatives?

14. After both federal and provincial income taxes are deducted, who would earn more money, an individual earning $85,000 in New Brunswick or an individual earning $79,000 in Nunavut?

Challenge, Critical Thinking, & Other Applications

15. Rawatha is the human resources manager for her firm located in Yukon. The salespeople for her organization are paid an annual base salary of $30,000 plus 8% straight commission on their annual sales. If the average salesperson sells $210,000 annually, what is the total annual federal and provincial income tax owing on the salesperson's gross taxable income?

16. Esmerelda works on the production line and is paid a piecework rate of $2.25 per unit produced. She is capable of producing an average of five units per hour, and works eight hours every weekday. Assuming a 52-week year, what is the annual federal income tax and provincial income tax owing if she lives in Prince Edward Island?

17. Mary Jane is paid biweekly at a rate of $34.68 per hour and works 36.25 hours every week. Assuming a 52-week year, what are the amounts of the federal income tax deduction and

provincial income tax deduction that must be removed from each paycheque if Mary Jane lives in Manitoba? (Hint: income tax would be deducted equally across all paycheques for the year.)

18. Felix was transferred from his head office in Ontario to Alberta. In accepting the transfer, his employer agreed to increase his salary of $88,000 by $7,000. What is the percent change in Felix's after-tax income?

19. If a wage earner paid $2,277.98 in total annual Ontario provincial income taxes, determine his annual gross taxable income.

20. Perform a comparison across all territories and provinces of the income earned after both federal and provincial income taxes are deducted for a Canadian who earns $56,738 in gross taxable income. Which province or territory has the highest income after deductions? Which has the lowest? What is the percent difference between the two?

3.3 Indexes

You read in the *Financial Post* that the S&P/TSX Composite Index is up 12.74 points while the NASDAQ is down 45.98, and you wonder what this means for your investment portfolio. The consumer price index rose 3.1% since last year, and you wonder if the raise your employer offered you keeps pace. You need to understand indexes to make sense of these numbers.

What Is An Index?

An **index** is a number used to compare two quantities sharing the same characteristic as measured under different circumstances. Usually, the comparison involves two different points of time. The two quantities are expressed as a rate of one another and are then converted to an index number using a base value (usually 100 or 1,000). The result is called an **index number**, which expresses the relationship between the two quantities.

The Formula

Index Numbers

Chosen Quantity is The Portion: This variable is the quantity to be compared against some other number. It is the quantity of interest in your calculation.

Base Quantity is The Base: This variable is the quantity that you want to compare the chosen value against. The result of taking the Chosen Quantity divided by the Base Quantity is a rate, since it involves taking a portion and dividing it by a base.

Base Value is The Conversion: Most indexes are expressed in terms of percentages; therefore, the base value is commonly 100. This is the default value to use unless otherwise stated.

Formula 4.3 – Index Numbers:
$$\text{Index Number} = \frac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$$

Index Number is a Modified Rate: The expression of the relationship between the chosen quantity and the base quantity in the unit of the base value. There are two common interpretations to this number.

1. Most index numbers represent a percentage (where the base value is 100); however, the percent sign is never written after the index number. Therefore, a calculation resulting in an index number of 117 is interpreted as the chosen quantity being 117% of the base quantity. This is also interpreted as a percent change in which the chosen quantity is 17% higher than the base quantity. Index numbers of this nature (i.e., with a base of 100) are rounded to one decimal in most cases, which is the rounding standard for this textbook.

2. If the base value is other than 100, then the index number is not a percentage and only expresses the ratio between the two numbers. For example, an index of 2,000 using a base value of 1,000 concludes that the chosen quantity is twice as large as the base quantity. Ultimately, this type of index number is convertible into a percentage using Formula 2.2 (Rate, Portion, Base), and the chosen quantity is expressed as 200% of the base quantity, or a 100% change or increase from the base quantity. If the value being represented is dollars, round this index to two decimals. If the value is in other units, a suitable rounding standard must be

How It Works

Follow these steps to calculate an index number:

Step 1: Identify the base and the base value for the index.

Step 2: Identify the value of the chosen quantity to be compared against the base value.

Step 3: Calculate the index number by applying Formula 4.3.

In the above steps, note that in the event that the index number is known and you are solving for a different unknown, identify the known variables in steps 1 and 2. Assume the price of tuition was $3,535.00 in 2009 and $3,978.67 in 2012. The goal is to express the 2012 tuition indexed against the 2009 tuition.

Step 1: The tuition in 2009 is the amount to be compared against; thus, Base Quantity = $3,535. No base value is specified; thus, use the default of Base Value = 100.

Step 2: The 2012 tuition is the quantity for which you wish to compute the index, resulting in Chosen Quantity = $3,978.67.

Step 3: The Index Number $= \frac{\$3,978.67}{\$3,535.00} \times 100 = \$112.6$. You can conclude that tuition in 2012 is 112.6% higher than 2009, or that tuition has increased by 12.6% over the three years.

Example 3.3 A: Indexing the Price of Gold

In January 2000, the average price of gold per troy ounce was $284.32 (in US dollars). In May 2011, the average price of gold was $1,494.89. Express the gold price in 2011 indexed against the gold price in 2000.

You need to calculate an index number that compares the 2011 gold price against the 2000 gold price.

What You Already Know	How You Will Get There
Step 1: Base Quantity = 2000 price = $284.32 Assume the standard Base Value = 100 Step 2: Chosen Quantity = 2011 price = $1,494.89	Step 3: Apply Formula 4.3.

Step 3:

Index Number $= \frac{\$1,494.89}{\$284.32} \times 100 = 525.8$

The May 2011 gold price has an index number of 525.8 when compared to the January 2000 price. This means that the price of gold has increased by 425.8% over the approximate 11-year period.

Specialty Indexes

While indexes can be created for any comparison of two quantities, two specialized indexes are widely used across the business and financial sectors of Canada: the consumer price index and the S&P/TSX Composite Index.

Consumer Price Index

One of the most closely watched price indexes in Canada is the **consumer price index**, or CPI. Historical values for this index (base year = 2002) are illustrated the graph on the left while the historical rates of inflation (1915-2012) are depicted in the graph on the right.[1]

[1] Values determined from Bank of Canada "Inflation Calculator" (all index values measured in May of each year). www.bankofcanada.ca/rates/related/inflation-calculator/?page_moved=1 (accessed November 26, 2012).

The CPI, calculated by Statistics Canada on a monthly basis, is used as a measure for estimating the rate of inflation or the cost of living (notice the slope of the CPI line increases or decreases in accordance with the change in the rate of inflation). The CPI measures the average price of the goods and services that a typical Canadian household commonly purchases, which is called the **market basket**. This basket contains about 600 items in categories such as food, housing, transportation, furniture, recreation, and clothing. The CPI is used for many purposes, such as increases in wages, pensions, salaries, prices, the Canada Pension Plan (CPP), and Old Age Security (OAS).
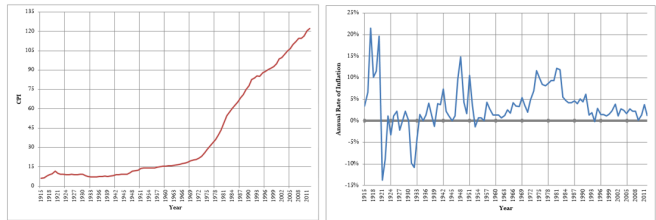
In the section opener, you need to know the CPI in discussing your raise with your employer. Since the CPI reflects the changing cost of living, you must ensure that any raise your employer offers covers, at the barest minimum, the increased costs of living. For example, if the CPI increased from 114.3 to 116.5 over the year, reflecting a 1.78% rate of inflation, then your raise needs to be at least 1.78% to keep you in the same financial position as last year.

S&P/TSX Composite Index

Another index reported daily and nationally across Canada is the Standard & Poor's/Toronto Stock Exchange Composite Index, or **S&P/TSX Composite Index**. This index captures the equity prices of approximately 200 of Canada's largest companies on the Toronto Stock Exchange as measured by their market capitalization,

representing approximately 70% of the Canadian market capitalization listed on the TSX. The historical values of the index are illustrated on the previous page. The exact number of companies in the index continually changes because of strict requirements on market capitalization, liquidity, and domicile that must be met. The index is a measure of the strength and direction of the

economy of Canada. For example, during the 2008 subprime mortgage crisis in the United States, the index retreated from a value of 14,714 to just over 8,123 in a very short time frame (demonstrating the close interconnectedness of Canada's economy with that of the United States). At this time, the Canadian economy entered its first recession in years.
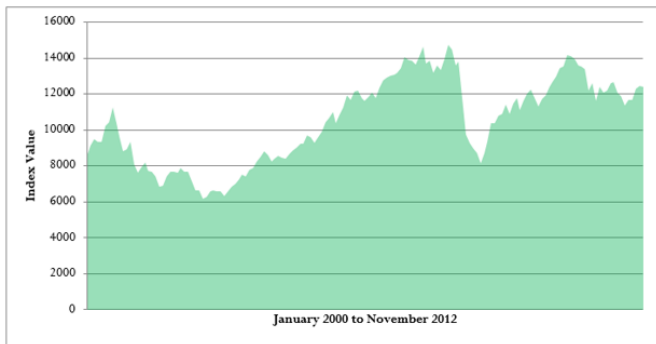
In the section opener, knowing the S&P/TSX Composite Index is invaluable for the meeting with your financial adviser to evaluate the health of your investment portfolio. For example, from May 2010 to May 2011 the index rose approximately 15.7%. If your investment portfolio rose by 9% over the same time frame (which on its own sounds impressive), you know from comparison to the index that your portfolio in fact had a subpar performance.

The Formula

The base year used in CPI calculations is mid-2002, and the base value is 100. The CPI is always rounded to one decimal. The base year used in S&P/TSX calculations is 1975, and the base value is 1,000. The S&P/TSX is always rounded to two decimals. Regardless of which index is being calculated, Formula 4.3 remains unchanged.

Often when you are working with these indexes, you want to compare how quantities have increased or decreased. This requires you to use Formula 3.1 (Percent Change) to calculate percent changes.

How It Works

No new steps or procedures are required when you work with the CPI or the S&P/TSX Composite Index. To calculate the value of an index, apply the index number steps. To calculate the percent change between two index values, apply the percent change steps.

For example, in July 2005 the CPI was 107.1. In March 2011 the CPI was 119.4. How much did the

cost of living rise from July 2005 to March 2011? To solve this problem, apply the percent change steps:

Step 1: New = 119.4 and Old = 107.1. You are looking for the Δ%.

Step 2: Substituting into Formula 3.1: Δ% = $\frac{119.4 - 107.1}{107.1} \times 100 = 11.4846\%$. You can conclude that the cost of living has risen by 11.4846% over the approximate six-year time frame.

Important Notes

From time to time, Statistics Canada updates the base period for the CPI and adjusts all CPI numbers accordingly. Previously, the base period was 1992 before being adjusted to 2002. Another update to the base period can probably be expected soon; however, this occurs retroactively since it takes time to recalculate all indexes and gather all of the latest data.

Example 3.3 B: How Much Money Do You Have?

If you had invested $15,000 into an S&P/TSX portfolio in 1975, how much money would you have in 2011 when the S&P/TSX Composite Index indicated a value of 13,607.25?

You need to determine the amount of money in the portfolio at the current time period of 2011. This amount represents the Chosen Quantity.

What You Already Know	How You Will Get There
Step 1: Since the money was invested in the base year (1975), the amount of the investment is the base quantity. Base Quantity = $15,000 Base Value = 1,000	Step 3: Apply Formula 4.3 and rearrange for Chosen Quantity.
Step 2: In this case, seek the Chosen Quantity with a known Index Number = 13,607.25	

Step 3:

$$13,607.25 = \frac{\text{Chosen Quantity}}{\$15,000} \times 1,000$$
$$204,108,750 = \text{Chosen Quantity} \times 1,000$$
$$\$204,108.75 = \text{Chosen Quantity}$$

If you had invested $15,000 in the S&P/TSX Composite Index back in 1975, your investment portfolio is now valued at $204,108.75, representing an increase of 1,260.725%.

Purchasing Power

The consumer price index is applied in determining the **purchasing power of a dollar**, which is the amount of goods and services that can be exchanged for a dollar. The result of the calculation is a percentage showing how much more or less product is received in return for every dollar spent. For example, recall that my parents could attend a movie for 25¢ in the 1950s. If my parents had lost that quarter under their mattress and found it today, the quarter no longer can purchase admission to the movie (nor much else!). Therefore, the purchasing power of that quarter has drastically declined.

The Formula

Purchasing Power

PPD is Purchasing Power of a Dollar: This percentage gives you a true understanding of how the ability of your money to purchase products has changed over time. The change is based on the base year for the consumer price index used in the calculation. Therefore, if the CPI is based in 2002, then the result of this formula expresses the change in your purchasing power since that year.

× 100 is Percent Conversion: Express the purchasing power of a dollar in percent format.

Formula 4.4 – Purchasing Power Of A Dollar:
$$PPD = \frac{\$1}{CPI / 100} \times 100$$

CPI is Consumer Price Index: The index value used is the CPI at the chosen point in time.

/ 100 is Decimal Conversion: Convert the CPI to decimal format before completing the division.

How It Works

Follow these steps to calculate the purchasing power of a dollar:

Step 1: Choose the point in time to perform the calculation. Identify the value of the CPI at that point in time as well as noting the base year upon which the CPI is determined (so you can properly interpret the calculated amount). Note that in the event that the purchasing power is known and you are solving for a different unknown, you must identify the known variables in this step.

Step 2: Calculate the purchasing power of a dollar by applying Formula 4.4.

Assume the CPI is 111.1 today. How has your purchasing power changed since 2002 (the base year for the CPI)?

Step 1: The CPI = 111.1 and the base year is 2002.

Step 2: Substitute into the formula: PPD = .. Therefore, your purchasing power is 90.009% what it was in 2002. In 2002, if you could buy 10 loaves of bread with your dollar, today you could buy

approximately nine loaves of bread with the same amount of money. In other words, your dollar purchases about 10% less product with the same amount of money.

Example 3.3 C: How Has Your Purchasing Power Changed?

In March 2006, the CPI was 108.6. By March 2011 it had risen to 119.4. As a percent change, how did your purchasing power change over that time frame?

You need to calculate the percent change, or Δ%, between the purchasing power of a dollar (PPD) in 2006 and 2011.

What You Already Know	How You Will Get There
Step 1: 2006 represents your purchasing power of a dollar in the past, and 2011 represents the present.	Step 2: Apply Formula 4.4 to calculate the PPD for both 2006 and 2011.
	Step 3: Apply Formula 3.1 to calculate the percent change.'

Old = PPD in 2006	New = PPD in 2011
CPI 2006 = 108.6	CPI 2011 = 119.4

The current CPI is based in 2002.

Step 2:

$$PPD\ 2006 = \frac{\$1}{108.6/100} \times 100 = 92.0810\%$$

$$PPD\ 2011 = \frac{\$1}{119.4/100} \times 100 = 83.7521\%$$

Step 3:

$$\Delta\% = \frac{0.837521 - 0.920810}{0.920810} \times 100 = -9.0452\%$$

Calculator Instructions

OLD	NEW	%CH	#PD
0.920810	0.837521	Answer: -9.045188	1

From March 2006 to March 2011, your purchasing power declined by 9.0452%. This means that if in March 2006 you were able to buy approximately 92 items with a certain sum of money, in March 2011 you were only able to buy about 83 items with the same amount of money.

Real Income

Another application of the consumer price index allows you to assess your real earnings. With the cost of living (inflation) always changing, it is difficult to assess how much more or less money you are truly making. **Real income** allows you to remove the effects of inflation from your income, allowing for a fair comparison of earnings at different points in time. For example, if your gross earnings last year were $50,000 and you received a $1,500 raise this year, how much more are you truly earning if the CPI changed from 114.6 to 116.6 over the same time frame?

The Formula

Real Income

RI is Real Income: This is the true amount of your income once inflation has been removed. The income represents the true amount of your earnings corresponding to the CPI base year. Thus, if the CPI is based in 2002, then these are your real earnings for 2002.

Nominal Income: The nominal income reflects your current gross earnings inclusive of any inflation since the base period of the CPI.

Formula 4.5 – Real Income: $$RI = \frac{\text{Nominal Income}}{CPI\,/\,100}$$

CPI is Consumer Price Index: The index value used is the CPI that corresponds to the same point in time at which you earned the nominal

/ 100 is Decimal Conversion: Convert the CPI to decimal format before completing the division.

How It Works

Follow these steps to calculate your real income:

Step 1: Identify the nominal income. Determine the CPI value that corresponds to the same time

frame as the nominal income and note the base year for interpretation purposes. In the event that the real income is known and you are solving for a different unknown, you must identify the known variables in this step.

Step 2: Calculate the real income by applying Formula 4.5.

Assume your gross earnings were $40,000 when the CPI was 111.7. Currently your income is $42,000 and the CPI is 113.3. Calculate your real income to assess how your income has changed.

Step 1: In this case, you have two nominal incomes to convert for comparison purposes. In the past, nominal income = $40,000 and CPI = 111.7. Currently, nominal income = $42,000 and CPI = 113.3.

Step 2: In the past, RI = $\frac{\$40,000}{111.7 \ / \ 100} = \$35,810.21$. Currently, RI = $\frac{\$42,000}{1.133 \ / \ 100} = \$37,069.73$. Comparing the amounts, you see that your real income expressedin 2002 dollars has risen by $37,069.73 – $35,810.21 = $1,259.52. Notice that your nominal income rose by $2,000. This means that of your $2,000 increase in income, you really are only making $1,259.82 more. The other $740.18 represents your increased cost of living.

Example 3.3 D: Did You Get a Raise?

Now that your boss has conducted your performance review, the time has come to discuss your raise. Currently, you earn $81,250 annually. Your boss has offered you a new salary of $83,000 annually. You know last year that the CPI was 104.1 and this year it sits at 106.6. Assess the salary offer.

To assess the offer, remove inflation from your nominal incomes, allowing for a fair comparison between the two salary amounts. Calculate the real income (RI) for each salary.

What You Already Know	How You Will Get There
Step 1: You know the following information about your old and new salaries:	Step 2: Apply Formula 4.5 to each salary.

Old Nominal Income = $81,250	Old CPI = 104.1
New Nominal Income = $83,000	New CPI = 106.6

Both CPI amounts are based on the year 2002.

Step 2:

$$\text{RI Last Year} = \frac{\$81{,}250}{104.1/100} = \$78{,}049.95$$

$$\text{RI This Year} = \frac{\$83{,}000}{106.6/100} = \$77{,}861.16$$

$$\$77{,}861.16 - \$78{,}049.95 = -\$188.79$$

Although it appears as if you are receiving a $1,750 wage increase, when you factor in the increased cost of living you are actually earning $188.79 less than you did last year. Perhaps you should discuss this matter further with your boss. Since the CPI had a percent change of 2.4015%, your income must rise to at least $83,201.25 for you to break even.

Exercises

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. Round index numbers to one decimal and base values to the nearest integer.

	Chosen Quantity ($)	Base Quantity ($)	Base Value	Index Number
1.	$559.99	$539.99	100	?
2.	$315,000.00	?	1,000	12,415.9
3.	$114.30	$112.50	?	101.6
4.	?	$248.75	1,000	1,548.9

For questions 5–9, solve for the unknown variables (identified with a ?) based on the information provided.

	Consumer Price Index (CPI)	Purchasing Power of a Dollar (PPD)
5.	107.9	?
6.	?	80%

	Nominal Income ($)	Consumer Price Index (CPI)	Real Income (RI)
7.	$45,000.00	120.0	?
8.	?	105.0	$29,523.81
9.	$86,000.00	?	$80,298.71

Applications

10. In Regina, a 4 L bag of 2% milk has an average price of $3.71. Prices in Toronto and Montreal are $4.55 and $5.40, respectively. Calculate an index of these prices using Regina as the base.
11. If Sabrina is currently earning $53,000 and the CPI changes from 105.9 to 107.6, how much money does she need to earn next year just to keep up with inflation?
12. If the CPI rises from 103.4 to 108.8, how much money at the start is required to have the same purchasing power as $100 at the end?
13. George currently earns $28,000 per year. As per his union salary grid, next year he will earn $32,500 per year. If the CPI increases from 104.0 to 106.1 over the same time frame, how much of his raise is real income?

14. Last year the purchasing power of a dollar was 84.3%. This year the purchasing power of a dollar is 81.4%. What is the percent change in the CPI between the two years?

Challenge, Critical Thinking, & Other Applications

15. An investor had $200,000 invested in an S&P/TSX Composite Index portfolio in 2009. By 2011, the index rose to 14,136.50. If the index in 2009 was 8,694.90, how much money would the investor have in her portfolio in 2011, if she was able to match the market performance?
16. Over the years, Hannah's income has changed from $36,000 to $40,000 to $45,000. If the CPI changed from 102.9 to 105.1 to 108.6 over the same time period, determine Hannah's percent change in real income from year to year.
17. The CPI rose from 102.6 in 2003 to 116.5 in 2010. The S&P/TSX Composite Index rose from 6,569.49 to 11,294.42 over the same time period. If an investor had $125,000 invested in 2003, how much of the growth (in dollars) for the portfolio over the seven years represents real growth?
18. An enterprise has 136 employees earning an average income of $42,250 per year. If the CPI rises from 103.7 to 107.2, by what amount do total wages rise if all employee wages are adjusted to match the CPI?
19. Using January 2003 as your base year with a base value = 100, compute a series of indexes for the S&P/TSX Composite Index from select years in 2003 to 2011. Interpret your results.

Year	2003	2005	2007	2009	2011
January index	6,569.49	9,204.05	13,034.12	8,694.90	13,551.91

20. Below is select information on the CPI from 2004 to 2010:

Year	2004	2006	2008	2010
July CPI	105.0	109.6	115.8	116.8

    a. The purchasing power of a dollar for each year. Interpret your results.
    b. If you were earning $34,500 in 2002, calculate the nominal income required each year to keep up with the changes in the CPI.

The Language of Business Mathematics

basic personal amount The amount of income for which the wage earner is granted a tax exemption.

commission An amount or a fee paid to an employee for performing or completing some form of transaction.

consumer price index A measure of the average price of a typical Canadian market basket, which is used to estimate inflation in Canada.

graduated commission A form of compensation where an employee is offered increasing rates of commission for higher levels of performance.

gross earnings The amount of money earned before any deductions are removed from a paycheque.

holiday earnings Earnings paid to an employee on a statutory holiday for which no work is performed.

hourly wage A variable compensation based on the time an employee has worked.

index A number used to compare two quantities sharing the same characteristic as measured under different circumstances.

index number The expression of a relationship between two quantities; it is a result of an index calculation.

market basket The average price of the goods and services that a typical Canadian household commonly purchases; used in calculating the consumer price index.

Key Takeaways

Section 3.1: Gross Earnings (Off to Work You Go)

- Calculating salary and hourly gross earnings
- Discussion of employment contract characteristics for salary and hourly earners
- Calculating commission gross earnings
- Calculating piecework gross earnings

Section 3.2: Personal Income Tax (The Taxman Taketh)

- Provision of 2013 federal and provincial/territorial tax brackets and corresponding tax rates
- Calculation of income taxes

Section 3.3: Indexes (The Times Are Changing)

- What is an index and how is it calculated?
- Specialty indexes, including the consumer price index and the S&P/TSX Composite Index
- Using the CPI to calculate the purchasing power of a

dollar
- Using the CPI to calculate real income

overtime Work time in excess of an employee's regular workday or regular workweek.

overtime or premium earnings Earnings determined by an employee's overtime rate of pay and that occur when regular hours are exceeded.

personal income tax A tax on earned income that is levied by both the federal and provincial/territorial governments.

piecework A form of compensation where an employee is paid on a per-unit basis.

progressive tax system A personal income tax system where the tax rate increases as the amount of income increases; however, the increased tax rates apply only to income amounts above a minimum threshold.

public holiday A provincially recognized day for which employees may or may not get a day of rest and may or may not receive pay depending on provincial employment standards.

purchasing power of a dollar The amount of goods and services that can be purchased with a dollar.

real income Income that has the effects of inflation removed from its amount.

regular earnings Earnings determined by an employee's regular rate of pay.

S&P/TSX Composite Index This index captures the equity prices of approximately 200 of Canada's largest companies on the Toronto Stock Exchange as measured by their market capitalization, representing approximately 70% of the Canadian market capitalization listed on the TSX.

salary A fixed compensation paid to a person on a regular basis for services rendered.

salary plus commission A form of compensation in which gross earnings combine a basic salary together with commissions on transactions.

statutory holiday A legislated day of rest with pay.

statutory holiday worked earnings Earnings paid to an employee on a statutory holiday at a premium rate for working on the statutory holiday.

straight commission A form of compensation where the employee's entire earnings are based on dollar transactions and calculated strictly as a percentage of the total transactions.

The Formulas You Need to Know

Symbols Used

CPI = consumer price index

GE = gross earnings

PPD = purchasing power of a dollar

RI = real income

Formulas Introduced

Formula 4.1: Salary and Hourly Gross Earnings:

GE= Regular Earnings + Overtime Earnings + Holiday Earnings + Statutory Holiday Worked Earnings (Section 4.1)

Formula 4.2 Annual Income Tax:

$\text{Income Tax} = \sum \left(\text{Eligible Income in Tax Bracket} \times \text{Tax Bracket Rate} \right)$ (Section 4.2)

Formula 4.3 Index Numbers: $\text{Index Number} = \frac{\text{Chosen quantity}}{\text{Base quantity}} \times \text{Base value}$ (Section 4.3)

Formula 4.4: Purchasing Power of a Dollar: (Section 4.3)

Formula 4.5: Real Income: $\text{RI} = \frac{\text{Nominal Income}}{\text{CPI} / 100}$ (Section 4.3)

Chapter 3: Human Resources and Economics Applications (answers to section exercises)

- ## 3.1 Gross Earnings

1. a. $5,000; b. $2,500; c. $2,307.69; d. $1,153.85
2. $1,277.21
3. $57,600
4. $6,110
5. $468.24
6. $188
7. $3,767.58
8. $1,443
9. $19,162.50
10. $16,860
11. $850
12. Offer #2 Best = $3,412.50; Exceeds Offer #1 = $312.50; Exceed Offer #3 = $67.50
13. $5,902.89
14. $1,454.38
15. $43,550.45
16. $168,000
17. 2%
18. Option 1 = 1.47%; Option 2 = 1.05%
19. $3,342.35
20. $1,755.25

- ## 3.2 Personal Income Tax

1. $1,777.20
2. $34,410.41
3. $18,735.35
4. $5,478.46
5. $8,417.62
6. $4,731.86
7. $8.701.20
8. $16,981.20
9. $12,121.55
10. $4,604.80
11. −$817.22
12. British Columbia is $4,025.28 better
13. Saskatchewan − $78,120.88 (LOW); Ontario = $81,278.61 (HIGH); NWT =$81,081.29;

Difference =\ $3,157.73

14. New Brunswick = $62,322.77 (MORE); Nunavut = $62,240.61
15. Federal =$5,808.87; Provincial =$2,692.38; Total = $8,501.25
16. Federal = $1,930.95; Provincial =$1,537.82; Total =$3,468.77
17. Federal = $380.56; Provincial =$262.50; Total = $643.06
18. 5.5097%
19. $46,846.57
20. Nunavut = $46,434.59 (HIGH); Quebec = $40,659.97 (LOW) which is−12.436%

- ## 3.3 Indexes

1. 103.7
2. $25,370.69
3. 100
4. $385.29
5. 92.6784%
6. 125.0
7. $37,500
8. $31,000
9. 107.1
10. Toronto=122.6; Montreal=145.6
11. $53,850.80
12. $105.22
13. $3,934.61
14. 3.6256%
15. $325,167.63
16. 8.7853% then 8.8743%
17. $72,968.21
18. $193,934.64
19. 100.0 (base year), 140.1, 198.4, 132.4, 206.3
20. a. 95.2381%, 91.2409%, 86.3558%, 85.6164%, b. $36,225; $37,812; $39,951; $40,296

4.

MARKETING APPLICATIONS

<div style="background:orange">

Chapter 4 Topics

4.1 Discounts

4.2 Markup

4.3 Markdown

4.4 Merchandising

</div>

When you buy an iPod, it is very important that the right price is set. The price should:

- be seen as fair by you the buyer,
- pay for the costs (plastics, battery, buttons, circuit boards, headset) and expenses (employees, factory, electricity, distribution) of making the iPod, and
- allow the seller's business to make some extra money as profit so that it can grow its business further.

If your business is selling a product, you will need to pay close attention to price adjustments because they affect profitability. Various discounts, like putting items on sale, may increase sales while lowering the amount of profit per transaction. How do you know where to set the balance to maximize profit overall?

As a student in a business program, consider this chapter essential to the success of any business. Whether your pricing strategy is high or low, the company must ensure that it can still pay its bills as a minimum requirement. And that requires careful juggling of many factors. If it fails to manage its pricing properly, the company will go bankrupt!

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every

day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of "deals." You can easily explain why the same product sells for two different prices at two different stores.

In this chapter, you must learn the language of marketers to perform merchandising mathematics involving product costs, expenses, prices, markups, markdowns, and ultimately profitability. Once the study of the various pricing components is complete, we will see how the various pieces of the pricing puzzle fit together into a cohesive merchandising environment.

4.1 Discounts

You mutter in exasperation, "Why can't they just set one price and stick with it?" Your mind boggles at all the competing discounts you encounter at the mall in your search for that perfect Batman toy for your nephew. Walmart is running their Rollback promotion and is offering a Batmobile for 25% off, regularly priced at $49.99. Toys R' Us has an outlet in the parking lot where the regular price for the same toy is $59.99, but all Batman products are being cleared out at 40% off. You head over to The Bay for a warehouse clearance event that has the same toy priced at $64.99 but at 35% off. It is also Bay Days, which means you can scratch and win a further 10% to 20% off the sale price. You go to Dairy Queen for a Blizzard to soothe your headache while you figure things out.

The **cost** of a product is the amount of money required to obtain the merchandise. If you are a consumer, the ticketed price tag on the product is your cost. If you are a reseller (also known as a middleman or intermediary), what you pay to your supplier for the product is your cost. If you are a manufacturer, then your cost equals all of the labour, materials, and production expenditures that went into creating the product.

A **discount** is a reduction in the price of a product. As a consumer, you are bombarded with discounts all the time. Retailers use various terms for discounts, including sales or clearance. If your business purchases a product from a supplier, any discount it receives lowers how much the business pays to acquire the product. When a business buys products, the price paid is the cost to the business. Therefore, a lower price means a lower cost.

If your business is the one selling the product, any discount offered lowers the selling price and reduces revenue per sale. Since the revenue must cover all costs and expenses associated with the product, the lower price means that the business reduces profits per sale. In business, it is common practice to express a discount as a percentage off the regular price.

How Distribution and Pricing Work
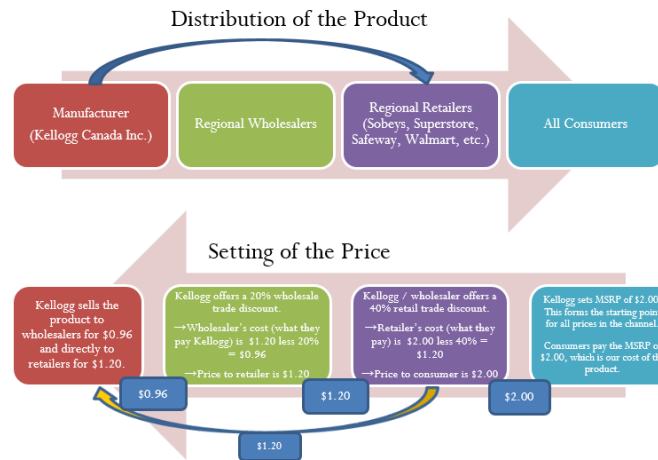
Start with distribution in the top half of the figure and work left to right. As an example, let's look at a manufacturer such as Kellogg Canada Inc. (which makes such products as Pop-Tarts, Eggo Waffles, and Rice Krispies). Kellogg's Canadian production plant is located in London, Ontario. To distribute its products to the rest of Canada, Kellogg Canada uses various regional wholesalers. Each wholesaler then resells the product to retailers in its local trade area; however, some retailers (such as the Real Canadian Superstore) are very large, and Kellogg Canada distributes directly to these organizations, bypassing the wholesaler as represented by the blue arrow. Finally, consumers shop at these retailers and acquire Kellogg products.

The relationship of distribution to pricing is illustrated in the bottom half of the figure, working right to left. For now, focus on understanding how pricing works; the mathematics used in the figure will be explained later in this chapter. Kellogg Canada sets a **manufacturer's suggested retail price**, known as the MSRP. This is a recommended retail price based on consumer market research. Since grocery retailers commonly carry thousands or tens of thousands of products, the MSRP helps the retailer to determine the retail price at which the product should be listed. In this case, assume a $2.00 MSRP, which is the price consumers will pay for the product.

The retailer must pay something less than $2.00 to make money when selling the product. Kellogg Canada understands its distributors and calculates that to be profitable most retailers must pay approximately 40% less than the MSRP. Therefore, it offers a 40% discount. If the retailer purchases directly from Kellogg, as illustrated by the yellow arrow, the price paid by the retailer to acquire the product is $2.00 less 40%, or $1.20. Smaller retailers acquire the product from a wholesaler for the same price. Thus, the retailer's cost equals the wholesaler's price (or Kellogg Canada's price if the retailer purchases it directly from Kellogg).

The wholesaler's price is $1.20. Again, Kellogg Canada, knowing that the wholesaler must pay something less than $1.20 to be profitable, offers an additional 20% discount exclusively to the wholesaler. So the price paid

by the wholesaler to acquire the product from Kellogg Canada is $1.20 less 20%, or $0.96. This $0.96 forms Kellogg Canada's price to the wholesaler, which equals the wholesaler's cost.

In summary, this discussion illustrates two key pricing concepts:

1. Companies higher up in the distribution channel pay lower prices than those farther down the channel. Companies receive discounts off the MSRP based on their level in the distribution system. This may result in multiple discounts, such as a wholesaler receiving both the retailer's discount and an additional discount for being a wholesaler.
2. One organization's price becomes the next organization's cost (assuming the typical distribution channel structure):
   Manufacturer's Price = Wholesaler's Cost
   Wholesaler's Price = Retailer's Cost
   Retailer's Price = Consumer's Cost

Types of Discounts

You will perform discount calculations more effectively if you understand how and why single pricing discounts and multiple pricing discounts occur. Businesses or consumers are offered numerous types of discounts, of which five of the most common are trade, quantity, loyalty, sale, and seasonal.

1. *Trade Discounts.* A **trade discount** is a discount offered to businesses only based on the type of business and its position in the distribution system (e.g., as a retailer, wholesaler, or any other member of the distribution system that resells the product). Consumers are ineligible for trade discounts. In the discussion of the figure, two trade discounts are offered. The first is a 40% retail trade discount, and the second is a 20% wholesale trade discount. Typically, a business that is higher up in the distribution system receives a combination of these trade discounts. For example, the wholesaler receives both the 40% retail trade discount and the 20% wholesale trade discount from the MSRP. The wholesaler's cost is calculated as an MSRP of $2.00 less 40% less 20% = $0.96.
2. *Quantity Discounts.* A **quantity discount** (also called a volume discount) is a discount for purchasing larger quantities of a certain product. If you have ever walked down an aisle in a Real Canadian Superstore, you probably noticed many shelf tags that indicate quantity discounts, such as "Buy one product for $2" or "Take two products for $3." Many Shell gas stations offer a Thirst Buster program in which customers who purchase four Thirst Busters within a three-month period get the fifth one free. If the Thirst Busters are $2.00 each, this is equivalent to buying five drinks for $10.00 less a $2.00 quantity discount.
3. *Loyalty Discounts.* A **loyalty discount** is a discount that a seller gives to a purchaser for repeat

business. Usually no time frame is specified; that is, the offer is continually available. As a consumer, you see this regularly in marketing programs such as Air Miles or with credit cards that offer cash back programs. For example, Co-op gas stations in Manitoba track consumer gasoline purchases through a loyalty program and mails an annual loyalty discount cheque to its customers, recently amounting to 12.5¢ per litre purchased. In business-to-business circles, sellers typically reward loyal customers by deducting a loyalty discount percentage, commonly ranging from 1% to 5%, from the selling price.

4. *Sale Discounts.* A **sale discount** is a temporary lowering of the price from a product's regular selling price. Businesses put items on sale for a variety of reasons, such as selling excess stock or attracting shoppers. You see such promotional events all the time: LED monitors are on sale at Best Buy; Blu-Ray discs are half off at Walmart; The Brick is having a door crasher event Saturday morning.

5. *Seasonal Discounts.* A **seasonal discount** is a discount offered to consumers and businesses for purchasing products out of season. At the business level, manufacturers tend to offer seasonal discounts encouraging retailers, wholesalers, or distributors to purchase products before they are in season. Bombardier Inc. manufactures Ski-Doos, which are sold in Canada from approximately November through March—a time of year when most of the country has snow and consumers would want to buy one. To keep production running smoothly from April through October, Bombardier could offer seasonal discounts to its wholesalers and retailers for the coming winter season. At a retail level, the examples are plentiful. On November 1 most retailers place their Halloween merchandise on seasonal discount to clear out excess inventory, and many retailers use Boxing Day (or Boxing Week) to clear their out-of-season merchandise.

Single Discounts

Let's start by calculating the cost when only one discount is offered. Later in this section you will learn how to calculate a cost involving multiple discounts.

Figuring out the price after applying a single discount is called a net price calculation. When a business calculates the net price of a product, it is interested in what you still have to pay, not in what has been removed. Note in Formula 4.1 below that you take 1 and subtract the discount rate to determine the rate owing. If you are eligible for a 20% discount, then you must pay 80% of the list price, as illustrated in the figure below:

20% Discount Interpretation

■ What remains of the original price (to be paid) --> 80%

■ What was taken off of the original price (the discount) --> 20%

100%
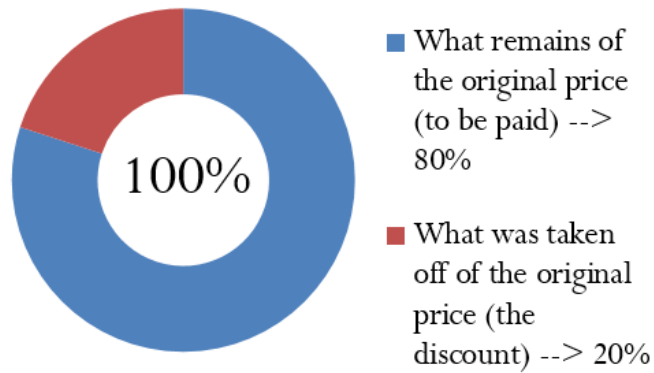
Formula 4.1 – Single Discount

$$N = L \times (1 - d)$$

N is the **Net Price:** The price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

L is the **List Price**: The normal or regular dollar price of the product before any discounts. It is the **Manufacturer's Suggested Retail Price (MRSP).**

d is the **Discount Rate:** The percentage (in decimal format) of the list price that is deducted.

Formula 4.1 includes rate, portion, and base, where the list price is the base, the $(1d)$ is the rate, and the net price represents the portion of the price to be paid.

Formula 4.2a and Formula 4.2b – Discount Amount

Formula 4.2a

$$D\$ = L \times d$$

Formula 4.2b

$$D\$ = L - N$$

$D\$$ is **Discount Amount**: Determine the discount amount in one of two ways, depending on what information is known:
If the **list price** and **discount rate** are known, apply Formula 4.2a**.**
If the **list price** and **net price** are known, apply Formula 4.2b.

L is **List Price**: The dollar amount of the price before any discounts.

N is **Net Price**: The dollar amount of the price after you have deducted all discounts.

d is **Discount Rate:** The percentage (in decimal format) of the list price to be deducted. This time, you are interested in figuring out the amount of the discount, therefore you do not take it away from 1.

Notice that Formula 4.1 requires the discount to be in a percentage (decimal) format; sometimes a discount is expressed as a dollar amount, though, such as "Save $5 today." Formulas 4.2a and 4.2b relate the discount dollar amount to the list price, discount percent, and net price. Choose one formula or the other depending on which variables are known.

How It Works

Follow these steps to calculate the net price involving a single discount. These steps are adaptable if the net price is a known variable and one of the other variables is unknown.

Step 1: Identify any known variables, including list price, discount rate, or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price using an appropriate formula.

Step 3: Calculate the net price.

- If the list price and discount are are known, apply Formula 4.1.
- If the list price and discount amount are known, apply Formula 4.2b and rearrange for N.

Assume a product sells for $10 and is on sale at 35% off the regular price. Calculate the net price for the product.

Step 1: The list price of the product is $L = \$10$. It is on sale with a discount rate of $d = 0.35$.

Step 2: List price is known, so this step is not needed.

Step 3: Applying Formula 4.1 results in a new price of

$$N = \$10 \times (1 - 0.35) = \$6.50.$$

Note that if you are interested in learning the discount amount, you apply Formula 4.2b to calculate:

$$D\$ = \$10 - \$6.50 = \$3.50.$$

The Net Price of a Product

$10.00 List Price

- Net Price (what you pay) --> $6.50
- Discount (what you save) --> 35% or $3.50

You can combine Formula 4.1 and either version of Formula 4.2 in a variety of ways to solve any single discount situation for any of the three variables. As you deal with increasingly complicated pricing formulas, your algebraic skills in solving linear equations and substitution become very important.

Many of the pricing problems take multiple steps that combine various formulas, so you need to apply the PUPP model systematically. In any pricing problem, you must understand which variables are provided and match them up to the known formulas. To get to your end goal, you must look for formulas in which you

know all but one variable. In these cases, solving for variables will move you forward toward solving the overall pricing problem.

If you find you cannot produce a formula with only one unknown variable, can you find two formulas with the same two unknowns? If so, recall from Chapter 1 that you can use your algebraic skills to find the roots of the two equations simultaneously. Alternatively, you can solve one formula for a variable then substitute it into the other formula, allowing you to isolate the remaining variable. Throughout the examples in this chapter you will see many applications of these algebraic skills.

Remember to apply the rounding rules:

1. Until you arrive at the final solution, avoid rounding any interim numbers unless you have some special reason to do so.
2. Round all dollar amounts to the nearest cent. If the dollar amount has no cents, you may write it either without the cents or with the ".00" at the end.
3. Round all percentages to four decimals when in percent format.

When working with single discounts, you are not always solving for the net price. Sometimes you must calculate the discount percent or the list price. At other times you know information about the discount amount but need to solve for list price, net price, or the discount rate. The triangle technique can remind you how to rearrange the formulas for each variable, as illustrated in the figure below.

Formula 4.1: Single Discount   **Formula 4.2a**: Discount Amount

Give It Some Thought:

1. Will you pay more than, less than, or exactly $10.00 for a product if you are told that you are paying:
   a. a net price of $10.00 when there is a discount of 25%?
   b. a list price of $10.00 when there is a discount of 25%?
2. If an item is subject to a 40% discount, will the net price be more than or less than half of the list price of the product?

Example 4.1A – Determining the Retailer's Net Price for a Pair of Jeans

A manufacturer that sells jeans directly to its retailers uses market research to find out it needs to offer a 25% trade discount. In doing so, the retailers will then be able to price the product at the MSRP of $59.99. What price should retailers pay for the jeans?

Plan:
Calculate how much a retailer should pay for the jeans after the regular price has been discounted to accommodate the trade discount. This is called the **net price** for the product, or N.

Understand:
Step 1: The list price and the discount rate are known.

$$L = \$59.99$$
$$d = 0.25$$

Step 2: List price is known, so skip this step.

Step 3: Apply Formula 4.1

Perform:
Step 3:

$$N = \$59.99 \times (1 - 0.25) = \$59.99 \times 0.75 = \$44.99$$

Present:

The manufacturer should sell the jeans to the retailers for $44.99.

Example 4.1B – Determining the List Price of a Jacket

Winners pays a net price of $27.50 for a winter jacket after receiving a retail trade discount of 45%. What was the MSRP of the jacket?

Plan:

Calculate the MSRP for the jacket before Winners received the retail trade discount to arrive at the net price. This is called the **list price** for the product, or L.

Understand:

Step 1: The net price and the discount rate are known:

$$N = \$27.50$$
$$d = 0.45$$

Step 2: List price is the unknown variable; skip this step.

Step 3: Apply Formula 4.1, rearranging for L.

Perform:

Step 3:

$$\$27.50 = L(1 - 0.45)$$
$$L = \$27.50 \div (1 - 0.45)$$
$$L = \$27.50 \div 0.55 = \$50.00$$

Present:

The MSRP, or list price, of the winter jacket is $50.00.

Example 4.1C – Determining the Discount Percent and Discount Amount

You are shopping at Mountain Equipment Co-op for a new environmentally friendly water bottle. The price tag reads $14.75, which is $10.24 off the regular price. Determine the discount rate applied.

Plan:

You need to find out how the sale price translates into the **discount rate**, or d.

Understand:

Step 1: The discount amount and net price are known:

$$D\$ = \$10.24$$
$$N = \$14.75$$

Step 2: Use Formula 4.2b to calculate the list price, rearranging for L.

Step 3: Convert the discount amount into a percentage by applying Formula 4.2a, rearranging for d.

Perform:

Step 2:

List price:

$$\$10.24 = L - \$14.75$$
$$L = \$24.99$$

Step 3:

Discount rate:

$$d = D\$ \div L$$
$$d = \frac{\$10.24}{\$24.99} = 0.409764$$
$$\text{or}$$
$$40.9764\%$$

Present:

The water bottle today has been reduced in price by the amount of $10.24. This represents a sale discount of 40.9674%.

Multiple Discounts

You are driving down the street when you see a large sign at Old Navy that says, "Big sale, take an additional 25% off already reduced prices!" In other words, products on sale (the first discount) are being reduced by an additional 25% (the second discount). Because Formula 4.1 handles only a single discount, you must use an extended formula in this case.

Businesses commonly receive more than one discount when they make a purchase. Consider a transaction in which a business receives a 30% trade discount as well as a 10% volume discount. First, you have to understand that this is not a 30% + 10% = 40% discount. The second discount is always applied to the net price after the first discount is applied. Therefore, the second discount has a smaller base upon which it is calculated. If there are more than two discounts, you deduct each subsequent discount from continually smaller bases. Formula 4.3 expresses how to calculate the net price when multiple discounts apply.

Formula 4.3 – Multiple Discounts

$$N = L \times (1 - d_1) \times (1 - d_2) \times \cdots \times (1 - d_n)$$

N is **Net Price:** The dollar amount of the price after all discounts have been deducted.

L is **List Price:** The dollar amount of the price before any discounts.

d_1 is **First Discount**, d_2 is **Second Discount**, d_n is **nth Discount**:
When there is more than one discount, you must extend beyond Formula 4.1 by multiplying another discount expression. These discounts are represented by the same d symbol; however, each discount receives a subscript to make its symbol unique. Therefore, the first discount receives the symbol of d_1, the second discount receives the symbol d_2, and so on. Recall that the symbol n represents the number of pieces of data (a count), so you can expand or contract this formula to the exact number of discounts being offered.

It is often difficult to understand exactly how much of a discount is being received when multiple discounts are involved. Often it is convenient to summarize the multiple discount percentages into a single percentage. This makes it easier to calculate the net price and aids in understanding the discount benefit. Simplifying multiple percent discounts into a single percent discount is called finding the **single equivalent discount**. Whether you apply the multiple discounts or just the single equivalent discount, you arrive at the same net price. The

conversion of multiple discount percentages into a single equivalent discount percent is illustrated in Formula 4.4.

Formula 4.4 – Single Equivalent Discount

$$d_{equiv} = 1 - (1 - d_1) \times (1 - d_2) \times \cdots \times (1 - d_n)$$

d_{equiv} (or just d) is the **single equivalent discount rate** that is equal to the series of multiple discounts. Recall that taking $(1 - d)$ calculates what you pay. Therefore, if you take 1, which represents the entire amount, and reduce it by what you pay, the rate left over must be what you did not pay. In other words, it is the **discount rate**.

d_1 is **First Discount**, d_2 is **Second Discount**, d_n is **nth Discount:**
This is the same notation as in Formula 4.3. Since there are multiple discounts, each discount receives a numerical subscript to give it a unique identifier. You can expand or contract the formula to the exact number of discounts being offered.

How It Works

Refer back to the steps in calculating net price. The procedure for calculating a net price involving a single discount extends to a more generic procedure involving multiple discounts. As with the single discount procedures, you can adapt the model if the net price is known and one of the other variables is unknown. Follow these steps to calculate the net price involving any number of discounts:

Step 1: Identify any known variables, including list price, discount rate(s), or discount amount.

Step 2: If the list price is known, skip this step. Otherwise, solve for list price.

- If only one discount is involved, apply Formula 4.2a.
- If more than one discount is involved, the discount amount represents the total discount amount received from all of the discounts combined. This requires you first to convert the multiple discount rates into an equivalent single discount rate using Formula 4.4 and then

to apply Formula 4.2a.

Step 3: Calculate the net price.

- If the list price and only a single known discount rate are involved, apply Formula 4.1.
- If the list price and multiple discount rates are known and involved, apply Formula 4.3.
- If the list price and the total discount amount are known, apply Formula 4.2b and rearrange for N.

Assume a product with an **MSRP** of $100 receives a trade discount of 30% and a volume discount of 10%. Calculate the net price.

Step 1: The list price and discounts are

$$L = \$100$$
$$d_1 = 0.30$$
$$d_2 = 0.10.$$

Step 2: List price is known, so skip this step.

Step 3: Apply Formula 4.3 to calculate the net price:

$$N = \$100 \times (1 - 0.30) \times (1 - 0.10) = \$63$$

The net price is $63, which is illustrated in the following diagram:

Pricing with Two Discounts

- 30% discount (of $100) --> $30
- 10% (of $70) discount --> $7
- The net price --> $63

MSRP $100

If you are solely interested in converting multiple discounts into a single equivalent discount,

you need only substitute into Formula 4.4. In the above example, the product received a trade discount of 30% and a volume discount of 10%. To calculate the single equivalent discount, apply Formula 4.4:

$$d_{equiv} = 1 - (1 - 0.30) \times (1 - 0.10)$$
$$d_{equiv} = 1 - (0.70)(0.90)$$
$$d_{equiv} = 1 - 0.63 = 0.37$$

Therefore, whenever discounts of 30% and 10% are offered together, the single equivalent discount is 37%. Whether it is the multiple discounts or just the single equivalent discount that you apply to the list price, the net price calculated is always the same.

Order of Discounts:

The order of the discounts *does not* matter in determining the net price. Remember from the rules of BEDMAS that you can complete multiplication in any order. Therefore, in the above example you could have arrived at the $63 net price through the following calculation:

$$\$100 \times (1 - 0.10) \times (1 - 0.30) = \$63$$

The order of the discounts *does* matter if trying to interpret the value of any single discount. If the trade discount is applied before the quantity discount and you are wanting to know the quantity discount amount, then the quantity discount needs to be second. Thus,

$$\$100 \times (1 - 0.30) = \$70$$
$$\$70 \times 0.10 = \$7$$

which is the amount of the quantity discount.

Price Does Not Affect Single Equivalent Discount:

Notice in Formula 4.4 that the list price and the net price are not involved in the calculation of the single equivalent discount. When working with percentages, whether you have a net price of $6.30 and a list price of $10, or a net price of $63 and a list price of $100, the equivalent percentage always remains constant at 37%.

A common mistake when working with multiple discounts is to add the discounts together to calculate the single equivalent discount. This mistaken single discount is then substituted into Formula 4.1 to arrive at the wrong net price. Remember that if two discounts of 30% and 10% apply, you cannot sum these discounts. The

second discount of 10% is applied on a smaller price tag, not the original price tag. To calculate the net price you must apply Formula 4.3.

If you happen to know any two of the net price (N), list price (L), or the total discount amount ($D\$$), then you could also use Formula 4.2 to solve for the single equivalent discount, d_{equiv}.

For example, if you know the net price is $63 and the total discount amount for all discounts is $37, you could use Formula 4.2b to figure out that the list price is $100, then convert the discount amount into a percentage using Formula 4.2a. This method will also produce a single equivalent discount of 37%.

Another method of calculating the single equivalent discount is to recognize Formula 4.2a as an application of Formula 3.1 involving percent change. The variable d is a discount rate, which you interpret as a negative percent change. The discount amount, D, is the difference between the list price (representing the Old price) and the net price (representing the New price after the discount). Therefore, Formula 4.2a can be rewritten as follows:

$$D\$ = L \times d$$

becomes

$$\text{New} - \text{Old} = \text{Old} \times \Delta\%$$

or

$$\frac{\text{New} - \text{Old}}{\text{Old}} = \Delta\%$$

Therefore, any question about a single equivalent discount where net price and list price are known can be solved as a percent change. Using our ongoing net price example, you have:

$$\frac{\$63 - \$100}{\$100} = -0.37$$
$$\text{or}$$
$$-37\%$$

This is a discount of 37%.

Give It Some Thought:

3. If you are offered discounts in the amount of 25%, 15%, 10%, and 5%, will your total discount percent be 55%, less than 55%, or more than 55%?

Example 4.1D – Retailer Purchasing Ski-Doos with Multiple Discounts

A retail dealership purchases some Expedition TUV Yeti II Ski-Doos to stock in its stores. Examining the merchandising terms of the manufacturer, Bombardier, the dealership notices that it would be eligible to receive a 35% trade discount, 15% volume discount, and 3% loyalty discount. Because it is June and Ski-Doos are out of season, Bombardier offers a seasonal discount of 12% for purchases made before June 30. If the **MSRP** for the Ski-Doo is $12,399.00 and the dealership purchases this item on June 15, what price would it pay?

Plan:

You are looking for the net price that the retail dealership will pay for the Ski-Doo, or N.

Understand:

Step 1: The retail dealership is eligible for all four discounts (it qualifies for the seasonal discount since it is purchasing before June 30). Therefore,

$$L = \$12,399.00$$
$$d_1 = 0.35$$
$$d_2 = 0.15$$
$$d_3 = 0.03$$
$$d_4 = 0.12$$

Step 2: You know the list price, so skip this step.

Step 3: Apply Formula 4.3.

Perform:

Step 3:

$$N = \$12,399 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$$
$$N = \$12,399 \times 0.65 \times 0.85 \times 0.97 \times 0.88 = \$5,847.54$$

Present:

After all four discounts, the retail dealership could purchase the Ski-Doo for $5,847.54.

Ski-Doo Pricing with Four Discounts

MSRP $12,399

- 35% discount (of $12,399) --> $4,339.65
- 15% discount (of $8,059.45*) --> $1,208.90
- 3% discount (of $6,850.45*) --> $205.51
- 12% discount (of $6,644.93*) --> $797.39
- What you pay (net price) --> $5,847.54

*Note: For illustrative purposes these numbers have been rounded to two decimals. In practice, we would need to retain all decimals throughout our calculations.
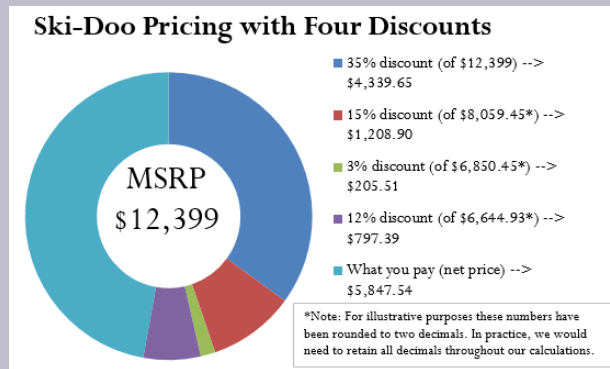
Example 4.1E – Reducing Multiple Discounts to a Single Equivalent Discount

The retail dealership in Example 4.1D purchases more products subject to the same discounts. It needs to simplify its calculations. Using the information from Example 4.1D, what single equivalent discount is equal to the four specified discounts?

Plan:

You are looking for a single equivalent discount that is equal to the four discount percentages, or d_{equiv} (or just d).

Understand:

You know the discount rates:

$$d_1 = 0.35$$
$$d_2 = 0.15$$
$$d_3 = 0.03$$
$$d_4 = 0.12$$

Apply Formula 4.4.

Perform:

$$d_{equiv} = 1 - (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$$
$$d_{equiv} = 1 - (0.65)(0.85)(0.97)(0.88)$$
$$d_{equiv} = 1 - 0.471614 = 0.528386$$
$$\text{or}$$
$$52.8386\%$$

Present:

The retail dealership can apply a 52.8386% discount to all the products it purchases.

Example 4.1F – Making a Smart Consumer Purchase

You are shopping on Boxing Day for an 80" HDTV. You have just one credit card in your wallet, a cashback Visa card, which allows for a 1% cash rebate on all purchases. While scanning flyers for the best deal, you notice that Visions is selling the TV for $5,599.99 including taxes, while Best Buy is selling it for $5,571.99 including taxes. However, because of a computer glitch Best Buy is unable to accept Visa today. Where should you buy your television?

Plan:

You want to know which store you should buy the television at. You must calculate the **net price** (N) for each of the stores.

Understand:

Step 1: You know the list price for each of the stores. You also know the discount available from Visa. Thus,

$$L_{BestBuy} = \$5,571.99 \text{ with no discounts since Visa cannot be used there}$$
$$L_{Visions} = \$5,599.99, d = 0.01, \text{ since you can use your Visa card there}$$

Step 2: List price is known, so skip this step.

Step 3: Apply Formula 4.1.

Perform:

Step 3:

Best Buy: No discounts apply, so the list price equals the net price.

$$N = \$5,571.99$$

Visions:

$$N = \$5,599.99 \times (1 - 0.01) = \$5,599.99 \times 0.99 = \$5,543.99$$

Present:

The net price for Visions is $5,543.99. You save $5,571.99 – $5,543.99 = $28.00 by purchasing your TV at Visions.

Example 4.1G – Understanding the Price

An advertisement claims that at 60% off, you are saving $18. However, today there is an additional 20% off. What price should you pay for this item? What percent savings does this represent?

Plan:

You are looking for how much you should pay after the discounts (N), and the single equivalent percentage that represents the two discounts (d_{equiv} or just d).

Understand:

Step 1: You know the discount amount for the first discount only, as well as the two discount rates:

$$D\$_1 = \$18$$
$$d_1 = 0.60$$
$$d_2 = 0.20$$

Step 2: Calculate the list price by applying Formula 4.2a and rearranging for L.

Step 3: To calculate the net price, apply Formula 4.3.

Step 4: To calculate the single equivalent discount, apply Formula 4.4.

Perform:

Step 2:

$$\$18.00 = L \times 0.60$$
$$L = \$18.00 \div 0.6 = \$30.00$$

Step 3:

$$N = \$30 \times (1 - 0.60) \times (1 - 0.20) = \$30 \times (0.40) \times (0.80) = \$9.60$$

Step 4:

$$d_{equiv} = 1 - (1 - 0.60) \times (1 - 0.20) = 1 - (0.40)(0.80) = 1 - 0.32 = 0.68$$

$$\text{or}$$
$$68\%$$

Present:

You should pay $9.60 for the item, which represents a 68% savings.

Give It Some Thought Answers

1. a. Exactly $10. The net price is the price after the discount.

    b. Less than $10. The discount needs to be removed from the list price.
2. More than half. A 40% discount means that you will pay 60% of the list price.

Exercises

Round all money to two decimals and percentages to four decimals in each of the following questions.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. "N/A" indicates that the particular variable is not applicable in the question.

	List Price or MRSP	First Discount	Second Discount	Third Discount	Net Price	Equivalent Single Discount Rate	Total Discount Amount
1.	$980.00	42%	N/A	N/A	?	N/A	?
2.	?	25%	N/A	N/A	$600.00	N/A	?
3.	$1,975.00	25%	15%	10%	?	?	?
4.	?	18%	4%	7%	$366.05	?	?

Applications

5. A wholesaler of stereos normally qualifies for a 35% trade discount on all electronic products purchased from its manufacturer. If the MSRP of a stereo is $399.95, what net price will the wholesaler pay?

6. Mary is shopping at the mall where she sees a sign that reads, "Everything in the store is 30% off, including sale items!" She wanders in and finds a blouse on the clearance rack. A sign on the clearance rack states, "All clearance items are 50% off." If the blouse is normally priced at $69.49, what price should Mary pay for it?

7. A distributor sells some shoes directly to a retailer. The retailer pays $16.31 for a pair of shoes that has a list price of $23.98. What trade discount percent is the distributor offering to its retailers?

8. A retailer purchases supplies for its head office. If the retailer pays $16.99 for a box of paper and was eligible for a 15% volume discount, what was the original MSRP for the box of paper?

9. Mountain Equipment Co-op has purchased a college backpack for $29 after discounts of 30%, 8%, and 13%. What is the MSRP for the backpack? What single discount is equivalent to the three discounts?

10. Walmart purchased the latest CD recorded by Selena Gomez. It received a total discount of $10.08 off the MSRP for the CD, which represents a discount percent of 42%.
a. What was the MSRP?
b.What was the net price paid for the CD?

11. Best Buy just acquired an HP Pavilion computer for its electronics department. The net price on the computer is $260.40 and Best Buy receives discounts of 40% and 38%.
a. What single discount is equivalent to the two discounts?
b. What is the list price?
c. What is the total discount amount?

12. TELUS retails a Samsung cellphone at the MSRP of $399.99. TELUS can purchase the phone from its supplier and receive a 20% trade discount along with a 5% volume discount.
a. What is the single equivalent percent discount?
b. What net price does TELUS pay for the phone?
c. How much of a discount in dollars does this represent?

13. A wholesaler offers the following discounts: 10% seasonal discount for all purchases made between March 1 and May 1, 15% cumulative quantity discount whenever more than 5,000 units are purchased in any month, 5% loyalty discount for customers who have made regular purchases every month for at least one year, and a 33% trade discount to any retailer. Ed's Retail Superstore makes a purchase of 200 watches, MSRP $10, from the wholesaler on April 29. This month alone, Ed's has ordered more than 5,000 watches. However, Ed's has purchased from the wholesaler for only the past six months. Determine the total price that Ed's should pay for the watches.

14. If a distributor is eligible for a 60% trade discount, 5% volume discount, and 3% seasonal discount, what single equivalent discount rate would it be eligible to receive? If the trade discount is applied first and equals a trade discount of $48, calculate the net price for the item.

Challenge, Critical Thinking, & Other Applications

As mentioned previously, discount percentages share a commonality with negative percent changes (Section 2.1). Use the formulas from this chapter to solve questions 15–17 involving percent change.

15. A human resource manager needs to trim labour costs in the following year by 3%. If current year labour costs are $1,231,498, what are the labour costs next year?

16. At an accounting firm, the number of accountants employed is based on the ratio of 1:400 daily manual journal entries. Because of ongoing increases in automation, the number of manual journal entries declines at a constant rate of 4% per year. If current entries are 4,000 per year, how many years and days will it take until the firm needs to lay off one accountant? (Hint: An accountant is laid off when the number of journal entries drops below 3,600.)

17. An economist is attempting to understand how Canada reduced its national debt from 1999

to 2008. In 1999, Canada's national debt was $554.143 billion. In 2008, the national debt stood at $457.637 billion. What percentage had the national debt been reduced by during this time period?

18. Sk8 is examining an invoice. The list price of a skateboard is $109.00, and the invoice states it received a trade discount of 15% and quantity discount of 10% as well as a loyalty discount. However, the amount of the loyalty discount is unspecified.

a. If Sk8 paid $80.88 for the skateboard, what is the loyalty discount percent?

b. If the loyalty discount is applied after all other discounts, what amount of loyalty dollars does Sk8 save per skateboard?

19. Currently, a student can qualify for up to six different tuition discounts at a local college based on such factors as financial need or corporate sponsorships. Mary Watson just applied to the college and qualifies for all six discounts: 20%, 15%, 23%, 5%, 3%, and 1%.

a. She is confused and wants the college to tell her what single discount percent she is receiving. What should the college tell her?

b. If her total list tuition comes to $6,435.00, how much should she pay?

20. Sumandeep is very loyal to her local hairstylist. Because she is loyal, her hairstylist gives her three different discounts: 10%, 5%, and 5%. These discounts amount to $14.08 in savings.

a. What was the list price her hairstylist charged her?

b. What amount did she pay her hairstylist?

c. If her hairstylist increases prices by 5%, what are the list price, net price, and total discount amount?

4.2 Markup

As you wait in line to purchase your Iced Caramel Macchiato at Starbucks, you look at the pricing menu and think that $4.99 seems like an awful lot of money for a frozen coffee beverage. Clearly, the coffee itself doesn't cost anywhere near that much. But then gazing around the café, you notice the carefully applied colour scheme, the comfortable seating, the high-end machinery behind the counter, and a seemingly well-trained barista who answers customer questions knowledgeably. Where did the money to pay for all of this come from? You smile as you realize your $4.99 pays not just for the macchiato, but for everything else that comes with it.

The process of taking a product's cost and increasing it by some amount to arrive at a selling price is called **markup.** This process is critical to business success because every business must ensure that it does not lose money when it makes a sale. From the consumer perspective, the concept of markup helps you make sense of

the prices that businesses charge for their products or services. This in turn helps you to judge how reasonable some prices are (and hopefully to find better deals).

The Components in a Selling Price

Before you learn to calculate markup, you first have to understand the various components of a selling price. Then, in the next section, markup and its various methods of calculation will become much clearer.

When your business acquires merchandise for resale, this is a monetary outlay representing a cost. When you then resell the product, the price you charge must recover more than just the product cost. You must also recover all the selling and operating expenses associated with the product. Ultimately, you also need to make some money, or profit, as a result of the whole process.

Most people think that marking up a product must be a fairly complex process. It is not. Formula 4.5 illustrates the relationship between the three components of cost, expenses, and profits in calculating the selling price.

Formula 4.5 – The Selling Price of a Product

$$S = C + E + P$$

S is **Selling Price:** Once you calculate what the business paid for the product (cost), the bills it needs to cover (expenses), and how much money it needs to earn (profit), you arrive at a selling price by summing the three components.

C is **Cost:** The cost is the amount of money that the business must pay to purchase or manufacture the product. If manufactured, the cost represents all costs incurred to make the product. If purchased, this number results from applying an appropriate discount formula from Section 4.1. There is a list price from which the business will deduct discounts to arrive at the net price. The net price paid for the product equals the cost of the product. If a business purchases or manufactures a product for $10 then it must sell the product for at least $10. Otherwise, it fails to recover what was paid to acquire or make the product in the first place—a path to sheer disaster!

E is **Expenses:** Expenses are the financial outlays involved in selling the product. Beyond just purchasing the product, the business has many more bills to pay, including wages, taxes, leases,

equipment, electronics, insurance, utilities, fixtures, décor, and many more. These expenses must be recovered and may be calculated as:

- A fixed dollar amount per unit.
- A percentage of the product cost. For example, if a business forecasts total merchandise costs of $100,000 for the coming year and total business expenses of $50,000, then it may set a general guideline of adding 50% ($50,000 ÷ $100,000) to the cost of a product to cover expenses.
- A percentage of the product selling price based on a forecast of future sales. For example, if a business forecasts total sales of $250,000 and total business expenses of $50,000, then it may set a general guideline of adding 20% ($50,000 ÷ $250,000) of the selling price to the cost of a product to cover expenses.

P is **Profit:** Profit is the amount of money that remains after a business pays all of its costs and expenses. A business needs to add an amount above its costs and expenses to allow it to grow. If it adds too much profit, though, the product's price will be too high, in which case the customer may refuse to purchase it. If it adds too little profit, the product's price may be too low, in which case the customer may perceive the product as shoddy and once again refuse to purchase it. Many businesses set general guidelines on how much profit to add to various products. As with expenses, this profit may be expressed as:

- A fixed dollar amount per unit.
- A percentage of the product cost.
- A percentage of the selling price.

How It Works

Follow these steps to solve pricing scenarios involving the three components:

Step 1: Four variables are involved in Formula 4.5. Identify the known variables. Note that you may have to calculate the product's cost by applying the single or multiple discount formulas (Formulas 4.1 and 4.3, respectively). Pay careful attention to expenses and profits to capture how you calculate these amounts.

Step 2: Apply Formula 4.5 and solve for the unknown variable.

Assume a business pays a net price of $75 to acquire a product. Through analyzing its finances, the business estimates expenses at $25 per unit, and it figures it can add $50 in profit. Calculate the selling price.

Step 1: The net price paid for the product is the product cost. The known variables are:

$$C = \$75$$
$$E = \$25$$
$$P = \$50$$

Step 2: According to Formula 4.5, the unit selling price is:

$$S = C + E + P = \$75 + \$25 + \$50 = \$150$$

In applying Formula 4.5 you must adhere to the basic rule of linear equations requiring all terms to be in the same unit. That is, you could use Formula 4.5 to solve for the selling price of an individual product, where the three components are the unit cost, unit expenses, and unit profit. When you add these, you calculate the unit selling price. Alternatively, you could use Formula 4.5 in an aggregate form where the three components are total cost, total expenses, and total profit. In this case, the selling price is a total selling price, which is more commonly known as total revenue. But you cannot mix individual components with aggregate components.

The most common mistake in working with pricing components occurs in the "Understand" portion of the PUPP model. It is critical to identify and label information correctly. You have to pay attention to details such as whether you are expressing the expenses in dollar format or as a percentage of either cost or selling price. Systematically work your way through the information provided piece by piece to ensure that you do not miss an important detail.

Give It Some Thought:

1. What three components make up a selling price? In what units are these components commonly expressed?
2. In what three ways are expenses and profits expressed?

3. What is the relationship between net price and cost?

Example 4.2A – Setting a Price on Fashion in Dollars

Mary's Boutique purchases a dress for resale at a cost of $23.67. The owner determines that each dress must contribute $5.42 to the expenses of the store. The owner also wants this dress to earn $6.90 toward profit. What is the regular selling price for the dress?

Plan:

You are looking for the regular selling price for the dress, or S.

Understand:

Step 1: The unit cost of the dress and the unit expense and the unit profit are all known:

$$C = \$23.67$$
$$E = \$5.42$$
$$P = \$6.90$$

Step 2: Apply Formula 4.5.

Perform:

Step 2:

$$S = \$23.67 + \$5.42 + \$6.90 = \$35.99$$

Present:

Mary's Boutique will set the regular price of the dress at $35.99.

Example 4.2B – Setting the Price Using Percentage of Cost

John's Discount Store just completed a financial analysis. The company determined that expenses average 20% of the product cost and profit averages 15% of the product cost. John's Discount Store purchases Chia Pets from its supplier for an MSRP of $19.99 less a trade discount of 45%. What will be the regular selling price for the Chia Pets?

Plan:
You are looking for the regular selling price for the Chia pets, or S.

Understand:
Step 1: The list price, discount rate, expenses, and profit are known:

$$L = \$19.99$$
$$d = 0.45$$
$$E = 20\% \text{ of cost, or } 0.20C$$
$$P = 15\% \text{ of cost, or } 0.15C$$

Although the cost of the Chia Pets is not directly known, you do know the **MSRP** (list price) and the trade discount. The cost is equal to the net price. Apply Formula 4.1.

Step 2: To calculate the selling price, apply Formula 4.5.

Perform:
Step 1:

$$N = \$19.99 \times (1 - 0.45) = \$19.99 \times 0.55 = \$10.99 = C$$

Step 2:

$$S = \$10.99 + 0.20C + 0.15C$$
$$S = \$10.99 + 0.20(10.99) + 0.15(\$10.99)$$
$$S = \$10.99 + \$2.20 + \$1.65 = \$14.84$$

Present:
John's Discount Store will sell the Chia Pet for $14.84.

The Price Components for a Chia Pet

$14.84

■ Cost -->
$10.99

■ Expenses -->
$2.20
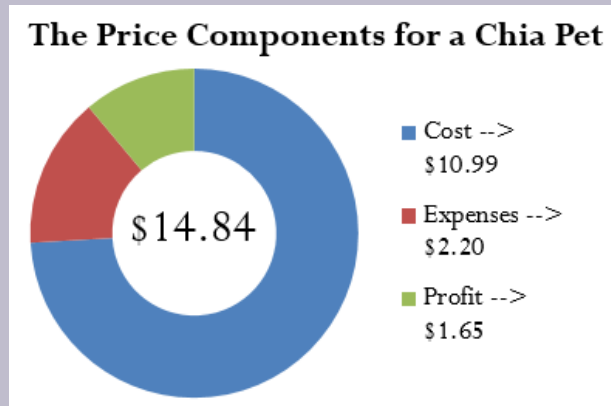
■ Profit -->
$1.65

Example 4.2C – Setting the Price Using Percentage of Selling Price

Based on last year's results, Benthal Appliance learned that its expenses average 30% of the regular selling price. It wants a 25% profit based on the selling price. If Benthal Appliance purchases a fridge for $1,200, what is the regular unit selling price?

Plan:
You are looking for the regular unit selling price for the fridge, or S.

Understand:
Step 1: The cost, expenses, and profit for the fridge are known:

$$E = 30\% \text{ of } S, \text{ or } 0.3S$$
$$P = 25\% \text{ of } S, \text{ or } 0.25S$$
$$C = \$1,200.00$$

Step 2: Apply Formula 4.5.

Perform:
Step 2:

$$S = \$1,200.00 + 0.3S + 0.25S$$

$$S = \$1,200 + 0.55S$$
$$S - 0.55S = \$1,200.00$$
$$0.45S = \$1,200.00$$
$$S = \$2,666.67$$

Present

Benthal Appliance should set the regular selling price of the fridge at $2,666.67.

Price Components for a Fridge

$2,666.67

■ Cost --> $1,200.00

■ Expenses -->
0.3($2,666.67) =
$800.00
■ Profit -->
0.25($2,666.67) =
$666.67
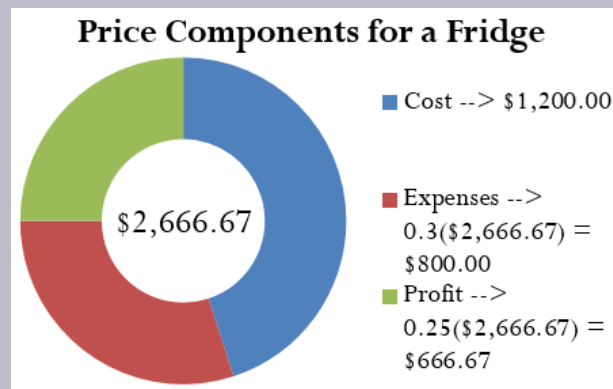
Example 4.2D – Using Selling Price to Figure Out the Cost

If a company knows that its profits are 15% of the selling price and expenses are 30% of cost, what is the cost of an MP3 player that has a regular selling price of $39.99?

Plan:

You are looking for the cost of the MP3 player, or C.

Understand:

Step 1: The expenses, profits, and the regular unit selling price are as follows:

S = $39.99

P = 15% of S, or $0.15S$

E = 30% of cost, or $0.3C$

Step 2: Apply Formula 4.5, rearranging for C.

Perform:

Step 2:

$$\$39.99 = C + 0.3C + 0.15(\$39.99)$$
$$\$39.99 = 1.3C + \$6.00$$
$$\$33.99 = 1.3C$$
$$\$26.15 = C$$

Present:

The cost of the MP3 Player is $26.15.

Price Components of an MP3 Player

$39.99

■ Cost --> $26.15

■ Expenses -->
0.3($26.15) = $7.84
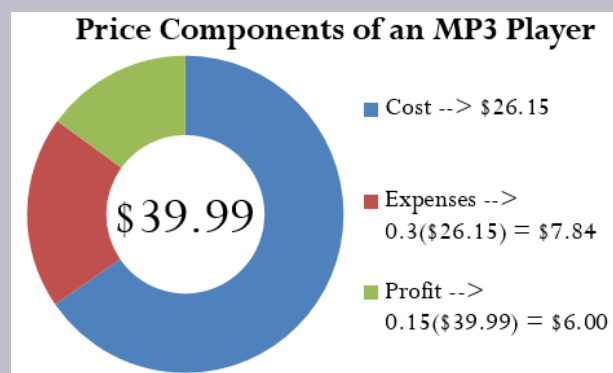
■ Profit -->
0.15($39.99) = $6.00

Example 4.2E – Determining Profitability

Peak of the Market considers setting the regular unit selling price of its strawberries at $3.99 per kilogram. If it purchases these strawberries from the farmer for $2.99 per kilogram and expenses average 40% of product cost, does Peak of the Market make any money?

Plan:

In asking whether Peak of the Market makes money, you are looking for the profit, or P.

Understand:

Step 1: The cost, expenses, and proposed regular unit selling price for the strawberries are as follows:

$$S = \$3.99$$
$$C = \$2.99$$
$$E = 40\% \text{ of cost, or } 0.4C$$

Step 2: Apply Formula 4.5, rearranging for P.

Perform:

Step 2:

$$\$3.99 = \$2.99 + 0.4(\$2.99) + P$$
$$\$3.99 = \$4.19 + P$$
$$-\$0.20 = P$$

Present:

The negative sign on the profit means that Peak of the Market would take a loss of $0.20 per kilogram if it sells the strawberries at $3.99. Unless Peak of the Market has a marketing reason or sound business strategy for doing this, the company should reconsider its pricing.

Calculating the Markup Dollars

Most companies sell more than one product, each of which has different price components with varying costs, expenses, and profits. Can you imagine trying to compare 50 different products, each with three different components? You would have to juggle 150 numbers! To make merchandising decisions more manageable and comparable, many companies combine expenses and profit together into a single quantity, either as a dollar amount or a percentage. This section focuses on the markup as a dollar amount.

One of the most basic ways a business simplifies its merchandising is by combining the dollar amounts of its expenses and profits together as expressed in Formula 4.6.

Formula 4.6 – Markup Amount

$$M\$ = E + P$$

$M\$$ is **Markup Amount:** Markup is taking the cost of a product and converting it into a selling price. The **markup amount** represents the dollar amount difference between the cost and the selling price.

E is **Expenses:** The expenses associated with the product.

P is **Profit:** The profit earned when the product sells.

Note that since the markup amount ($M\$$) represents the expenses (E) and profit (P) combined, you can substitute the variable for markup amount into Formula 4.5 to create Formula 4.7, which calculates the regular selling price.

Formula 4.7 – Selling Price Using Markup

$$S = C + M\$$$

S is **Selling Price:** The regular **selling price** of the product.

C is **Cost:** The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

$M\$$ is **Markup Amount:** From Formula 4.6, this is the single number that represents the total of the expenses and profit.

How It Works

Follow these steps when you work with calculations involving the markup amount:

Step 1: You require three variables in either Formula 4.6 or Formula 4.7. At least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 4.6 or Formula 4.7 for the unknown variable.

Recall from Example 4.2D that the MP3 player's expenses are $7.84, the profit is $6.00, and the cost is $26.15. Calculate the markup amount and the selling price.

Step 1: The known variables are

$$E = \$7.84$$
$$P = \$6.00$$
$$C = \$26.15.$$

Step 2: According to Formula 4.6, the markup amount is the sum of the expenses and profit, or

$$M\$ = \$7.84 + \$6.00 = \$13.84.$$

Applying Formula 4.7, add the markup amount to the cost to arrive at the regular selling price, resulting in

$$S = \$26.15 + \$13.84 = \$39.99$$

You might have already noticed that many of the formulas in this chapter are interrelated. The same variables appear numerous times but in different ways. To help visualize the relationship between the various formulas and variables, many students have found it helpful to create a markup chart, as shown below.

This chart demonstrates the relationships between Formulas 4.5, 4.6, and 4.7. It is evident that the selling price (the green line) consists of cost, expenses, and profit (the red line representing Formula 4.5); or it can consist of cost and the markup amount (the blue line representing Formula 4.7). The markup amount on the blue line consists of the expenses and profit on the red line (Formula 4.6).

Example 4.2F – Markup as a Dollar Amount

A cellular retail store purchases an iPhone with an MSRP of $779 less a trade discount of 35% and volume discount of 8%. The store sells the phone at the MSRP.

a. What is the markup amount?

b. If the store knows that its expenses are 20% of the cost, what is the store's profit?

Plan:

First of all, you need to calculate the markup amount ($M\$$). You also want to find the store's profit (P).
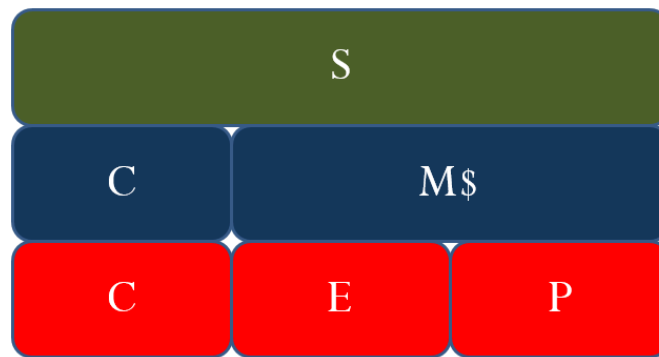
Understand:

Step 1: The smartphone MSRP and the two discounts are known, along with the expenses and selling price:

$$L = \$270$$
$$d_1 = 0.35$$
$$d_2 = 0.08$$
$$E = 20\% \text{ of cost, or } 0.2C$$
$$S = \$779$$

Calculate the cost of the iPhone by applying Formula 4.3.

Step 2: Calculate the markup amount using Formula 4.7.

Calculate the profit by applying Formula 4.6, rearranging for P.

<u>Perform:</u>

Step 1:

$$N = \$779.00 \times (1 - 0.35) \times (1 - 0.08)$$
$$N = \$779.00 \times 0.65 \times 0.92 = \$465.84 = C$$

Step 2:

$$\$779.00 = \$465.84 + M\$$$
$$\$313.16 = M\$$$

$$\$313.16 = 0.2(\$465.84) + P$$
$$\$313.16 = \$93.17 + P$$
$$\$219.99 = P$$

<u>Present:</u>

The markup amount for the iPhone is $313.16. When the store sells the phone for $779.00, its profit is $219.99.

The Price Components of an Apple iPhone

$779

■ Cost --> $465.84

■ Expenses --> $93.17

■ Profit --> $219.99
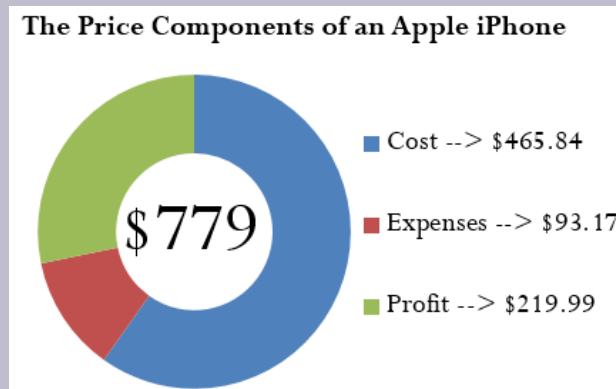
Calculating the Markup Percent

It is important to understand markup in terms of the actual dollar amount; however, it is more common in business practice to calculate the markup as a percentage. There are three benefits to converting the markup dollar amount into a percentage:

1. Easy comparison of different products having vastly different price levels and costs, to help you see how each product contributes toward the financial success of the company. For example, if a chocolate bar

has a 50¢ markup included in a selling price of $1, while a car has a $1,000 markup included in a selling price of $20,000, it is difficult to compare the profitability of these items. If these numbers were expressed as a percentage of the selling price such that the chocolate bar has a 50% markup and the car has a 5% markup, it is clear that more of every dollar sold for chocolate bars goes toward list profitability.

2. Simplified translation of costs into a regular selling price—a task that must be done for each product, making it helpful to have an easy formula, especially when a company carries hundreds, thousands, or even tens of thousands of products. For example, if all products are to be marked up by 50% of cost, an item with a $100 cost can be quickly converted into a selling price of $150.

3. An increased understanding of the relationship between costs, selling prices, and the list profitability for any given product. For example, if an item selling for $25 includes a markup on selling price of 40% (which is $10), then you can determine that the cost is 60% of the selling price ($15) and that $10 of every $25 item sold goes toward list profits.

You can translate the markup dollars into a percentage using two methods, which express the amount either as a percentage of cost or as a percentage of selling price:

- *Method 1: Markup as a Percentage of Cost.*
  This method expresses the markup rate using cost as the base. Many companies use this technique internally because most accounting is based on cost information. The result, known as the **markup on cost percentage**, allows a reseller to convert easily from a product's cost to its regular unit selling price.
- *Method 2: Markup as a Percentage of Selling Price.*
  This method expresses the markup rate using the regular selling price as the base. Many other companies use this method, known as the **markup on selling price percentage**, since it allows for quick understanding of the portion of the selling price that remains after the cost of the product has been recovered. This percentage represents the list profits before the deduction of expenses and therefore is also referred to as the list profit margin.

The markup on cost percentage is expressed in Formula 4.8, while the markup on selling price percentage is expressed in Formula 4.9.

Formula 4.8 – Markup On Cost Percentage

$$MoC\% = \frac{M\$}{C} \times 100$$

$MoC\%$ is **Markup on Cost Percentage:** This is the percentage by which the cost of the product needs to be increased to arrive at the selling price for the product.

$M\$$ is **Markup Amount:** The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.

C is **Cost:** The amount of money needed to acquire or manufacture the product. If the product is being acquired, the cost is the same amount as the net price paid.

× 100 is **Percent Conversion**: The markup on cost is always a percentage.

Formula 4.9 – Markup On Selling Price Percentage

$$MoS\% = \frac{M\$}{S} \times 100$$

$MoS\%$ is **Markup on Selling Price Percentage:** This is the percentage of the selling price that remains available as list profits after the cost of the product is recovered.

M is **Markup Amount**: The total dollars of the expenses and the profits; this total is the difference between the cost and the selling price.

S is **Selling Price**: The regular selling price of the product.

× 100 is **Percent Conversion**: The markup on cost is always a percentage.

How It Works

Refer back to the steps in calculating markup. The steps you must follow for calculations involving markup percent are almost identical to those for working with markup dollars:

Step 1: Three variables are required in either Formula 4.8 or Formula 4.9. For either formula, at least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

Step 2: Solve either Formula 4.8 or Formula 4.9 for the unknown variable.

Continuing to work with Example 4.2D, recall that the cost of the MP3 player is $26.15, the markup amount is $13.84, and the selling price is $39.99. Calculate both markup percentages.

Step 1: The known variables are

$$C = \$26.15$$
$$M\$ = \$13.84$$
$$S = \$39.99.$$

Step 2: To calculate the markup on cost percentage, apply Formula 4.8:

$$MoC\% = \frac{\$13.84}{\$26.15} \times 100 = 52.9254\%$$

In other words, you must add 52.9254% of the cost on top of the unit cost to arrive at the regular unit selling price of $39.99.

To calculate the markup on selling price percentage, apply Formula 4.9:

$$MoS\% = \frac{\$13.84}{\$39.99} \times 100 = 34.6087\%$$

In other words, 34.6087% of the selling price represents list profits after the business recovers the $26.15 cost of the MP3 player.

Businesses are very focused on profitability. Your Texas Instruments BAII Plus calculator is programmed with the markup on selling price percentage. The function is located on the second shelf above the number three. To use this function, open the window by pressing 2nd 3. You can scroll between lines using your ↑ and ↓ arrows. There are three variables:

- CST is the cost. Use the symbol C.
- SEL is the selling price. Use the symbol S.
- MAR is the markup on selling price percentage. Use the symbol $MoS\%$.

As long as you know any two of the variables, you can solve for the third. Enter any two of the three variables (you need to press ENTER after each), making sure the window shows the output you are seeking, and press CPT.

Merchandising involves many variables. Nine formulas have been established so far, and a few more are yet to be introduced. Though you may feel bogged down by all of these formulas, just remember that you have encountered most of these merchandising concepts since you were very young and that you interact with retailers and pricing every day. This chapter merely formalizes calculations you already perform on a daily basis, whether at work or at home. The calculation of discounts is no different than going to Walmart and finding your favourite CD on sale. You know that when a business sells a product, it has to recoup the cost of the product, pay its bills, and make some money. And you have worked with percentages since elementary school.

Do not get stuck in the formulas. Think about the concept presented in the question. Change the scenario of the question and put it in the context of something more familiar. Ultimately, if you really have difficulties then look at the variables provided and cross-reference them to the merchandising formulas. Your goal is to find formulas in which only one variable is unknown. These formulas are solvable. Then ask yourself, "How does knowing that new variable help solve any other formula?"

You do not need to get frustrated. Just be systematic and relate the question to what you already know.

The triangle method simplifies rearranging both Formulas 4.8 and 4.9 to solve for other unknown variables as illustrated in the figure below.

Formula 4.8: Markup on Cost   **Formula 4.9:** Markup on Selling Price

Sometimes you need to convert the markup on cost percentage to a markup on selling price percentage, or vice versa. Two shortcuts allow you to convert easily from one to the other:

$$MoC\% = \frac{MoS\%}{1-MoS\%}$$

$$MoS\% = \frac{MoC\%}{1+MoC\%}$$

Notice that these formulas are very similar. How do you remember whether to add or subtract in the denominator? In normal business situations, the $MoC\%$ is always larger than the $MoS\%$. Therefore, if you are converting one to the other you need to identify whether you want the percentage to become larger or smaller.

- To calculate $MoC\%$, you want a larger percentage. Therefore, make the denominator smaller by subtracting $MoS\%$ from 1.
- To calculate $MoS\%$, you want a smaller percentage. Therefore, make the denominator larger by adding $MoC\%$ to 1.

Give It Some Thought:

Answer the following true/false questions.

4. The markup on selling price percentage can be higher than 100%.

5. The markup dollar amount can be more than the selling price.

6. The markup on cost percentage can be higher than 100%.

7. The markup on cost percentage in most business situations is higher than the markup on selling price percentage.

8. If you know the markup on cost percentage and the cost, you can calculate a selling price.

9. If you know the markup on selling price percentage and the cost, you can calculate a selling price.

Example 4.2G – Markup as a Percentage

A large national retailer wants to price a Texas Instruments BAII Plus calculator at the MSRP of $39.99. The retailer can acquire the calculator for $17.23.

a. What is the markup on cost percentage?

b. What is the markup on selling price percentage?

Plan:

You have been asked to solve for the markup on cost percentage $(MoC\%)$ and the markup on selling price percentage $(MoS\%)$.

Understand:

Step 1: The regular unit selling price and the cost are provided:

$$S = \$39.99$$
$$C = \$17.23$$

You need the markup dollars. Apply Formula 4.7, rearranging for $M\$$.

Step 2: To calculate markup on cost percentage, apply Formula 4.8.

To calculate markup on selling price percentage, apply Formula 4.9.

Perform:

Calculator Instructions:

CST	SEL	MAR
17.23	39.99	Answer: 56.9142

Step 1:

$$\$39.99 = \$17.23 + M\$$$
$$\$22.76 = M\$$$

Step 2:

$$MoC\% = \tfrac{\$22.76}{\$17.23} \times 100 = 132.0952\%$$
$$MoS\% = \tfrac{\$22.76}{\$39.99} \times 100 = 56.9142\%$$

Present:

The markup on cost percentage is 132.0952%. The markup on selling price percentage is 56.9142%.

Break-Even Pricing

In running a business, you must never forget the "bottom line." In other words, if you fully understand how your products are priced, you will know when you are making or losing money. Remember, if you keep losing money you will not stay in business for long! 15% of new businesses will not make it past their first year, and 49% fail in their first five years. This number becomes even more staggering with an 80% failure rate within the first decade.[1] Do not be one of these statistics! With your understanding of markup, you now know what it takes to break even in your business. **Break-even** means that you are earning no profit, but you are not losing money either. Your profit is zero.

If the regular unit selling price must cover three elements—cost, expenses, and profit—then the regular unit selling price must exactly cover your costs and expenses when the profit is zero. In other words, if Formula 4.5 is modified to calculate the selling price at the break-even point (S_{BE}) with P=0, then:

$$S_{BE} = C + E$$

This is not a new formula. It just summarizes that at break-even there is no profit or loss, so the profit (P) is eliminated from the formula.

How It Works

The steps you need to calculate the break-even point are no different from those you used to calculate the regular selling price. The only difference is that the profit is always set to zero.

Recall Example 4.2D that the cost of the MP3 player is $26.15 and expenses are $7.84.

---

1. Statistics Canada, "Failure Rates for New Firms," The Daily, February 16, 2000, www.statcan.gc.ca/daily-quotidien/000216/dq000216b-eng.htm.

The break-even price (S_{BE}) is

$$\$26.15 + \$7.84 = \$33.99$$

This means that if the MP3 player is sold for anything more than $33.99, it is profitable; if it is sold for less, then the business does not cover its costs and expenses and takes a loss on the sale.

Example 4.2H – Knowing Your Break-Even Price

John is trying to run an eBay business. His strategy has been to shop at local garage sales and find items of interest at a great price. He then resells these items on eBay. On John's last garage sale shopping spree, he only found one item—a Nintendo Wii that was sold to him for $100. John's vehicle expenses (for gas, oil, wear/tear, and time) amounted to $40. eBay charges a $2.00 insertion fee, a flat fee of $2.19, and a commission of 3.5% based on the selling price less $25. What is John's minimum list price for his Nintendo Wii to ensure that he at least covers his expenses?

Plan:
You are trying to find John's break-even selling price (S_{BE}).

Understand:
Step 1: John's cost for the Nintendo Wii and all of his associated expenses are as follows:

$$C = \$100.00$$
$$E \text{ (vehicle)} = \$40.00$$
$$E \text{ (insertion)} = \$2.00$$
$$E \text{ (flat)} = \$2.19$$
$$E \text{ (commission)} = 3.5\%(S_{BE} - \$25.00)$$

You have four expenses to add together that make up the E in the formula.

Step 2: Formula 4.5 states $S = C + E + P$. Since you are looking for the break-even point, then P is set to zero and $S_{BE} = C + E$.

Perform:

Step 1:

$$E = \$40.00 + \$2.00 + \$2.19 + 3.5\%(S_{BE} - \$25.00)$$
$$E = \$44.19 + 0.035(S_{BE} - \$25.00) = \$44.19 + 0.035S_{BE} - \$0.875$$
$$E = \$43.315 + 0.035S_{BE}$$

Step 2:

$$S_{BE} = \$100 + \$43.315 + 0.035S_{BE}$$
$$S_{BE} = \$143.315 + 0.035S_{BE}$$
$$S_{BE} - 0.035S_{BE} = \$143.315$$
$$0.965S_{BE} = \$143.315$$
$$S_{BE} = \$148.51$$

Present:

At a price of $148.51 John would cover all of his costs and expenses but realize no profit or loss. Therefore, $148.51 is his minimum price.

Give It Some Thought Answers

1. Cost, expenses, and profit. They are expressed either per unit or as a total.
2. A specific dollar amount, a percentage of cost, or a percentage of the selling price.
3. The net price paid for a product is the same as the cost of the product.
4. False. The markup amount is a portion of the selling price and therefore is less than 100%.
5. False. The markup amount plus the cost equals the selling price. It must be less than the selling price.
6. True. A cost can be doubled or tripled (or increased even more) to reach the price.
7. True. The base for markup on cost percentage is smaller, which produces a larger percentage.
8. True. You could combine Formulas 4.7 and 4.8 to arrive at the selling price.
9. True. You could convert the $MoS\%$ to a $MoC\%$ and solve as in the previous question.

Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–8, solve for the unknown variables (identified with a ?) based on the information provided.

	Regular Unit Selling Price	Cost	Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price
1.	?	$188.42	$48.53	$85.00	?	?	?	?
2.	$999.99	?	30% of C	23% of C	?	?	?	?
3.	?	?	?	10% of S	$183.28	?	155%	?
4.	$274.99	?	20% of S	?	?	?	?	35%
5.	?	?	45% of C	?	$540.00	$1,080.00	?	?
6.	?	$200 less 40%	?	15% of S	?	?	68%	?
7.	?	?	$100.00	?	$275.00	?	?	19%
8.	?	?	15% of C	12% of S	?	$253.00	?	?

Applications

9. If a pair of sunglasses sells at a regular unit selling price of $249.99 and the markup is always 55% of the regular unit selling price, what is the cost of the sunglasses?

10. A transit company wants to establish an easy way to calculate its transit fares. It has determined that the cost of a transit ride is $1.00, with expenses of 50% of cost. It requires $0.75 profit per ride. What is its markup on cost percentage?

11. Daisy is trying to figure out how much negotiating room she has in purchasing a new car. The car has an MSRP of $34,995.99. She has learned from an industry insider that most car dealerships have a 20% markup on selling price. What does she estimate the dealership paid for the car?

12. The markup amount on an eMachines desktop computer is $131.64. If the machine regularly retails for $497.25 and expenses average 15% of the selling price, what profit will be earned?

13. Manitoba Telecom Services (MTS) purchases an iPhone for $749.99 less discounts of 25% and 15%. MTS's expenses are known to average 30% of the regular unit selling price.
a. What is the regular unit selling price if a profit of $35 per iPhone is required?
b. What are the expenses?
c. What is the markup on cost percentage?
d. What is the break-even selling price?

14. A snowboard has a cost of $79.10, expenses of $22.85, and profit of $18.00.
a. What is the regular unit selling price?
b. What is the markup amount?
c. What is the markup on cost percentage?
d. What is the markup on selling price percentage?
e. What is the break-even selling price? What is the markup on cost percentage at this break-even price?

Challenge, Critical Thinking, & Other Applications

15. A waterpark wants to understand its pricing better. If the regular price of admission is $49.95, expenses are 20% of cost, and the profit is 30% of the regular unit selling price, what is the markup amount?

16. Sally works for a skateboard shop. The company just purchased a skateboard for $89.00 less discounts of 22%, 15%, and 5%. The company has standard expenses of 37% of cost and desires a profit of 25% of the regular unit selling price. What regular unit selling price should Sally set for the skateboard?

17. If an item has a 75% markup on cost, what is its markup on selling price percentage?

18. A product received discounts of 33%, 25%, and 5%. A markup on cost of 50% was then applied to arrive at the regular unit selling price of $349.50. What was the original list price for the product?

19. Mountain Equipment Co-op (MEC) wants to price a new backpack. The backpack can be purchased for a list price of $59.95 less a trade discount of 25% and a quantity discount of 10%. MEC estimates expenses to be 18% of cost and it must maintain a markup on selling price of 35%.
a. What is the cost of backpack?
b. What is the markup amount?
c. What is the regular unit selling price for the backpack?
d. What profit will Mountain Equipment Co-op realize?
e. What happens to the profits if it sells the backpack at the MSRP instead?

20. Costco can purchase a bag of Starbucks coffee for $20.00 less discounts of 20%, 15%, and 7%. It then adds a 40% markup on cost. Expenses are known to be 25% of the regular unit selling price.
a. What is the cost of the coffee?
b. What is the regular unit selling price?
c. How much profit will Costco make on a bag of Starbucks coffee?
d. What markup on selling price percentage does this represent?
e. Repeat questions (a) through (d) if the list price changes to $24.00.

4.3 Markdown

Flashy signs in a retail store announce, "40% off, today only!" Excitedly you purchase three tax-free products with regular price tags reading $100, $250, and $150. The cashier processing the transaction informs you that your total is $325. You are about to hand over your credit card when something about the total makes you pause. The regular total of all your items is $500. If they are 40% off, you should receive a $200 deduction and pay only $300. The cashier apologizes for the mistake and corrects your total.

Although most retail stores use automated checkout systems, these systems are ultimately programmed by human beings. A computer system is only as accurate as the person keying in the data. A study by the Competition Bureau revealed that 6.3% of items at various retail stores scanned incorrectly. The average error spread is up to 13% around the actual product's price![2] Clearly, it is important for you as a consumer to be able to calculate markdowns.

Businesses must also thoroughly understand markdowns so that customers are charged accurately for their purchases. Businesses must always comply with the Competition Act of Canada, which specifically defines legal pricing practices. If your business violates this law, it faces severe penalties.

The Importance of Markdowns

A **markdown** is a reduction from the regular selling price of a product resulting in a lower price. This lower price is called the **sale price** to distinguish it from the selling price.

---

2. Competition Bureau, Fair Business Practices Branch, Price Scanning Report, Table B, page 5, 1999, www.competitionbureau.gc.ca/epic/site/cb-bc.nsf/en/01288e.html.

Many people perceive markdowns as a sign of bad business management decisions. However, in most situations this is not true. Companies must always attempt to forecast the future. In order to stock products, a reseller must estimate the number of units that might sell in the near future for every product that it carries. This is both an art and a science. While businesses use statistical techniques that predict future sales with a relative degree of accuracy, consumers are fickle and regularly change shopping habits. Markdowns most commonly occur under four circumstances:

1. ***Clearing Out Excess or Unwanted Inventory***. In these situations, the business thought it could sell 100 units; however, consumers purchased only 20 units. In the case of seasonal inventory, such as Christmas items on Boxing Day, the retailer wishes to avoid packing up and storing the inventory until the next season.
2. ***Clearing Out Damaged or Discontinued Items***. Selling a damaged product at a discount is better than not selling it at all. When products are discontinued, this leaves shelf space underused, so it is better to clear the item out altogether to make room for profitable items that can keep the shelves fully stocked.
3. ***Increasing Sales Volumes***. Sales attract customers because almost everyone loves a deal. Though special marketing events such as a 48 hour sale reduce the profitability per unit, by increasing the volume sold these sales can lead to a greater profit overall.
4. ***Promoting Add-On Purchases***. Having items on sale attracts customers to the store. Many times customers will not only purchase the item on sale but also, as long as they are on the premises, grab a few other items, which are regularly priced and very profitable. Like many others, you may have walked into Target to buy one item but left with five instead.

Markdowns are no different from offering a discount. Recall from Section 4.1 that one of the types of discounts is known as a sale discount. The only difference here lies in choice of language. Markdowns are common, so you will find it handy to adapt the discount formulas to the application of markdowns, replacing the symbols with ones that are meaningful in merchandising. Formula 4.1, introduced in Section 4.1, calculates the net price for a product after it receives a single discount:

$$N = L \times (1 - d)$$

Formula 4.10 adapts this formula for use in markdown situations.

Formula 4.10 – The Sale Price Of A Product

$$S_{onsale} = S \times (1 - d)$$

S_{onsale} is **Sale Price:** The sale price is the price of the product after reduction by the markdown percent. Conceptually, the sale price is the same as the net price.

S is **Selling Price**: The regular selling price of the product before any discounts. The higher price is the list price. In merchandising questions, this dollar amount may or may not be a known variable. If the selling price is unknown, you must calculate it using an appropriate formula or combination of formulas from either Section 4.1 or Section 4.2.

d is **Markdown Rate**: A markdown rate is the same as a sale discount rate. Therefore you use the same discount rate symbol from Section 4.1 to represent the percentage (in decimal format) by which you reduce the selling price. As in Formula 4.1, note that you are interested in calculating the sale price and not the amount saved. Thus, you take the markdown rate away from 1 to find out the rate owing.

In markdown situations, the selling price and the sale price are different variables. The sale price is always less than the selling price. In the event that a regular selling price has more than one markdown percent applied to it, you can extend Formula 4.10 in the same manner that Formula 4.3 calculated multiple discounts.

If you are interested in the markdown amount in dollars, recall that Formula 4.2 calculates the discount amount in dollars. Depending on what information is known, the formula has two variations:

Formula 4.2a: $D\$ = L \times d$
Formula 4.2b: $D\$ = L - N$

Formulas 4.11a and 4.11b adapt these formulas to markdown situations.

Formula 4.11a and 4.11b – Markdown Amount

Formula 4.11a

$$D\$ = S \times d$$

Formula 4.11b

$$D\$ = S - S_{onsale}$$

$D\$$ is **Markdown Amount**: You determine the markdown amount using either formula depending on what information is known. If you know the selling price and markdown percent, apply Formula 4.11a. If you know the selling price and sale price, apply Formula 4.11b.

S is **Selling Price**: The regular selling price before you apply any markdown percentages.

d is **Markdown Rate**: the percentage of the selling price to be deducted (in decimal format). In this case, because you are interested in figuring out how much the percentage is worth, you do not take it away from 1 as in Formula 4.10.

S_{onsale} is **Sale Price**: The price after you have deducted all markdown percentages from the regular selling price.

The final markdown formula reflects the tendency of businesses to express markdowns as percentages, facilitating easy comprehension and comparison. Recall Formula 4.9 from Section 4.2, which calculated a markup on selling price percent:

$$MoS\% = \frac{M\$}{S} \times 100$$

Formula 4.12 adapts this formula to markdown situations.

Formula 4.12 – Markdown Percentage

$$d = \frac{D\$}{S} \times 100$$

d is **Markdown Percentage**: You always deduct a markdown amount from the regular selling price of the product. Therefore, you always express the markdown percent as a percentage of the selling price. Use the same symbol for a discount rate, since markdown rates are synonymous with sale discounts.

$D\$$ is **Markdown Amount**: The total dollar amount deducted from the regular selling price.

S is **Selling Price**: The regular selling price of the product before any discounts.

100 is **Percent Conversion**: The markdown is always expressed as a percentage.

How It Works

Follow these steps to calculate a markdown:

Step 1: Across all three markdown formulas, the four variables consist of the selling price (S), sale price (S_{onsale}), markdown dollars ($D\$$), and markdown rate (d). Identify which variables are known. Depending on the known information, you may have to calculate the selling price using a combination of discount and markup formulas.

Step 2: Apply one or more of Formulas 4.10, 4.11a, 4.11b, and 4.12 to calculate the unknown variable(s). In the event that multiple markdown rates apply, extend Formula 4.10 to accommodate as many markdown rates as required.

Recall from Section 4.2 the example of the MP3 player with a regular selling price of $39.99.

Assume the retailer has excess inventory and places the MP3 player on sale for 10% off. What is the sale price and markdown amount?

Step 1: The selling price and markdown percent are S = $39.99 and d = 0.10, respectively.

Step 2: Apply Formula 4.10 to calculate the sale price, resulting in

$$S_{onsale} = \$39.99 \times (1 - 0.10) = \$35.99.$$

You could use either of Formulas 4.11a or 4.11b to calculate the markdown amount since the selling price, sale price, and markdown percent are all known. Arbitrarily choosing Formula 4.11a, you calculate a markdown amount of

$$D\$ = \$39.99 \times 0.10 = \$4.00$$

Therefore, if the retailer has a 10% off sale on the MP3 players, it marks down the product by $4.00 and retails it at a sale price of $35.99.

Just as in Section 4.2, avoid getting bogged down in formulas. Recall that the three formulas for markdowns are not new formulas, just adaptations of three previously introduced concepts. As a consumer, you are very experienced with endless examples of sales, bargains, discounts, blowouts, clearances, and the like. Every day you read ads in the newspaper and watch television commercials advertising percent savings. This section simply crystallizes your existing knowledge. If you are puzzled by questions involving markdowns, make use of your shopping experiences at the mall!

Three of the formulas introduced in this section can be solved for any variable through algebraic manipulation when any two variables are known. Recall that the triangle technique helps you remember how to rearrange these formulas, as illustrated here.

Formula 4.10: The Sale Price of a Product

Formula 4.11a & 4.12:
Markdown Amount and Rate

Example 4.3A – Determining the Sale Price and Markdown Amount

The MSRP for the "Guitar Hero: World Tour" video game is $189.99. Most retail stores sell this product at a price in line with the MSRP. You have just learned that a local electronics retailer is selling the game for 45% off. What is the sale price for the video game and what dollar amount is saved?

Plan:

There are two unknown variables. The first is the video game's sale price (S_{onsale}). The second is the markdown amount $(D\$)$ that is realized at that sale price.

Understand:

Step 1: The regular selling price for the video game and the markdown rate are known:

$$S = \$189.99$$
$$d = 0.45$$

Step 2: Calculate the sale price by applying Formula 4.10.

Calculate the markdown amount by applying Formula 4.11b.

Perform:

Step 2:

$$S_{onsale} = \$189.99 \times (1 - 0.45)$$
$$= \$189.99 \times 0.55 = \$104.49$$

$$D\$ = \$189.99 - \$104.49 = \$85.50$$

Present:

The sale price for the video game is $104.49. When purchased on sale, "Guitar Hero: World Tour" is $85.50 off of its regular price.

Example 4.3B – Markdown Requiring Selling Price Calculation

A reseller acquires an Apple iPad for $650. Expenses are planned at 20% of the cost, and profits are set at 15% of the cost. During a special promotion, the iPad is advertised at $100 off. What is the sale price and markdown percent?

Plan:

The unknown variables for the iPad are the sale price (S_{onsale}) and the markdown rate (d).

Understand:

Step 1: The pricing elements of the iPad along with the markdown dollars are known:

$$C = \$650$$
$$E = 0.2C$$
$$P = 0.15C$$
$$D\$ = \$100$$

Calculate the selling price of the product by applying Formula 4.5.

Step 2: Calculate the markdown percent by applying Formula 4.12.

Calculate the sale price by applying Formula 4.11b, rearranging for S_{onsale}.

Perform:

Step 1:

$$S = \$650 + 0.2(\$650) + 0.15(\$650) = \$877.50$$

Step 2:

$$d = \frac{\$100}{\$877.50} \times 100 = 11.396\%$$
$$\$100 = \$877.50 - S_{onsale}$$
$$S_{onsale} = \$777.50$$

Present:

When the iPad is advertised at $100 off, it receives an 11.396% markdown and it will retail at a sale price of $777.50.

Never-Ending Sales

Have you noticed that some companies always seem to have the same item on sale all of the time? This is a common marketing practice. Recall the third and fourth circumstances for markdowns. Everybody loves a sale, so markdowns increase sales volumes for both the marked-down product and other regularly priced items.

For example, Michaels has a product line called the Lemax Village Collection, which has seasonal display villages for Christmas, Halloween, and other occasions. When these seasonal product lines come out, Michaels initially prices them at the regular unit selling price for a short period and then reduces their price. For Michaels, this markdown serves a strategic purpose. The company's weekly flyers advertising the Lemax Village Collection sale attract consumers who usually leave the store with other regularly priced items.

If an item is on sale all the time, then businesses plan the pricing components with the sale price in mind. Companies using this technique determine the unit profitability of the product at the sale price and not the regular selling price. They adapt Formula 4.5 as follows:

$$S = C + E + P$$

becomes

$$S_{onsale} = C + E + P_{onsale}$$

where P_{onsale} represents the planned profit amount when the product is sold at the sale price. This is not a new formula, just a new application of Formula 4.5.

How It Works

Under normal circumstances, when businesses set their selling and sale prices they follow a three-step procedure:

1. Determine the product's cost, expenses, and profit amount.
2. Set the regular selling price of the product.
3. If a markdown is to be applied, determine an appropriate markdown rate or amount and set the sale price.

However, when a product is planned to always be on sale, businesses follow these steps instead to set the sale price and selling price:

Step 1: Set the planned markdown rate or markdown dollars. Determine the pricing components such as cost and expenses. Set the profit so that when the product is marked down, the profit amount is achieved. Alternatively, a planned markup on cost, markup on selling price, or even markup dollars may be set for the sale price.

Step 2: Calculate the sale price of the product. If cost, expenses, and profit are known, apply the adapted version of Formula 4.5. Alternatively, adapt and apply any of the other markup formulas (Formulas 4.6 through 4.9) with the understanding that the result is the sale price of the product and not the regular selling price.

Step 3: Using the known markdown rate or markdown amount, set the regular selling price by applying any appropriate markdown formula (Formulas 4.10 through 4.12).

Assume for the Michael's Lemax Village Collection that most of the time these products are on sale for 40% off. A particular village item costs $29.99, expenses are $10.00, and a planned profit of $8.00 is achieved at the sale price. Calculate the sale price and the selling price.

Step 1: The known variables at the sale price are

$$C = \$29.99$$
$$E = \$10.00$$
$$P = \$8.00$$
$$d = 0.40.$$

Step 2: Adapting Formula 4.5, the sale price is

$$S_{onsale} = C + E + P_{onsale} = \$29.99 + \$10.00 + \$8.00 = \$47.99.$$

This is the price at which Michael's plans to sell the product.

Step 3: However, to be on sale there must be a regular selling price. Therefore, if the 40% off results in a price of $47.99, apply Formula 4.10 and rearrange to get the selling price:

$$S = \$47.99 \div (1 - 0.40) = \$79.98.$$

Therefore, the product's selling price is $79.98, which, always advertised at 40% off, results in a sale price of $47.99. At this sale price, Michael's earns the planned $8.00 profit.

You may ask, "If the product is always on sale, what is the importance of establishing the regular price?" While this textbook does not seek to explain the law in depth, it is worth mentioning that pricing decisions in Canada are regulated by the Competition Act. With respect to the discussion of never-ending sales, the Act does require that the product be sold at a regular selling price for a reasonable period of time or in reasonable quantity before it can be advertised as a sale price.

If you revisit the Michael's example, note in the discussion that the village initially needs to be listed at the regular selling price before being lowered to the sale price.

Give It Some Thought:

1. If a product has a markup on cost of 40% and a markdown of 40%, will it sell above or below cost?
2. What happens to the profit if a product that is always on sale actually sells at the regular selling price?
3. Under normal circumstances, arrange from smallest to largest: regular selling price, cost, and sale price.

Example 4.3C – Setting the Price in a Never-Ending Sale

An electronics retailer has 16GB USB sticks on sale at 50% off. It initially priced these USB sticks for a short period of time at regular price, but it planned at the outset to sell them at the sale price. The company plans on earning a profit of 20% of the cost when the product is on sale. The unit cost of the USB stick is $22.21, and expenses are 15% of the cost.
a. At what price will the retailer sell the USB stick when it is on sale?
b. To place the USB stick on sale, it must have a regular selling price. Calculate this price.
c. If the USB stick is purchased at the regular selling price during the initial time period, how much profit is earned?

Plan:
a. This company plans on always having the product on sale, so the pricing needs to be set for the sale price, or S_{onsale}.
b. You need the regular selling price, or S.
c. Solve for the profit at the regular selling price, or P.

Understand:
Step 1: You know the unit cost, the retailer's associated expenses, its planned profit at the sale price, and the markdown rate:

$$d = 0.50$$
$$C = \$22.21$$
$$E = 0.15C$$
$$P_{onsale} = 0.2C$$

Step 2: To solve part (a), apply the adapted version of Formula 4.5:

$$S_{onsale} = C + E + P_{onsale}$$

Step 3: After you know the sale price, solve part (b) by applying Formula 4.10, rearranging for S.

Step 4: Solving part (c) requires applying Formula 4.5.

Perform:
Step 2:

$$S_{onsale} = \$22.21 + 0.15(\$22.21) + 0.2(\$22.21)$$
$$= \$22.21 + \$3.33 + \$4.44 = \$29.98$$

Step 3:

$$\$29.98 = S \times (1 - 0.5)$$
$$\$29.98 = S \times 0.5$$
$$\$59.96 = S$$

Step 4:

$$\$59.96 = \$22.21 + \$3.33 + P$$
$$\$59.96 = \$25.54 + P$$
$$\$34.42 = P$$

<u>Present:</u>

The USB stick is on sale for $29.98, letting the company achieve its profit of $4.44 per unit. During the initial pricing period, the USB stick sells for $59.96 (its regular selling price). If a consumer actually purchases a USB stick during the initial pricing period, the electronics store earns a profit of $34.42 per unit (which is a total of the $4.44 planned profit plus the planned markdown amount of $29.96).

Give It Some Thought Answers

1. Below cost, since the 40% markdown is off of the selling price, which is a larger value.
2. The profit will be increased by the markdown amount.
3. Cost, sale price, regular selling price.

Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

	Regular Selling Price	Markdown Amount	Markdown Percent	Sale Price
1.	$439.85	?	35%	?
2.	?	$100.00	?	$199.95
3.	$1,050.00	?	?	$775.00
4.	$28,775.00	$3,250.00	?	?
5.	?	?	33%	$13,199.95
6.	?	$38.33	12%	?

Applications

7. A pair of Nike athletic shoes is listed at a regular selling price of $89.99. If the shoes go on sale for 40% off, what is the sale price?

8. During its special Bay Days, The Bay advertises a Timex watch for $39.99 with a regular price of $84.99. Calculate the markdown percent and markdown amount.

9. For spring break you are thinking about heading to Tulum, Mexico. In planning ahead, you notice that a one-week stay at the Gran Bahia Principe Tulum, regularly priced at $2,349 for air and six nights all inclusive, offers an early-bird booking discount of $350. What markdown percentage is being offered for booking early?

10. A Heritage Infusio deep frying pan is advertised at 70% off with a sale price of $39.99. What is the frying pan's regular selling price, and what markdown amount does this represent?

11. A mass merchandiser uses its Lagostina cookware product line as a marketing tool. The cookware is always on sale at an advertised price of 45% off. The cost of the cookware is $199.99, expenses are $75, and the planned profit at the sale price is $110. Calculate the sale price and selling price for the cookware.

12. Quicky Mart regularly sells its Red Bull sports drink for $2.99 per can. Quicky Mart noticed that one of its competitors down the street sells Red Bull for $1.89. What markdown percentage must Quicky Mart advertise if it wants to match its competitor?

13. A hardware store always advertises a Masterdesigner 75-piece screwdriver set at 80% off for a sale price of $17.99.

a. If the cost of the set is $10 and expenses are 30% of the sale price, what is the planned profit when the product is on sale?

b. What profit is earned if the product actually sells at its regular selling price?

14. A campus food outlet is advertising a "Buy one, get one 25% off" deal. The 25% off comes off the lower-priced item. If you purchase a chicken dinner for $8.99 and your friend gets the burger combo for $6.99, what is the markdown percentage on the total price?

15. Blast'em Stereos purchases a stereo system for $1,900 less two discounts of 40% and 18%. The store uses this product to draw customers to the store and always offers the stereo on sale at 25% off. When the stereo is on sale, it plans on expenses equaling 30% of the cost and a profit of 20% of the sale price.

a. What is the sale price for the stereo?

b. How much profit does Blast'em make when the stereo sells at the sale price?

c. By law, this stereo must sell at the regular selling price for a period of time before going on sale. What is the regular selling price?

d. What profit does Blast'em earn if a customer purchases the stereo during this initial period?

Challenge, Critical Thinking, & Other Applications

16. Frigid Boards purchases one of its snowboards for $395 less a retail trade discount of 15% and a loyalty discount of 4%. Its markup on selling price percentage on all snowboards is 21%. At the end of the season, any leftover snowboards are marked down by 10%. What is the sale price for the snowboard?

17. An HP LaserJet printer has an MSRP of $399.95. It is subject to trade discounts of 30% and 23%. The LaserJet is a featured item for a computer store and is always on sale. The store plans to sell the LaserJet for a sale price that allows it to cover expenses equalling 15% of cost and realize a profit of $35.00.

a. What is the sale price?

b. If the MSRP is the regular unit price of the printer, what rate of markdown can the computer store advertise?

c. What markup on selling price percentage is realized at the sale price?

18. The Brick advertises that when you purchase a queen-size Tempur-Pedic mattress set for $2,499.97 it will give you a 51" 3-D plasma television with a 3-D starter kit included. The value of this gift is $1,199.99. What markdown percent does this represent?

19. A Maytag 27 cubic foot refrigerator retails for $2,400.00 at Landover Appliance Centre. The company, which is celebrating its 30th anniversary this coming weekend, features the fridge for 30% off. The markup on selling price percentage on the fridge at the regular unit selling price is 53%.
a. What is the sale price?
b. At the sale price, what is the markup on selling price percentage?
c. If the expenses are 15% of the regular selling price, what is the profit when the fridge is on sale?

20. Dreger Jewelers is selling a diamond bracelet. It uses this bracelet in its promotions and almost always has it on sale. The cost of the bracelet is $2,135 less discounts of 20% and 30%. When the bracelet is on sale for 25% off, the expenses are 15% of cost and the profit is 20% of cost.
a. What is the sale price?
b. What is the bracelet's regular selling price?
c. If the bracelet sells at the regular selling price, what are the markup amount and the markup on cost percent?

4.4 Merchandising

Running a business requires you to integrate all of the concepts in this chapter. From discounts to markups and markdowns, all the numbers must fit together for you to earn profits in the long run. It is critical to understand how pricing decisions affect the various financial aspects of your business and to stay on top of your numbers.

Merchandising does not involve difficult concepts, but to do a good job of it you need to keep track of many variables and observe how they relate to one another. Merchandising situations in this section will apply all of the previously discussed pricing concepts. Next, the concept of maintained markup helps you understand the combined effect of markup and markdown decisions on list profitability. Finally, you will see that coupons and rebates influence expenses and profitability in several ways beyond the face value of the discounts offered.

The Complete Pricing Picture

Up to this point, you have examined the various components of merchandising as separate topics. Your study of basic product pricing has included the following:

1. Taking an MSRP or list price and applying discounts to arrive at a product's cost.
2. Marking up a product by adding expenses and profits to the cost to arrive at a regular selling price.

3. Marking down a product by applying a discount and arriving at a sale price.
4. Working with various percentages in both markup and markdown situations to either simplify calculations or present a clearer pricing picture.

Now is the time to tie all of these merchandising concepts together. Section 4.2 introduced the concept of a markup chart to understand the relationship between various markup components. The figure below extends this figure to include all of the core elements of merchandising.

The figure reveals the following characteristics:

- You calculate the cost of a product by applying a discount formula. The net price, N, is synonymous with cost, C.
- When you mark a product down, not only is the profit reduced but the markup dollars are also reduced. The following relationships exist when a product is on sale:

$$M\$_{onsale} = E + P_{onsale}$$
OR
$$M\$_{onsale} = M\$ - D\$$$

$$P_{onsale} = P - D\$$$

- At break-even (remember, P or P_{onsale} is zero) a sale product has the following relationships:

$$M\$_{onsale} = E$$
$$S_{onsale} = C + E$$
$$D\$ = P$$

None of these relationships represent new formulas. Instead, these reflect a deeper understanding of the relationships between the markup and markdown formulas.

How It Works

Follow these steps to solve complete merchandising scenarios:

Step 1: It is critically important to correctly identify both the known and unknown merchandising variables that you are asked to calculate.

Step 2: For each of the unknown variables in step 1, examine Formulas 4.1 through 4.12. Locate which formulas contain the unknown variable. You must solve one of these formulas to arrive at the answer. Based on the information provided, examine these formulas to determine which formula may be solvable. Write out this formula, identifying which components you know and which components remain unknown.

For example, assume that in step 1 you were asked to solve for the expenses, or E. In looking at the formulas, you find this variable appears only in Formulas 4.5 and 4.6, meaning that you must use one of these two formulas to calculate the expenses. Suppose that, in reviewing the known variables from step 1, you already have the markup amount and the profit. In this case, Formula 4.6 would be the right formula to calculate expenses with.

Step 3: Note the unknown variables among all the formulas written in step 2. Are there common unknown variables among these formulas? These common variables are critical variables. Solving for these common unknowns is the key to completing the question. Note that these unknown variables may not directly point to the information you were requested to calculate, and they do not resolve the merchandising scenario in and of themselves. However, without these variables you cannot solve the scenario.

For example, perhaps in step 1 you were requested to calculate both the markup on cost percent ($MoC\%$) and the markup on selling price percent ($MoS\%$). In step 2, noting that these variables are found only in Formulas 4.8 and 4.9, you wrote them down. Examination of these formulas revealed that you know the cost and selling price variables. However, in both formulas the markup dollars remain unknown. Therefore, markup dollars is the critical unknown variable. You must solve for markup dollars using Formula 4.7, which will then allow you to calculate the markup percentages.

Step 4: Apply any of the merchandising formulas from Formulas 4.1 to 4.12 to calculate the unknown variables required to solve the formulas. Your goal is to identify all required variables and then solve the original unknown variables from step 1.

Before proceeding, take a few moments to review the various concepts and formulas covered earlier in this chapter. A critical and difficult skill is now at hand. As evident in steps 2 through 4 of the solving process, you must use your problem-solving skills to figure out which formulas to use and in what order. Here are three suggestions to help you on your way:

- Analyze the question systematically. Conscientiously use the Plan, Understand, Perform, and Present model.
- If you are unsure of how the pieces of the puzzle fit together, try substituting your known variables into the various formulas. You are looking for
  a. Any solvable formulas with only one unknown variable or
  b. Any pair of formulas with the same two unknowns, since you can solve this system using your algebraic skills of solving two linear equations with two unknowns (covered in Chapter 1).
- Merchandising has multiple steps. Think through the process. When you solve one equation for an unknown variable, determine how knowing that variable affects your ability to solve another formula. As mentioned in step 3, there is usually a critical unknown variable. Once you determine the value of this variable, a domino effect allows you to solve any other remaining formulas.

Example 4.4A – A Merchandising Situation

A skateboard shop stocks a Tony Hawk Birdhouse Premium Complete Skateboard. The MSRP for this board is $82. If the shop is going to stock the board, it must order at least 25 units. The known information is as follows:

- The supplier offers a retail trade discount of 37% with a quantity discount of 12% for orders that exceed 15 units.
- Typical expenses associated with selling this premium skateboard amount to 31% of the

regular selling price.

- The skateboard shop wants each of its products to have profitability of 13% of the regular selling price.
- In the event that the skateboards do not sell by the end of the season, the skateboard shop always holds an end-of-season clearance sale where it marks down its products to the break-even selling price.

Using this information about the skateboard, determine the following:

a. Cost

b. Regular unit selling price

c. Markup on cost percentage

d. Markup on selling price percentage

e. Sale price

f. Markdown percentage

<u>Plan:</u>

In order, you are looking for C, S, $MoC\%$, $MoS\%$, S_{onsale}, and $d(markdown)$.

<u>Understand:</u>

Step 1: The list price, two discounts (note the store qualifies for both since it meets the minimum quantity criteria), expense and profit information, as well as information on the sale price are known:

$$L = \$82.00$$
$$d_1 = 0.37$$
$$d_2 = 0.12$$
$$E = 0.31S$$
$$P = 0.13S$$
$$S_{onsale} = \text{break-even price}$$

The unknown variables are C, S, $MoC\%$, $MoS\%$, S_{onsale}, and $d(markdown)$.

Step 2:

a. Cost and net price are synonymous. Net price appears in Formulas 4.1, 4.2b, and 4.3. Since you know list price and two discounts, calculate cost through Formula 4.3.

b. Selling price appears in Formulas 4.5, 4.7, 4.9, 4.10, 4.11a, and 4.11b. Known variables include expenses and profit. You will know the cost once you have solved part (a). Choose Formula 4.5.

c. Markup on cost percent appears only in Formula 4.8. You will know the cost once you have solved part (a). $M\$$ is an unknown variable.

d. Markup on selling price percent appears only in Formula 4.9. You will know the selling price once you have solved parts (a) and (b). $M\$$ is an unknown variable.

e. A break-even price uses an adapted Formula 4.5. You will know the cost once you have solved part (a) and determined the expenses based on the selling price in part (b).

f. Markdown percent appears in Formulas 4.10, 4.11a, and 4.12. You will know the regular price once you have solved part (b). You will know the sale price once you have solved part (e). Choose Formula 4.12.

Step 3:

From the above rationalization, the only unknown variables are cost and markup dollars. Once you know these two variables you can complete all other formulas. Calculate cost as described in step 2a. To calculate markup dollars, use Formulas 4.6, 4.7, 4.8, and 4.9. Once you know the cost and selling price from parts 2(a) and 2(b), you can apply Formula 4.7, rearranging for markup dollars.

<u>Perform:</u>

Calculator Instructions:

For part (d) only, use 2nd Profit

CST	SEL	MAR
45.46	81.18	Answer: 44

a.

$$C = N = \$82 \times (1 - 0.37) \times (1 - 0.12) = \$45.46$$

b.

$$S = \$45.46 + 0.31S + 0.13S$$
$$0.56S = \$45.46$$
$$S = \$81.18$$

c. As determined in step 3, you need markup dollars first.

$$\$81.18 = \$45.46 + M\$$$
$$\$35.72 = M\$$$

Now solve:

$$MoC\% = \tfrac{\$35.72}{\$45.46} \times 100 = 78.5746\%$$

d.

$$MoS\% = \tfrac{\$35.72}{\$81.18} \times 100 = 44\%$$

e.

$$S_{onsale} = S_{BE} = \$45.46 + 0.31(\$81.18) = \$70.63$$

f.

$$\$70.63 = \$81.18 \times (1 - d)$$
$$0.87 = 1 - d$$
$$0.13 = d$$

Present:

The cost of the skateboard is $45.46. The regular selling price for the skateboard is $81.18. This is a markup on cost percent of 78.5746% and a markup on selling price percent of 44%. To clear out the skateboards, the sale price is $70.63, representing a markdown of 13%.

Maintained Markup

A price change results in a markup change. Whenever a selling price is reduced to a sale price, it has been demonstrated that the amount of the markup decreases (from $M\$$ to $M\$_{onsale}$) by the amount of the discount (D). This means that some of the products are sold at full markup and some at a partial markup. For businesses, the strategic marking up of a product must factor in the reality that they do not sell every product at the same level of markup. Therefore, if a business must achieve a specified level of markup on its products, it must set its price such that the average of all markups realized equals its markup objective. The average markup that is realized in selling a product at different price points is known as the **maintained markup.**

To understand the formula, assume a company has 100 units for sale. From past experience, it knows that 90% of the products sell for full price and 10% sell at a sale price because of discounts, damaged goods, and so on. At full price, the product has a markup of $10, and 90 units are sold at this level, equalling a total markup of $900. On sale, the product is discounted $5, resulting in a markup of $5 with 10 units sold, equalling a total markup of $50. The combined total markup is $950. Therefore, the maintained markup is $950 ÷ 100 = $9.50 per unit. If the company's goal is to achieve a $10 maintained markup, this pricing structure will not accomplish that goal. Formula 4.13 summarizes the calculations involved with maintained markup.

Formula 4.13 – Maintained Markup

$$MM = \frac{M\$(n_1)+(M\$-D\$)(n_2)}{n_1+n_2}$$

MM is **Maintained Markup:** The maintained markup is the average markup in dollars that is realized after factoring in the quantity of units sold and differing levels of markup. In its simplest form, this formula is an applied version of Formula 2.4 on weighted average discussed in Chapter 2.

$M\$$ is **Markup Amount**: The markup amount represents the amount of markup realized at the regular selling price for the product. If you do not know this amount directly, calculate it using any of the markup formula (Formulas 4.6 through 4.9).

n_1 is **Full Markup Output**: This variable represents the number of units sold at full markup (the regular selling price). It could be an actual or forecasted number of units sold. This variable acts as a weight: If you do not know actual unit information, then you can instead assign a forecasted weight or percentage.

n_2 is **Partial Markup Output**: This variable represents the number of units sold at the reduced markup (the sale price). If you do not know unit information, use a forecasted weight or percentage instead.

$D\$$ is **Markdown Amount:** The markdown amount represents the amount by which the regular selling price has been reduced to arrive at the sale price. If you do not know this amount directly, calculate it using any of the markdown formulas (Formulas 4.11a, 4.11b, or 4.12).

Formula 4.13 assumes that only one price reduction applies, implying that there is only one regular price and one sale price. In the event that a product has more than one sale price, you can expand the formula to include more terms in the numerator. To expand the formula for each additional sale price required:

1. Extend the numerator by adding another $(M\$D\$)(n_x)$, where the $D\$$ always represents the total difference between the selling price and the corresponding sale price and n_x is the quantity of units sold at that level of markup.
2. Extend the denominator by adding another n_x as needed.

For example, if you had the regular selling price and two different sale prices, Formula 4.13 appears as:

$$MM = \frac{M\$(n_1)+(M\$-D\$_1)(n_2)+(M\$-D\$_2)(n_3)}{n_1+n_2+n_3}$$

How It Works

Follow these steps to calculate the maintained markup:

Step 1: Identify information about the merchandising scenario. You must either know markup dollars at each pricing level or identify all merchandising information such as costs, expenses, profits, and markdown information. Additionally, you must know the number of units sold at each price level. If unit information is not available, you will identify a weighting based on percentages.

Step 2: If step 1 already identifies the markup dollars and discount dollars, skip this step. Otherwise, you must use any combination of merchandising formulas (Formulas 4.1 to 4.12) to calculate markup dollars and discount dollars.

Step 3: Calculate maintained markup by applying Formula 4.13.

Give It Some Thought:

1. A company's pricing objective is to maintain a markup of $5. It knows that some of the products sell at regular price and some of the products sell at a sale price. When setting the regular selling price, will the markup amount be more than, equal to, or less than $5?

Example 4.4B – What Level of Markup Is Maintained?

A grocery store purchases a 10 kg bag of flour for $3.99 and resells it for $8.99. Occasionally the bag of flour is on sale for $6.99. For every 1,000 units sold, 850 are estimated to sell at the regular price and 150 at the sale price. What is the maintained markup on the flour?

Plan:

You are looking to calculate the maintained markup (MM).

Understand:

Step 1: The cost, selling price, and sale price information along with the units sold at each price are known:

$$C = \$3.99$$
$$S = \$8.99$$
$$S_{onsale} = \$6.99$$
$$n_1 = 850$$
$$n_2 = 150$$

Step 2: To calculate markup dollars, apply Formula 4.7, rearranging for M. To calculate markdown dollars, apply Formula 4.11b.

Step 3: Calculate maintained markup by applying Formula 4.13.

Perform:

Step 2:

Markup amount:

$$\$8.99 = \$3.99 + C$$
$$C = \$5.00$$

Markdown amount:

$$D\$ = \$8.99 - \$6.99 = \$2.00$$

Step 3:

$$MM = \frac{\$5(850) + (\$5 - \$2)(150)}{850 + 150} = \$4.70$$

Present:

On average, the grocery store will maintain a $4.70 markup on the flour.

Example 4.4C – Setting Prices to Achieve the Maintained Markup

Maplewood Golf Club has set a pricing objective to maintain a markup of $41.50 on all 18-hole golf and cart rentals. The cost of providing the golf and rental is $10. Every day, golfers before 3 p.m. pay full price while "twilight" golfers after 3 p.m. receive a discounted price equalling $30 off. Typically, 25% of the club's golfers are "twilight" golfers. What regular price and sale price should the club set to meet its pricing objective?

Plan:

The golf club needs to calculate the regular selling price (S) and the sale price (S_{onsale}).

Understand:

Step 1: The cost information along with the maintained markup, markdown amount, and weighting of customers in each price category are known:

$$C = \$10$$
$$MM = \$41.50$$
$$D\$ = \$30$$
$$n_1 = 100\% - 25\% = 75\%$$
$$n_2 = 25\%$$

Reverse steps 2 and 3 since you must solve Formula 4.13 first and use this information to calculate the selling price and sale price.

Step 3: Apply Formula 4.13, rearranging for $M\$$.

Step 2: Calculate the selling price by applying Formula 4.7.

Calculate the sale price by applying Formula 4.11b, rearranging for S_{onsale}.

Perform:

Step 3:

$$\$41.50 = \frac{M\$(0.75)+(M\$-30)(0.25)}{0.75+0.25}$$
$$\$41.50 = 0.75M\$ + 0.25M\$ - \$7.50$$
$$\$49 = M\$$$

Step 2:

$$S = \$10 + \$49 = \$59$$

$$\$30 = \$59 - S_{onsale}$$
$$S_{onsale} = \$29$$

Present:

In order to achieve its pricing objective, Maplewood Golf Club should set a regular price of $59 and a "twilight" sale price of $29.

The Pricing Impact of Coupons and Mail-in Rebates

Consumers sometimes need a nudge to help them decide on a purchase. Have you ever been thinking about buying a product but put it off for a variety of reasons? Then, while browsing online one day, you suddenly come across a coupon. You print it off and hurry down to your local store, because the coupon expires in the next few days. You just participated in a price adjustment.

A **marketing price adjustment** is any marketing activity executed by a member of the distribution channel for the purpose of altering a product's price. This includes an alteration to the regular unit selling price or a sale price. For example, Sony could distribute a rebate giving its consumers $50 off the price of one of its televisions. Alternatively, Superstore could run a special promotion where customers receive $30 off if they purchase at least $250 in merchandise.

When making price adjustments, businesses must understand the full impact on the bottom line. Many inexperienced businesses distribute coupons for $1 off, assuming that the only cost to be accounted for is the $1. This is untrue for most price adjustments.

These marketing price adjustments reduce the prices that you pay as a consumer. Certainly, some price adjustments are more advantageous than others. Although they come in many types, this section focuses on two of the most widely used price adjustments: manufacturer coupons and mail-in rebates.

Manufacturer Coupons

A **coupon** is a promotion that entitles a consumer to receive a certain benefit, most commonly in the form of a price reduction. Coupons are typically offered by manufacturers to consumers.

The decision of a manufacturer to issue a coupon affects the expenses associated with the product in three ways.

1. ***Coupon Redemption Expense.*** The **coupon redemption expense** is the dollar amount price reduction that the coupon offers, also known as the face value of the coupon. The manufacturer must pay out this amount to the reseller as a reimbursement for redeeming the coupon, increasing the expenses accordingly. For example, if a coupon is for $3 off, then the coupon redemption expense is $3.

2. ***Coupon Handling Expense.*** Most commonly, retailers are the resellers that redeem manufacturer coupons. The resulting **coupon handling expense** for the manufacturer is known as a handling fee or handling charge, payable to the retailer. The manufacturer offers financial compensation to the retailer for redeeming the coupon, filling out the paperwork, and submitting it to manufacturer for reimbursement. Ultimately, the retailer should not be out any money for participating in the promotion. For example, in the small print a coupon may indicate that there is a $0.09 handling fee. This amount is payable to the retailer upon the manufacturer's receipt of the redeemed coupon.

3. ***Coupon Marketing Expense***. Coupons do not just magically appear in consumer hands. Companies must pay for the creation, distribution, and redemption of the coupons, which is known as the **coupon marketing expense**:

   ◦ Coupons are designed, turned into print-ready artwork, and printed. This incurs design, labour, and material costs.
   ◦ Coupons are commonly distributed separately from the product in media such as magazines, flyers, and newspapers. This requires the purchase of ad space.
   ◦ Coupons are commonly redeemed through coupon clearinghouses that handle coupon redemption requests. Manufacturers must pay the fees charged by the clearinghouse.
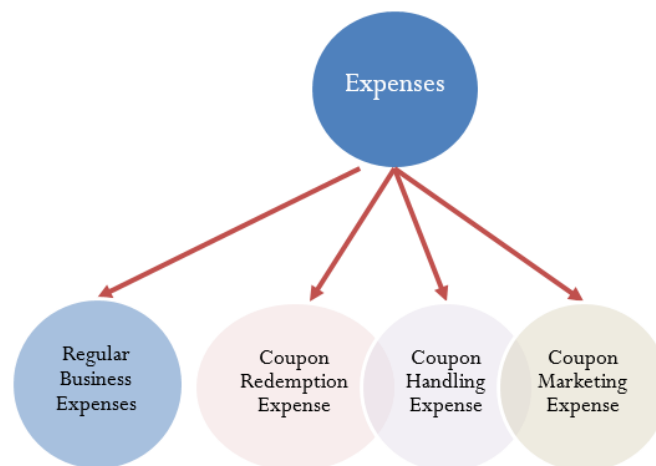
The manufacturer issuing the coupon estimates how many coupons it expects consumers will redeem and assigns the average of the coupon marketing costs to each unit based on the forecasted volume.

It should be evident that the decision to issue coupons results in expenses much higher than the face value of

the coupon. All three of the expenses listed above are added to the product's regular expenses. The end result is that the unit expenses increase while unit profits decrease.

As a consumer, you can appreciate the immediacy of the price reduction a coupon offers you at the cash register. Unlike with mail-in rebates, there is nothing further you need to do.

Mail-In Rebates

Have you ever purchased an item that came with a mail-in rebate but not submitted the rebate because it was too much hassle? Was your rebate refused because you failed to meet the stringent requirements? If you answered yes to either of these questions, then you have joined the ranks of many other consumers. In fact, marketers know you will either not mail in your rebates (some estimates are that 50% to 98% of mail-in rebates go unredeemed), or you will fail to send the required supporting materials. From a pricing point of view, this means that more product is sold at the regular price! That is why marketers use mail-in rebates.

Mail-in rebates are similar to coupons; however, there is a distinct difference in how the price adjustment works. A **mail-in rebate** is a refund at a later date after the purchase has already been made, whereas coupons instantly reward consumers with a price reduction at the cash register. Mail-in rebates commonly require consumers to fill in an information card as well as meet a set of stringent requirements (such as cutting out UPC codes and providing original receipts) that must be mailed in to the manufacturer at the consumer's expense (see the photo for an example). After a handling time of perhaps 8 to 12 weeks and if the consumer has met all requirements, the manufacturer issues a cheque in the amount of the rebate and sends it to the consumer.

With coupons there are three associated expenses: coupon redemption, handling, and marketing. In the case of rebates, since the participants in the channel are not involved with the rebate redemption, no handling expenses are paid out to the retailers.

1. *Rebate Redemption Expense.* The **rebate redemption expense** is the dollar amount price reduction that the rebate offers, also known as the face value of the rebate. This amount must be paid out to the consumer, commonly in the form of a cheque. For example, if a rebate is for $50, then this is the amount sent to the consumer. This increases expenses, but often not by the full $50. Recall that though many consumers purchase the product because the rebate is offered, not all of them redeem the rebate. To calculate the unit expense many companies average the rebate redemption amount based on the number of rebates they expect to fulfill. Take, for example, a company that issues a $50 rebate on the purchase of a surround sound system. From past experience, it knows that only 5% of the rebates are actually redeemed properly and thus paid out. Therefore, the increase in expenses is not $50, but 5% of $50, or $2.50 per unit sold under the rebate program.

2. ***Rebate Marketing Expense.*** Many mail-in rebates today are distributed either on or in the packaging of the product, or they can be located online. This tends to minimize (but not eliminate) the marketing expenses associated with creation, distribution, and redemption. The **rebate marketing expense** is commonly averaged across the increase in sales that results from the rebate issuance and not the number or rebates redeemed. Therefore, if a company has $100,000 in rebate marketing expenses, a direct increase in sales by 20,000 units, and only 1,000 rebates redeemed, the rebate marketing expense is $100,000 ÷ 20,000 = $5.00 per unit under the rebate program (note that the rebates redeemed is irrelevant to this calculation).
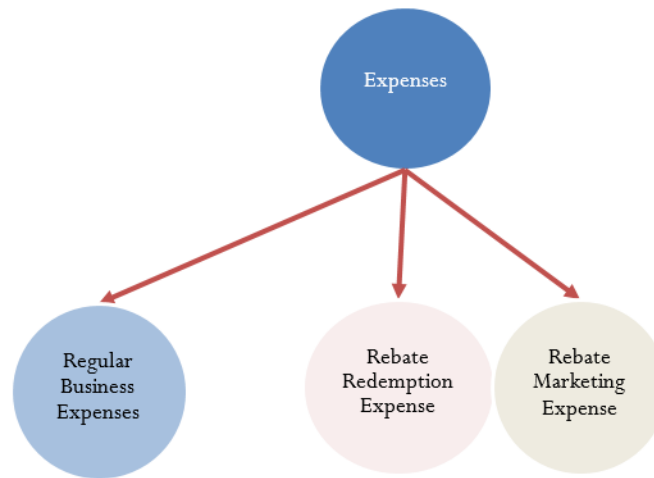
With coupons and rebates, the focus is on how these marketing tools affect expenses and profit. You do not require any new formula; instead, Formula 4.5 is adapted to reflect the pricing impact of these tactics.

Formula 4.5: The Selling Price of a Product Under a Coupon or Rebate:

$$S = C + E(adjusted) + P_{onsale}$$

S is **Selling Price:** The manufacturer's regular selling price of the product is unchanged for both rebates and coupons. Mail-in rebates do not change the manufacturer's selling price to the wholesaler. Coupons lower the retail price and not the manufacturer's price.

C is **Cost:** The cost of the product to the manufacturer remains constant when marketers use coupons and rebates.

$E(adjusted)$ is **Expenses:** When coupons and mail-in rebates are used, the manufacturer not only incurs regular operating expenses but must adjust this number upward to reflect the redemption, handling, and marketing expenses associated with these price adjustments. Therefore, products sold under a coupon program or rebate program always have higher expenses.

P_{onsale} is **Profit:** If the price and cost remain unchanged while the expenses have increased, then the profit per unit must be reduced. Using a coupon or rebate is essentially another form of putting the product on sale. You know that if a product is on sale, then the profits are reduced. The amount of this reduction will be equal to the amount by which the expenses increase.

How It Works

Follow these steps to calculate the selling price or any of its components when working with coupons or rebates:

Step 1: Identify the known components involved in the selling price, such as the cost, unit expense, profit, or selling price. Use any formulas needed to calculate any of these components. Also identify information about the coupon or rebate, including redemption, handling, and marketing expenses.

Step 2: Calculate the unit expense associated with the marketing pricing tactic.

- **Coupons.** The unit redemption expense is equal to the face value of the coupon. The unit handling expense is the promised handling fee of the coupon. The unit marketing expense is a simple average of the total marketing expense divided by the expected number of coupon redemptions. Calculate the total coupon expenses by summing all three elements.
- **Rebates.** The unit redemption expense is equal to the face value of the coupon multiplied by the redemption rate of the rebate. The unit marketing expense is equal to a simple average of the total marketing expense divided by the expected increase in sales as a result of the rebate. Calculate the total rebate expenses by summing both elements.

Step 3: Apply the required merchandising formulas to calculate the unknown variable(s). The modified Formula 4.5 presented earlier will most likely play prominently in these calculations.

Quick Coupon Example

Assume a manufacturer coupon is issued with a face value of $5. The handling charge is $0.15 per coupon. The marketing expenses are estimated at $150,000, and the company forecasts that 100,000 coupons will be redeemed. The regular profit on the product is $20. How much will the unit expenses of the product increase, and what is the unit profit under the coupon program?

Step 1: You know the following information:

$$P = \$20$$
$$\text{Face Value} = \$5$$
$$\text{Handling Charge} = \$0.15$$
$$\text{Total Marketing Expenses} = \$150,000$$
$$\text{Forecasted Coupon Redemption} = 100,000.$$

Step 2: The increase in expenses is a sum of the redemption, handling, and marketing expenses. The redemption expense is $5. The handling expense is $0.15. To get the marketing expense per unit, take the total expense and divide it by the number of redemptions expected. Therefore $\$150,000 \div 100,000 = \$1.50$.

The total increase in expenses is then $\$5.00 + \$0.15 + \$1.50 = \$6.65$.

Step 3: If the expenses rise by $6.65 with the same cost and selling price, then the profits fall by the same amount.

The unit profit on any product sold using the coupon is $\$20.00 - \$6.65 = \$13.35$.

Quick Rebate Example

Assume a manufacturer mail-in rebate is issued with a face value of $20. The marketing expenses are estimated at $300,000 and the company plans on 10% of the mail-in rebates being redeemed. It is forecasted that sales will rise by 37,500 units as a result of the mail-in rebate. The regular profit on the product is $25. How much will the unit expenses of the product increase, and what is the unit profit under the rebate program?

Step 1: You know the following information:

$$P = \$25$$
$$\text{Face Value} = \$20$$
$$\text{Total Marketing Expenses} = \$300,000$$
$$\text{Rebate Redemption Rate} = 10\%$$
$$\text{Increase in Sales} = 37,500.$$

Step 2: The increase in expenses is a sum of the redemption and marketing expenses. The redemption expense is $20.00 multiplied by the 10% redemption rate, or $2. To calculate the marketing expense per unit, take the total expense and divide it by the increase in sales. Therefore $\$300,000 \div 37.500 = \$8$.

The total increase in expenses is then $\$2 + \$8 = \$10$.

Step 3: If the expenses rise by $10 with the same cost and selling price, then the profits fall by the same amount. The unit profit on any product sold using the coupon is $\$25 - \$10 = \$15$.

When calculating the per-unit expense, be careful to distinguish how to handle the various associated expenses for both manufacturer coupons and mail-in rebates, as summarized in the table below.

Marketing Tactic	Redemption Expense	Handling Expense	Marketing Expense
Coupon	Face value of the coupon	A flat rate per coupon	The total expense divided by the number of coupons redeemed
Mail-In Rebate	Face value of the mail-in rebate multiplied by the redemption rate	N/A	The total expense divided by the increase in sales attributed to the mail-in rebate

It pays to be smart when you deal with coupons and mail-in rebates as a consumer. Two important considerations are timing and hassle.

1. **Timing.** Coupon amounts are usually smaller than rebate amounts. You may be offered a $35 coupon (sometimes called an "instant rebate," but should not be confused with rebates) at the cash register, or a $50 mail-in rebate. If you choose the former, you get the price savings immediately. If you choose the latter, you must spend the money up front and then wait 8 to 12 weeks (or more) to receive your rebate cheque.

2. **Hassle.** Mail-in rebates are subject to very strict rules for redemption. Most require original proofs of

purchase such as your original cash register receipts, UPC codes, model numbers, and serial numbers (which are difficult to find sometimes). You must photocopy these for your records in the event there is a rebate problem. Specific dates of purchase and deadlines for redemption must be met. Some rebates, particularly computer software upgrades, even require you to send your old CD-ROM in with your purchase, substantially raising your postage and envelope costs. Does all that extra time and cost justify the rebate? For example, Benylin ran a sales promotion offering an instant coupon for $1; alternatively you could purchase two products and mail-in two UPC codes for a $3 rebate. When factoring in costs for postage, envelope, photocopying, and time, the benefit from the mail-in rebate is marginal at best.

Give It Some Thought:

2. What happens to the unit profit when a coupon or rebate is issued under the following scenarios?
a. The cost and selling price remain unchanged.
b. The cost increases and the selling price remains unchanged.
c. The cost remains the same and the selling price is reduced.
d. The cost remains the same and the selling price increases.

Example 4.4D – The Impact of a Coupon

When the DVD edition of the *Bee Movie* was released in retail stores, DreamWorks Animation executed a coupon marketing program. Assume the unit cost of the DVD is $2.50, the normal expenses associated with the DVD amount to $1.25, and that DreamWorks sells it directly to retailers for $10. The coupon offered a $3 savings on the DVD to consumers and promised the retailer an $0.08 handling fee per coupon redeemed. The estimated costs of creating, distributing, and redeeming the coupons are $285,000, and the projected number of coupons redeemed is 300,000.

a. What is the unit profit of the DVD when it sold without the coupon?

b. What unit profit did DreamWorks realize for those DVDs sold under its coupon promotion?

Plan:

You need to first calculate the profit (P) when no coupon was available. Second, you will calculate the impact of the coupon on the profit when the DVD is discounted under the coupon program (P_{onsale}).

Understand:

Step 1: The unit cost of the DVD along with its normal associated expenses and regular unit selling price are known. The coupon details are:

$$C = \$2.50$$
$$E = \$1.25$$
$$S = \$10.00$$

Face Value = $3

Handling Fee = $0.08

Total Marketing Expense = $285,000

Number of Redeemed Coupons = 300,000

Step 2: When the coupon is issued, the expenses associated with the DVD rise to accommodate the coupon redemption expense, the handling expense, and the marketing expenses.

Step 3: Calculate the profit normally realized on the DVD by applying Formula 4.5, rearranging for P.

Recalculate the profit under the coupon program by adapting Formula 4.5, rearranging for P_{onsale}.

Perform:

Step 2:

Unit coupon expense = $\$3 + \$0.08 + (\$285,000 \div 300,000) = \$4.03$

Step 3:

$$\$10.00 = \$2.50 + \$1.25 + P$$
$$\$6.25 = P$$

$$\$10.00 = \$2.50 + (\$1.25 + \$4.03) + P_{onsale}$$
$$\$10.00 = \$7.78 + P_{onsale}$$
$$\$2.22 = P_{onsale}$$

Present:

Without the coupon promotion a *Bee Movie* DVD generates $6.25 in profit. Under the coupon promotion, DreamWorks achieves a reduced profit per DVD of $2.22.

Example 4.4E: The Impact of a Mail-In Rebate

Computer manufacturers commonly issue rebates on items such as printers and monitors. Assume LG decided to distribute its new 22" widescreen LCD monitor (MSRP: $329.99) with a prepackaged mail-in rebate included inside the box. The rebate is $75. The company predicts a 25% rebate redemption rate. Its unit cost to produce the monitor is $133.75, with expenses equalling 35% of cost. It sells the monitor directly to retailers for a regular unit selling price of $225. The increased marketing expenses for promotion and rebate redemption equal $8.20 per monitor. What is the profit for a monitor sold through the rebate program?

Plan:
You are looking for the profit (P_{onsale}) for a monitor sold under the rebate program. This profit must factor in the rebate face value, redemption rate, and marketing expenses.

Understand:
Step 1: The unit cost, unit expenses, and regular unit selling price are known. The rebate details are:

$$C = \$133.75$$
$$E = 0.35C$$
$$S = \$225.00$$
Face Value = $75
Redemption rate = 25%
Unit Marketing Expenses = $8.20

Step 2: When the rebate is issued, the expenses associated with the monitor rise to accommodate the rebate redemption expenses and the marketing expenses.

Step 3: Apply an adjusted Formula 4.5, rearranging for P_{onsale}.

Perform:

Step 2: Unit rebate expense = $\$75 \times 25\% + \$8.20 = \$18.75 + \$8.20 = \$26.95$

Step 3:

$$\$225.00 = \$133.75 + (\$46.81 + \$26.95) + P_{onsale}$$
$$\$225.00 = \$207.51 + P_{onsale}$$
$$\$17.49 = P_{onsale}$$

Present:

The rebate redemption expense for the LG monitor is $18.75. LG averages a profit of $17.49 per monitor sold under its mail-in rebate program.

Give It Some Thought Answers

1. More than $5. In order to average to $5, the markup at regular price must be higher than $5 and the markup at sale price must be lower than $5.

2. a. Profit decreases by the total unit expenses associated with the coupon or rebate.

b. Profit decreases by the sum of the total unit expenses associated with the coupon or rebate and the amount of the increased cost together.

c. Profit decreases by the sum of the total unit expenses associated with the coupon or rebate and increases by the amount of the price increase.

d. Profit decreases by the total unit expenses associated with the coupon or rebate and the amount of the increased cost together.

Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

	Regular Unit Selling Price	Cost	Unit Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price	Markdown Amount	Markdown Percent	Sale Price
1.	$100.00	?	24% of C	?	?	?	100%	50%	?	33%	?
2.	?	?	?	$20.00	$30.00	?	?	?	$13.00	?	$37.00
3.	?	$56.25	?	10% of S	?	?	33.3333%	?	?	?	$64.99
4.	$99.99	$43.00	?	?	?	$78.50	?	13%	?	10%	?

For questions 5–8, solve for the unknown variables (identified with a ?) based on the information provided. N/A indicates that the particular variable is not applicable in the question.

	Regular Unit Selling Price	Unit Cost	Regular Unit Expenses	Coupon or Rebate Handling Expense per Unit	Coupon or Rebate Redemption Expense per Unit	Coupon or Rebate Marketing Expense per Unit	Profit w/o Coupon or Rebate per Unit	Profit with Coupon or Rebate per Unit
5.	$9.85	$4.73	35% of C	$0.25	$2.00	$0.75	?	?
6.	$47.65	$23.50	15% of S	N/A	$10.00	$3.50	?	?
7.	$6.30	?	$0.95	$0.45	?	$1.15	$3.05	$0.45
8.	?	$38.25	40% of C	N/A	?	$2.25	$21.40	$4.15

Applications

9. A box of glossy inkjet photo paper costs $9.50. The markup on cost is 110%. The profit is 31.5% of the cost. The paper occasionally goes on sale for 20% off.
a. What are the regular unit selling price, expenses, profit, markup amount, and markup on selling price percentage?
b. When the paper goes on sale, what are the sale price, markdown amount, markup dollars, and markup on selling price percentage?

10. John's Bakery is thinking of distributing a $0.50 coupon for one of its least popular brands of bread in hopes of increasing its sales. The bread costs $0.43 to produce, and expenses are 10% of the regular unit selling price. The bread retails for $1.99 per loaf. John's Bakery estimates that the coupon marketing expenses are $0.25 per loaf sold, and coupon handling expenses would be an additional $0.10 per loaf sold.

a. What is John's Bakery profit when the loaf is sold at the regular unit selling price without the coupon?

b. If John's Bakery uses the coupon, what is the profit?

c. If the bread still doesn't sell well, John's Bakery will clear it out while still allowing consumers to use the coupon. What is the break-even selling price for a loaf in this situation?

11. An airline needs to maintain an average markup of $230 on a daily return flight from Toronto to Vancouver. The Boeing 737-800 has a capacity of 166 passengers, and the cost per passenger is $175. Typically 90% of tickets are sold at full fare while 10% are sold on standby at a sale price equalling $200 off the regular price. What regular and sale prices should the airline set to maintain the average markup?

12. A mail-in rebate form is printed on the side of every High Liner box of breaded fish. It allows the consumer to mail in the UPC along with the form and receive a $2 rebate. If a 700 g box of fish is normally sold to grocery stores for $3.50, costs are 40% of the regular unit selling price, and expenses are 20% of cost, what is the expected profit under the rebate program? Assume that only 25% of rebates are actually redeemed and that rebate marketing expenses are $0.15 per box.

13. Lay's Potato Chips is thinking of distributing a coupon on its 454 g bag of regular crinkled chips. Currently, the cost of producing a bag of chips is $0.49, with typical expenses of $0.23 per bag. Lay's sells the bag to retailers for $1.75 per bag. The coupon is in the amount of $0.50, while retailers are offered $0.12 for handling expenses. Lay's forecasts an increase in sales of 300,000 bags with the coupon, incurring total marketing expenses of $100,000. Will running this coupon promotion be profitable for Lay's? What is its profit (or loss) per unit if Lay's goes forward with the coupon?

14. LG is thinking of placing a $30 mail-in rebate on its 19" widescreen computer monitor. The cost to produce a monitor is $67.40, with typical expenses amounting to 50% of the cost. The monitor is sold to retailers for $119.00. Based on previous rebate programs, LG knows that only 40% of those who purchased a monitor with a rebate will actually redeem it. The marketing expense associated with the rebate totals $350,000, resulting in increased sales of 50,000 monitors. Is the rebate program profitable for LG? What is its profit (or loss) per monitor if it proceeds with the rebate offer?

15. A car dealership acquired a used Ford Mustang for $15,000. Its expenses on used vehicles are 15% of the cost, and it prices the vehicle with a 30% markup on cost percentage. After two months, if the car doesn't sell, it marks the car down by 10%.

a. What are the expenses?

b. What is the dollar amount of the markup?

c. What is the regular selling price?

d. What is the markup on selling price percentage?

e. What is the sale price?

f. What is the amount of the markdown?

g. What is the profit earned if the car is sold at the sale price?

h. What is the maintained markup supposing the dealership was able to acquire eight used Ford Mustangs at the same cost, where five cars are sold at regular price and three are sold at the sale price?

Challenge, Critical Thinking, & Other Applications

16. Target is trying to determine whether it would be more profitable to use a coupon or a mail-in rebate when promoting a new supersoaker water gun. The toy costs $14.94 and retails for $29.95. Expenses are 30% of cost. The coupon is for $4, handling expenses are $0.50 per unit, and the marketing expenses are $1 per unit. The rebate is for $7.75, with 45% of the rebate forms expected to be redeemed. Rebate marketing expenses are $1.25 per unit.

a. If Target forecasts that sales will increase 10,000 units under the coupon program and 8,000 units under the rebate program, should it use the coupon or the mail-in rebate?

b. How much more profit is earned based on your recommended option?

17. A hair salon has limited shelf space to carry shampoo products. The store's pricing policies have a markup on cost of 220%, and expenses equal 20% of cost. Two suppliers have approached the salon with the following offers:

Supplier #1: The shampoo has a list price of $29 with discounts of 60% and 20%. Over a six-month period, 150 bottles are expected to be sold at full price and 50 bottles at a 25% markdown during a special promotion.

Supplier #2: The shampoo has a list price of $37 with discounts of 70% and 10%. Over a six-month period, 120 bottles are expected to be sold at full price and 75 bottles at a 20% markdown during a special promotion.

Based on this information, recommend which supplier the salon should purchase the shampoo from. Show calculations to support your answer.

18. The Home Depot purchases garden sheds for $1,000 less discounts of 40% and 20% and then sells them at a 100% markup on cost percentage. Seasonally, these sheds sell for full price initially followed by a reduced price and then are blown out at 50% off the regular price. Past experience

indicates that 60% of the sheds sell at full price, 30% at the reduced price, and 10% at the blowout price. What should the reduced price be set at to achieve a maintained markup of $360?

19. CCM sells ice skates to Sport Chek for $179.99 less a trade discount of 40% and a quantity discount of 15%. The ice skates cost $27.60 to manufacture, and associated expenses are 35% of the regular unit selling price. CCM runs a mail-in rebate program on the ice skates where consumers receive $50 on a pair of skates. CCM estimates that 40% of purchasers will redeem the rebate. Rebate marketing expenses are estimated at $6 per pair sold. What is CCM's profitability under the rebate program?

20. Canadian Tire purchases a camping tent from its supplier at an MSRP of $79.99 less discounts of 40% and 25%. It plans to sell the tent at the MSRP, and its expenses are 20% of the selling price. At the end of the season, remaining tents are sold at 30% off.
a.What are the expenses?
b. What is the profit?
c. What is the dollar amount of the markup?
d. What is the markup on selling price percentage?
e. What is the markup on cost percentage?
f. What is the sale price?
g. What is the amount of the markdown?
h. What is the profit earned when tents are sold at the sale price?
i. What is the maintained markup if 90% of tents are sold at the regular price and 10% of tents are sold at the sale price?

The Formulas You Need to Know

Symbols Used

C = cost

d = discount rate or markdown rate

$d_1, d_2, ... d_n$ = multiple discount rates, where the subscript represents each discount up to a count of n discounts

$D\$$ = discount amount or markdown amount

Key Takeaways

Section 4.1: Figuring Out the Cost: Discounts (How Much?)

- The relationship between distribution and pricing
- Some of the types of

discounts available to businesses and consumers
- How to calculate the net price when only one discount is involved
- How to calculate the net price when multiple discounts are involved
- Converting multiple discounts into single discounts

Section 4.2: Markup: Setting the Regular Price (Need to Stay in Business)

- The three components that compose a selling price
- Calculating the markup in dollars and the relationship to pricing components
- Calculating the markup as a percentage under two different approaches
- Determining the price point where all costs are paid but no profits are earned—the break-even point

Section 4.3: Markdown: Setting the Sale Price (Everybody Loves a Sale)

- How to take a regular selling price and make it into a sale price by applying

E = expenses

L = list price

$M\$$ = markup amount

MM = maintained markup

$MoC\%$ = markup on cost percentage

$MoS\%$ = markup on selling price percentage

n = number of pieces of data, which in this chapter is the level of output

N = net price

P = profit

P_{onsale} = planned profit amount when a product is sold at the sale price

S = regular selling price

S_{BE} = the selling price at the break-even point

S_{onsale} = sale price

Formulas Introduced

Formula 4.1 Single Discount: $N = L \times (1 - d)$ (Section 4.1)

Formula 4.2a Discount Amount: $D\$ = L \times d$ (Section 4.1)

Formula 4.2b Discount Amount: $D\$ = L - N$ (Section 4.1)

Formula 4.3 Multiple Discounts: $N = L \times (1 - d_1) \times (1 - d_2) \times \cdots \times (1 - d_n)$ (Section 4.1)

Formula 4.4 Single Equivalent Discount: $d_{equiv} = 1 - (1 - d_1) \times (1 - d_2) \times \cdots \times (1 - d_n)$ (Section 4.1)

Markup Formulas

Formula 4.5 The Selling Price of a Product: $S = C + E + P$ (Section 4.2)

Formula 4.6 Markup Amount: $M\$ = E + P$ (Section 4.2)

Formula 4.7 The Selling Price of a Product Using the Markup Amount: $S = C + M\$$ (Section 4.2)

Formula 4.8 Markup on Cost Percentage: $MoC\% = \frac{M\$}{C} \times 100$ (Section 4.2)

Formula 4.9 Markup on Selling Price Percentage: $MoS\% = \frac{M\$}{S} \times 100$ (Section 4.2)

Formula 4.13 Maintained Markup: $MM = \frac{M\$(n_1) + (M\$ - D\$)(n_2)}{n_1 + n_2}$ (Section 4.4)

Markdown Formulas

Formula 4.10 The Sale Price of a Product: $S_{onsale} = S \times (1 - d)$ (Section 4.3)

Formula 4.11a Markdown Amount: $D\$ = S \times d$ (Section 4.3)

Formula 4.11b Markdown Amount: $D\$ = S - S_{onsale}$ (Section 4.3)

Formula 4.12 Markdown Percentage: $d = \frac{D\$}{S} \times 100$ (Section 4.3)

markdowns
- How to plan the merchandising of products that are always on sale
- Section 4.4: Merchandising (How Does It All Come Together?)
- Combining discounts, markup, and markdown into complete merchandising situations
- Calculating the maintained markup when a product was sold at both a regular selling price and a sale price
- How coupons and mail-in rebates impact product pricing

Section 4.4: Merchandising (How Does It All Come Together?)

- Combining discounts, markup, and markdown into complete merchandising situations
- Calculating the maintained markup when a product was sold at both a regular selling price and a sale price
- How coupons and mail-in rebates impact product pricing

Technology

Calculator

The following calculator functions were introduced in this chapter:

Markup on Selling Price Percentage

2nd Profit to access this feature.

Enter two of the three variables by pressing Enter after each input and using ↑ and ↓ to scroll through the display. The variables are:

CST = The cost of the item

SEL = The selling price of the item

MAR = The markup on selling price percentage (in % format)

Press CPT on the unknown (when it is on the screen display) to compute.

Chapter 4 Review Exercises

Round all money to two decimals and percentages to four decimals for each of the following exercises.

Mechanics

1. The Gap is purchasing an Eminem graphic T-shirt for its stores. If the list price for the T-shirt is $34.50 and the Gap can receive a 40% trade discount, what net price will the Gap pay?

2. Nike has been approached by a retailer who wants to carry their Shox brand. The retailer needs a trade discount of 30% and its wholesaler needs 15%. The retailer also wants a 5% quantity discount. If the MSRP for the shoes is $195.00, what is the wholesale (net) price that Nike will charge?

3. If the regular price of a product is $775.00 and it is eligible to receive discounts of 14%, 8%, and 5%,

what single discount percentage is equal to the multiple discount percentages? Show calculations that show how the single and multiple discounts produce the same net price.

4. Subway wants to price a new 12 inch sandwich. The cost of all the ingredients is $1.23. Expenses are 200% of cost, and Subway wants to earn a profit of 187% of cost. What is the regular selling price of the new sandwich?

5. If the cost of a Bose Wave music system is $27.82 and the list price is $49.99, determine the markup amount. Express the markup both as a percentage of cost and percentage of selling price.

6. Rogers Communications is thinking about having a special promotion on its Digital One Rate 250 Plan, which offers 250 minutes of cellular calling. If the current price is $85.00 per month and Rogers wants to put it on special for $75.00 per month, what is the percentage markdown?

7. If a $1 coupon is distributed for a product that normally has a profit of $4 per unit, how much is the profitability reduced if coupon handling expenses are $0.30 per unit and coupon marketing expenses are $1.10 per unit? What is the profit under the coupon program?

8. At a local retailer, a swimsuit costs $10 and sells for $30. At the end of the season, the swimsuit price is reduced by 30%. If 820 swimsuits are sold at the regular price and 145 swimsuits are sold at the sale price, what is the maintained markup?

Applications

9. An iMac has an MSRP of $774.99 with available trade discounts of 20% to the retailer and an additional 15% to the wholesaler.
a. What price should a retailer pay for the product? What is the dollar value of the discount?
b. What price should a wholesaler pay for the product? How much less than the retailer's price is this?
c. Overall, what single discount percent does the wholesaler receive?

10. The marketing manager for Tim Hortons is attempting to price a cup of coffee. She knows that the cost of the coffee is $0.14 per cup, and expenses are 30% of the regular selling price. She would like the coffee to achieve an 88.24% markup on selling price.
a. What is the regular selling price for a cup of coffee?
b. What is the profit per cup?
c. What is the markup on cost percentage?

11. A grocery store head office is thinking of distributing a $3 in-store coupon for one of its 454 g bags of coffee. The cost of the coffee to one of the franchised retail stores is $6.34, and it is usually sold to consumers for $12.99. A retail store incurs expenses of 25% of the regular selling price to stock,

shelve, and sell the coffee. The franchisees are reimbursed $0.25 per coupon redeemed for handling expenses, and the head office estimates that it incurs marketing expenses amounting to $1.35 per coupon. Based strictly on the profit associated with the sale of this coffee, should the head office proceed with this promotion?

12. Jonathan needs a new business suit for a presentation in his communications course. He heads to Moores and finds a suit on sale for 33% off. The regular price of the suit is $199.50.
a. What is the sale price?
b. What is the dollar amount of the markdown?

13. Birchwood Honda needs to clear out its Honda generators for an end-of-season sale. If the dealership pays $319.00, has expenses of 15% of cost, and profits of 30% of the regular unit selling price, what markdown percentage can the dealership advertise if it wants to break even during the sale?

14. eMachines has decided to issue a $50 mail-in rebate on one of its best-selling desktop computers. Retailers purchase the machines directly from eMachines for a list price of $449.00 less a 40% trade discount. eMachines has profits of 35% of its regular unit selling price and expenses of 20% of its regular unit selling price.
a. What is the profit per computer during the rebate, assuming rebate marketing expenses of $13.50 per unit and an expected redemption rate of 35%?
b. How much lower is the profit with the rebate than without the rebate?
c. If 2,500 computers are sold at regular price and 1,025 computers are sold through the rebate program, what is the maintained markup?

Challenge, Critical Thinking, & Other Applications

15. A retailer purchasing the Halo 3 video game is eligible to receive a trade discount, quantity discount, and loyalty discount. If the net price of the game is $22.94, the list price is $39.99, the trade discount is 25%, and the quantity discount is 15%, what percentage is the loyalty discount?

16. An article in a business paper on October 22, 2008, reported that the benchmark S&P/TSX Composite Index ended down 558.92 points for the day, or 5.7%. It went on further to state that the index is down 30% for the year 2008. Rounding all point values to two decimals, answer the following:
a. What were the point values on October 21 and 22, respectively?
b. What was the point value at the start of the year 2008?

17. Whistler Blackcomb Ski Resort purchased 400 pairs of skis for sale in its retail store. It sells the skis at an MSRP of $299.99. It is eligible to receive discounts of 35% and 20% on its purchase.

Expenses average 15% of cost. If it sells 300 pairs of skis at the regular unit selling price and the other 100 skis are sold during a clearance sale for 20% off, what is the total amount of profit that Whistler Blackcomb earns? What is the maintained markup?

18. A company is examining its options for selling a product listed at $359.97. Both expenses and profits are 20% and 22% of the regular unit selling price, respectively. At this price, the company forecasts sales of 6,000 units. Consider the following two scenarios:
Scenario A: If the company holds a sale and marks down the product by 10%, its market research predicts that sales will rise to 11,225 units.
Scenario B: If the company issues a $60.00 mail-in rebate, it will have increased marketing expenses of $6.35 per unit. Only 45% of rebates are expected to be redeemed. Market research predicts that sales will rise to 10,500 units.
Should the company leave the product at its regular price, conduct the sale, or issue the mail-in rebate?

19. Indigo is stocking a new hardcover book. The book has a cover price of $44.99 and Indigo is eligible to receive discounts of 25%, 8%, 4%, and 1%. Expenses are 20% of cost. The book sells for the cover price.
a. What is the profit per book?
b. Some new books are launched at 30% off the cover price. From a strictly financial perspective, would it be wise for Indigo to use this approach for this book? Why or why not?

20. Mary is shopping for a new George Foreman grill. While shopping at Polo Park Shopping Centre, she came across the following offers at three different retail stores:
Offer #1: Regular price $149.99, on sale for 30% off.
Offer #2: Regular price $169.99, on sale for 15% off plus an additional 30% off.
Offer #3: Regular price $144.95, on sale for 10% off plus an additional 15% off. Mary also gets a 5% loyalty discount at this store.
Which is the best offer for Mary?

Chapter 4 Case Study: Making a Profitable Product Selection

The Situation

Lightning Wholesale is looking at changing its ski product line for 2014. It wants to carry only two brands of skis. It has received four offers from four different ski manufacturers. The details of the offers are in the first table. It will select the two brands that offer the highest total profit. The accounting department has analyzed 2013 sales and provided the income statement summarized in the second table, which specifies the percentage of each category that is attributed to the sporting goods department.

The Data

	Nordica	Fischer	Ogasaka	Atomic
List Price:	$799.95	$914.99	$829.95	$839.99
Discounts:				
Retail	40%	40%	40%	40%
Wholesale	5%	11%	6%	7%
Seasonal	3%	5%		
Sale	5%		5%	10%
Loyalty	3%		5%	
Quantity	3%	4%	5%	2%

Lightning Wholesale Income Statement

	Year (2013)	Allocation of Dollars to Sporting Goods
Sales Revenue:	**$57,77**	
	7	20%
Cost of Goods Sold:	**$50,44**	
	5	19%
Operating Expenses:		
Sales and Marketing	$3,234	18%
General and Administrative	$348	14%
Payroll	$1,416	19%
Total Operating Expenses:	**$4,998**	
Operating Income:	**$2,334**	

*All dollar figures are in thousands

Important Information

- All of Lightning Wholesale's retail customers plan to sell the skis at the list price.
- Lightning Wholesale prices all of its skis on the assumption that all of its retailers use the standard industry markup of 40% of the regular selling price.
- Skis fall into the sporting goods category for Lightning Wholesale.
- Expenses are assigned to products based on the percentage of cost method for each category.
- Forecasted sales for each brand (if carried by Lightning Wholesale) in 2014 are as follows:
  Nordica = 380 units
  Fischer = 270 units
  Ogasaka = 310 units
  Atomic = 300 units

Your Tasks

1. Based strictly on financial considerations, recommend which two brands Lightning Wholesale should carry in 2014.
   a. Using the appropriate table, calculate the net price (cost) of each brand of skis.
   b. To determine the expenses for each brand of skis:

Using the appropriate table, calculate the dollar values assigned to sporting goods for each row by multiplying the company total by the allocation percentage.

Total the dollar amount of the expenses.

Convert the total expenses into a percentage of cost.

Calculate the expenses for each brand using the percentage of cost.

c. Using the appropriate table, calculate the wholesale price (the amount at which Lightning Wholesale will sell the skis to a retailer).

d. For each brand, calculate the profit per unit.

Calculate the total profitability for each brand.

2. At the end of the year, if any inventories are left over, Lightning Wholesale clears out all ski products at break-even.

a. Determine the break-even prices for the two recommended brands.

b. Determine the markdown percentage that can be advertised for each recommended brand.

Chapter 4: Marketing Applications (answers to section exercises)

- ## 4.1 Discounts

1. D$ = $411.60; N = $568.40
2. L = $800; D$ = $200
3. d = 42.625%; N = $1,133.16; D$ = $841.84
4. d = 26.7904%; L=$500; D$ =$133.95
5. $259.97
6. $24.32
7. 31.985%
8. $19.99
9. d = 43.972%; L = $51.76
10. L =\ 24; $N = 13.92$¡/li¿ ¡li class=”import-Normal”¿$d = 62.8$ = $439.60
11. d   =   26%;   N   = $303.99;   D$ =

96 $< /li >< liclass = "import − Normal" >$1,026¡/li¿ ¡li class="import-Normal"¿d $= 63.14$¡li class="import-Normal"¿$1, 194, 553.06 < /li >< liclass = "import − Normal" > 2years216days < /li >< liclass = "import − Normal" > Reducedby17.4154 < liclass = "import − Normal" > a.3 < liclass = "import−Normal" > a.52.2328 < liclass = "import−Normal" > a.74.99;$ b. $60.91; c.L =78.74;$ N $= 63.96; D =14.78 < /li >< /ol >< ul >< li >< h3 >< strong > 4 < /strong >< strong > .2 < /strong >< strong > Markup < /strong >< /h3 >< /li >< /ul >< ol >< liclass = "import−Normal" > S =321.95;$ SBE $=236.95; M=133.53; MoC < liclass = "import−Normal" > C =653.59;$ SBE $=849.67; M=346.41; MoC < liclass = "import−Normal" > C =118.25;$ S$=301.53; E =153.13;$ MoS¡li class="import-Normal"¿M$=96.25;$ P$=41.25; C =178.74;$ SBE $=223.74 < /li >< liclass = "import−Normal" > C =744.83;$ S$=1, 284.83; P =204.83;$ MoC¡li class="import-Normal"¿M$= 81.60; S =201.60;$ E$=51.36; MoS < liclass = "import − Normal" > P =175;$ S$=1, 447.37; C =1,172.37;$ MoC¡li class="import-Normal"¿C$=220; S =287.50;$ M$=67.50;$ MoS¡li class="import-Normal"¿112.50 $< /li >< liclass = "import − Normal" > 125 < liclass = "import − Normal" >27,996.79$¡/li¿ ¡li class="import-Normal"¿57.05 $< /li >< liclass = "import − Normal" > a.733.03;$ b. $219.91; c.53.3151 < liclass = "import − Normal" > a.119.95;$ b. $40.85; c.51.6435 < liclass = "import−Normal" >20.82$¡/li¿ ¡li class="import-Normal"¿102.40 $< /li >< liclass = "import−Normal" > 42.8571 < liclass = "import − Normal" >488.09$¡/li¿ ¡li class="import-Normal"¿a. $40.47; b.21.79;$ c. $62.26; d.14.51;$ e. $12.20; 2.31$ reduction¡/li¿ ¡li class="import-Normal"¿a. $12.65; b.17.71;$ c. $0.63; d.MoS < /ol >< ul >< li >< h3 >< strong > 4 < /strong >< strong > .3 < /strong >< strong > Markdown < /strong >< /h3 >< /li >< /ul >< ol >< liclass = "import−Normal" >< spanstyle = "font−size : NaNpt; color : ; text − decoration : none" > D=153.95; < /span > Sonsale =285.90$¡/li¿ ¡li class="import-Normal"¿S $= 299.95; d = 33.3389 < liclass = "import − Normal" > D = 275; d = 26.1905 < liclass = "import − Normal" > S < sub > onsale < /sub >=25,525;$ d$=11.2945$¡li class="import-Normal"¿S $= 19, 701.42; D = 6, 501.47 < /li >< li >< spanstyle = "font − size : 1rem" > S =319.42;$ ¡/span¿S¡sub¿onsale¡/sub¿ $=281.09 < /li >< li > S < sub > onsale < /sub >< spanstyle = "font−size : 1rem" >=53.99$¡/span¿¡/li¿ ¡li¿D$=45;$ d $= 52.9474$¡li¿d $= 14.9$¡li¿S $= 133.30; D = 93.31 < /li >< li > S < sub > onsale < /sub >< spanstyle = "font − size : 1rem" >=384.99;$ ¡/span¿S$=699.98 < /li >< li > d = 36.7893 < /li >< li > a.P < sub > onsale < /sub >=2.59;$ b. $P = 74.55; < /li >< li > d = 10.9512 < li > a.S < sub > onsale < /sub >=1,519.05;$ b. P¡sub¿onsale ¡/sub¿$=303.81; c.S =2,025.40;$ d. $P = 810.16 < /li >< li > S < sub > onsale < /sub >< spanstyle =$

15

12. a. M\$ = \$10.45; S = \$19.95; P = \$2.99; E = \$7.46; MoS% = 52.381%, 9b. Sonsale=\$15.96; D\$=\$3.99; M\$onsale =\$6.46; MoS%onsale =40.4762%

13. a. P = $1.36; b. P_{onsale} =$0.51; c. SBE = $1.48
14. S=$425; S_{onsale} =$225
15. P_{onsale} =$1.17
16. P_{onsale} =$0.08; The promotion is profitable and generates a total profit of $0.08 × 300,000 =$24,000
17. P_{onsale} =-$1.10; The promotion is not profitable and generates a loss of $1.10 × 50,000 =$55,000
18. a. E = $2,250; b. M$ = $4,500; c. S = $19,500; d. MoS% = 23.0769%; e. Sonsale=$17,550; f. D$ = $1,950; g. P_{onsale} =$250; h. MM = $3,768.75
19. a. Use the coupon; b. Increase in profit = $3,980
20. Choose Supplier 2 since $75.85 more profit is earned.
21. S_{onsale} =$720
22. P_{onsale} =$6.06
23. a. E = $16.00; b. P = $27.99; c. M$ = $43.99; d. MoS% = 54.9944%, e. MoC% = 122.1944%, f. S_{onsale} =$55.99; g. D$ =$24.00; h. P_{onsale} =$3.99; i. MM = $41.59

5.

ACCOUNTING APPLICATIONS

Chapter 5 Topics

5.1 Sales Taxes

5.2 Property Taxes

5.3 Exchange Rates and Currency Exchange

The price tag on the product is not necessarily what you pay for it. In 1789, Benjamin Franklin might have said it best when he wrote, "In this world nothing can be said to be certain, except death and taxes."[1] Almost everything is taxed, whether a physical good, a service, or even property.

For instance, while shopping on eBay, you find a product listed in an eBay store located in your home province. Another product you need is listed by a seller in the United States. You need both products and the prices including shipping costs are reasonable, so for each you click the "Buy It Now" option. The Canadian invoice lists the product's pre-tax price along with two more lines indicating GST and PST amounts, all totalled at the bottom. The US invoice lists the price of the item in US currency. When paying this seller, PayPal converts the price from US currency to Canadian currency using an exchange rate that does not quite match the rate you have seen published in the newspaper, and you wonder why. If you go with this US seller, you know that the Canada Border Services Agency (customs) will charge for GST.

This scenario highlights three of this chapter's key topics: taxes, currency exchange, and invoicing.

---

1. Benjamin Franklin. Letter to Jean-Baptiste Leroy dated November 13, 1789.

1. If there is one thing you encounter in all aspects of your life, it is sales tax, which provides revenue to governments in Canada and abroad. In addition, anyone involved in the ownership of real estate must deal with property taxes. Corporate offices, production facilities, and warehouses sit on business-owned real estate, which is charged property taxes. Residential homeowners pay annual taxes to their municipalities. Even if you just rent property like an apartment, the apartment building owner is charged property taxes, which are passed on to tenants within the rental charges.

2. Canadian businesses and consumers operate in a global economy in which they commonly complete international transactions. Any payment will incorporate an exchange rate ensuring that the amount paid in Canadian currency is equivalent to the amount charged in the foreign currency.

3. Many transactions are completed through an invoicing procedure. Along with a list of the items purchased and their respective prices, invoices show terms of payment, consequences of failing to make the payment, and additional charges such as taxes and shipping.

There is no avoiding sales taxes, property taxes, currency exchange, and invoicing. As a consumer, you generally need to pay these costs. As a business, you not only pay these costs but you need to charge your customers appropriately and collect taxes on behalf of the government. This chapter explores the mathematics behind these concepts.

5.1 Sales Taxes

On your recent cross-Canada road trip, you purchased from many different Tim Hortons' stores. At each store, your products retailed for $6.99. When you review your credit card receipts after returning home from your trip, you notice that you paid different totals everywhere. In Alberta, they only added GST and your combo cost $5.34. In British Columbia, they added both PST and GST, resulting in total cost of $7.83. In Ontario, they added something called HST, resulting in a total cost of $7.90. You find it interesting that the same combo came to different totals as you travelled across Canada.

Three Sales Taxes

A **sales tax** is a percent fee levied by a government on the supply of products. In Canada, there are three types of sales taxes: the goods and services tax (GST), provincial sales tax (PST), and the harmonized sales tax (HST). In this section you will learn the characteristics of each of these taxes and then the mathematics for calculating any sales tax.

Goods & Services Tax (GST)

The goods and services tax, better known as **GST**, is a national federal tax of 5% that applies to the purchase of most goods and services in Canada. Every province and territory has GST. The consumer ultimately bears the burden of this sales tax.

Businesses must collect GST on most of their sales and pay GST on most purchases in the daily course of operations. However, when remitting these taxes, businesses claim a credit with the federal government to recover the GST they paid on eligible purchases. The net result is that businesses do not pay the GST on these eligible purchases. While this may outrage some people, the logic is simple. If a business pays the GST, it becomes a cost of the business, which is then passed on to consumers as it is incorporated into retail prices. When the consumer purchases the product, the consumer would be charged the GST again! In essence, a consumer would be double-taxed on all purchases if businesses paid the GST.

Some goods and services are exempt from GST. While there are many complexities and nuances to the exemptions, generally items that are deemed necessities (such as basic groceries), essential services (such as health, legal aid, and childcare), and charitable activities are nontaxable. You can find a complete listing of exemptions on the Canada Revenue Agency website at www.cra.gc.ca.

Provincial Sales Tax (PST)

Province/Territory	PST
British Columbia	7%
Alberta	0%
Saskatchewan	5%
Manitoba	8%
Quebec	9.975%
Northwest Territories	0%
Yukon Territory	0%
Nunavut	0%

Provincial sales taxes, or **PST**, are provincially administered sales taxes that are determined by each individual provincial or territorial government in Canada. The table here lists the current PST rates in Canada.

Similar to GST, PST applies to the purchase of most goods and services in the province, and consumers bear the burden. For the same reasons as with GST, businesses typically pay the PST on purchases for non-resale items (such as equipment and machinery) and do not pay the PST on resale items. Businesses are responsible for collecting PST on sales and remitting the tax to the provincial government. Individual provincial websites list the items and services that are exempt from PST.

Harmonized Sales Tax (HST)

The harmonized sales tax, or HST, is a combination of GST and PST into a single number. Since most goods and services are subjected to both taxes anyway, HST offers a simpler method of collecting and remitting the

sales tax—a business has to collect and remit only one tax instead of two. Because there are pros and cons to HST, not all provinces use this method of collection, as summarized in the table below.

Pros

· Items that are previously PST payable to a business are now refunded, lowering input costs and lowering consumer prices

· Results in overall lower corporate taxes paid

· Increases the competitiveness of businesses and results in job creation

· Businesses only remit one tax and not two, resulting in financial and auditing savings

Cons

· Many items such as utilities, services, and children's clothing that are ineligible for PST become taxed at the full HST rate

· Consumer cost of living increases

· Tax-exempt items see prices rise because HST is being applied to services and goods such as transportation and gasoline

Province/Territory	HST
Ontario	13%
New Brunswick	13%
Nova Scotia	14%
Newfoundland and Labrador	13%
Prince Edward Island	14%

> ### The Formula
>
> With respect to sales taxes, you usually calculate two things:
>
> 1. The dollar amount of the sales tax.
> 2. The price of a product including the sales tax.

Calculating the Sales Tax Amount

A sales tax is a percent rate calculated on the base selling price of the product. Therefore, if you are interested solely in the amount of the sales tax (the portion owing), apply Formula 2.2 on Rate, Portion, Base: $\text{Rate} = \frac{\text{Portion}}{\text{Base}}$

becomes $\text{Tax Rate} = \frac{\text{Tax Amount}}{\text{Price before Taxes}}$

Rearranging this formula to solve for the tax amount gives the following:

Tax Amount = Tax Rate × Price before Taxes (which is the same as Portion = Rate × Base)

This is not a new formula. It applies the existing formula from Chapter 2.

Calculating a Price Including Tax

When calculating a selling price including the tax, you take the regular selling price and increase it by the sales tax percentage. This is a percent change calculation using Formula 3.1 from Chapter 3:

The Old price is the price before taxes, the percent change is the sales tax percentage, and you need to calculate the New price including the tax. Rearranging the formula for New, you have which you factor and rewrite as *or t*his is a widespread application of the percent change formula. Since it can be tedious to keep rearranging Formula 3.1, Formula 5.1 expresses this relationship.

Selling Price Including Tax

S_{tax} **is Selling Price Including Tax:** A subscript is added to the regular symbol of S for selling price to denote that this is the price that sums the selling price and sales tax amount together.

S is Selling Price Without Tax: The regular selling price before taxes.

Formula 7.1 - Selling Price Including Tax: $$S_{tax} = S + (S \times Rate)$$

Rate is Tax Rate: This is the sales tax in any form of GST, HST, or PST expressed in its decimal format. Round the result of (S × Rate) to two decimals. If two taxes are involved in the tax-inclusive price (such as GST and PST), you cannot combine the rates together into a single rate. For example, Manitoba has 5% GST and 8% PST. This is not necessarily equivalent to 13% tax since each tax is rounded to two decimals separately and then summed. If you use a single rate of 13%, you may miscalculate by a penny. Instead, expand the formula to include two separate (S × Rate) calculations:

$$S_{tax} = S + (S \times GST\ Rate) + (S \times PST\ Rate)$$

Ensure that you round off each calculation in brackets to two decimals before adding.

Key Takeaways

S_{tax} is Selling Price Including Tax: A subscript is added to the regular symbol S for Selling Price to denote that this is the price that sums the selling price and sales tax amount together.

S is Selling Price Without Tax: The regular selling price before taxes.

Rate is Tax Rate: This is the sales tax in any form of GST, HST, or PST expressed in its decimal format. Round the result of (S x Rate) to two decimals. If two taxes are involved in the tax-inclusive price (such as
GST and PST), you cannot combine the rates together into a single rate. For example, Manitoba has 5% GST
and 8% PST. This is not necessarily equivalent to 13% tax since each tax is rounded to two decimals separately
and then summed. If you use a single rate of 13%, you may miscalculate by a penny. Instead, expand

the
formula to include two sepaate (S x Rate) calculations:
S_{tax} = S (S x GST Rate) (S x PST Rate)

How It Works

Follow these steps to perform calculations involving sales taxes:

Step 1: Identify the pricing information. In particular, pay careful attention to distinguish whether the price is before taxes (S) or inclusive of taxes (Stax). Also identify all applicable sales taxes, including GST, PST, and HST.

Step 2: Apply Formula 5.1 to solve for the unknown variable.

Step 3: If you need to find sales tax amounts, apply Formula 2.2 and rearrange for portion. Ensure that for the base you use the price before taxes.

Assume a $549.99 product is sold in British Columbia. Calculate the amount of the sales taxes and the price including the sales taxes.

Step 1: The price before taxes is S = $549.99. In British Columbia, GST is 5% and PST is 7% (from the PST Table).

Step 2: To calculate the price including the sales taxes, apply Formula 5.1:

Stax = $549.99 ($549.99 × 5%) ($549.99 × 7%) = $615.99

Step 3: Applying the rearranged Formula 2.2 for the GST, you calculate:

GST Tax Amount = 5% × $549.99 = $27.50

Applying the same formula for the PST, you calculate:

PST Tax Amount = 7% × $549.99 =$38.50.

(Note: You may notice that you could just pull these amounts from the interim calculations in step 2). Therefore, on a $549.99 item in British Columbia, $27.50 in GST and $38.50 in PST are owing, resulting in a price including sales taxes of $615.99.

Important Notes

To calculate a price including a single sales tax rate, use the percent change function (Δ%) on the calculator. You can review the full instructions for this function at the end of Chapter 3. When using this function, OLD is the price before taxes, NEW is the price after taxes, and %CH is the single tax rate in its percentage format. You do not use the #PD variable, which therefore defaults to 1. Note that if more than one tax rate applies on the same base, you cannot use this calculator function because of the possibility of a rounding error, as explained above.

Paths To Success

You will often need to manipulate Formula 5.1. Most of the time, prices are advertised without taxes and you need to calculate the price including the taxes. However, sometimes prices are advertised including the taxes and you must calculate the original price of the product before taxes. When only one tax is involved, this poses no problem, but when two taxes are involved (GST and PST), combine the taxes into a single amount before you solve for S.

Give It Some Thought

On any given product selling for the same price, put the following provinces in order from highest price to lowest price including taxes (GST and PST, or HST): Alberta, Saskatchewan, British Columbia, Ontario, Prince Edward Island.

Solutions:

PEI (14% HST), Ontario (13% HST), British Columbia (5% GST 7% PST), Saskatchewan (5% GST 5% PST), Alberta (5% GST, no PST)

Example 5.1 A: Calculating Sales Taxes Across Canada

[table id=25 /]

Example 5.1 B: Calculating Taxes on a Tax-Inclusive Price

[table id=26 /]

The GST/HST Remittance

When a business collects sales taxes, it is a go-between in the transaction. These sales tax monies do not belong to the business. On a regular basis, the business must forward this money to the government. This payment is known as a **tax remittance**.

The Formula

Generally speaking, a business does not pay sales taxes. As a result, the government permits a business to take all eligible sales taxes that it paid through its acquisitions and net them against all sales taxes collected from sales. The end result is that the business is reimbursed for any eligible out-of-pocket sales tax that it paid. Formula 5.2 expresses this relationship.

GST/HST Remittance

Remit is GST/HST Remittance: This is the dollar amount of the remittance.
- If this amount is positive, it means that the business collected more tax than it paid out; the company must remit this balance to the government.
- If this amount is negative, it means that the business paid out more taxes than it collected; the government must refund this balance to the company.

Formula 7.2 – GST/HST Remittance: $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$

Tax Collected and Tax Paid: Both of these parts of the formula apply Formula 2.2, representing the total amount of sales tax (portion) from all taxable amounts (base) at the appropriate sales tax rate (rate). The taxes collected are based on total tax-eligible revenues. The taxes paid are based on the total tax-eligible acquisitions.

Tax Collected and Tax Paid: Both of these parts of the formula apply Formula 2.2, representing the total amount of sales tax (portion) from all taxable amounts (base) at the appropriate sales tax rate (rate). The taxes collected are based on total tax-eligible revenues. The taxes paid are based on the total tax-eligible acquisitions

How It Works

Follow these steps to complete a GST/HST remittance:

Step 1: Identify the total amounts of tax-eligible revenues and acquisitions upon which the sales tax is collected or paid, respectively. Identify the applicable sales tax rate of the GST or HST.

Step 2: Calculate the total taxes collected by applying Formula 2.2, where the sales tax is the rate and the total revenue is the base. Solve for portion.

Step 3: Calculate the total taxes paid by applying Formula 2.2, where the sales tax is the rate and the total acquisitions are the base. Solve for portion.

Step 4: Apply Formula 5.2 to calculate the tax remittance.

Assume a business has paid GST on purchases of $153,000. It has also collected GST on sales of $358,440. Calculate the GST remittance.

Step 1: Identifying the variables, you have:

Total Revenue =$358,440

Total Acquisitions = $153,000   GST Tax Rate = 5%

Step 2: Calculate taxes collected by applying Formula 2.2, where:

GST collected = 5% × $358,440 = $17,922.

Step 3: Calculate taxes paid by applying Formula 2.2, where:

GST paid = 5% × $153,000 = $7,650.

Step 4: To calculate the remittance, apply Formula 5.2 and calculate Remit = $17,922 − $7,650 = $10,272. The business should remit a cheque for $10,272 to the government.

Paths To Success

A shortcut can help you calculate the GST/HST Remittance using Formula 5.2. If you do not need to know the actual amounts of the tax paid and collected, you can net GST/HST–eligible revenues minus acquisitions and multiply the difference by the tax rate:

Remit = (Revenues − Acquisitions) × Rate

In the example above, Remit = ($358,440 − $153,000) x 5% = $10,272. If this calculation produces a negative number, then the business receives a refund instead of making a remittance.

Example 5.1 C: Calculating GST/HST Remittance

[table id=27 /]

Exercises

Mechanics

You are purchasing a new BlackBerry at the MSRP of $649.99. Calculate the price including taxes in the following provinces or territories:

1. Northwest Territories
2. New Brunswick
3. Nova Scotia
4. British Columbia

The Brick is advertising a new Serta mattress nationally for a price of $899.99 including taxes. What is the price before taxes and the sales tax amounts in each of the following provinces?

5. Ontario
6. Saskatchewan
7. Audiophonic Electronics is calculating its HST remittance in Prince Edward Island. For each of the following months, calculate the HST remittance or refund on these HST-eligible amounts.

Month	Purchases	Sales
January	$48,693	$94,288
February	$71,997	$53,639

8. Airwaves Mobility is calculating its GST remittance in Alberta. For each of the following quarters, calculate the GST remittance or refund on these GST-eligible amounts.

Quarter	Purchases	Sales
Winter	$123,698	$267,122
Spring	$179,410	$158,905
Summer	$216,045	$412,111
Fall	$198,836	$175,003

Applications

9. Elena lives in Nova Scotia and has relatives in Alberta, Saskatchewan, and Quebec. She gets together with them often. She wants to purchase a new aerobic trainer and would like to pay the lowest price. If a family member buys the item, Elena can pick it up at one of their regular family gatherings. The price of the trainer for each province is listed below:

Province	Regular Selling Price before Taxes
Nova Scotia	$1,229.50
Alberta	$1,329.95
Saskatchewan	$1,274.25
Quebec	$1,219.75

a. Where should Elena have the aerobic trainer purchased and how much would she pay?

b. How much money would she save from her most expensive option?

10.　Mary Lou just purchased a new digital camera in Nunavut for $556.49 including taxes. What was the price of the camera before taxes? What amount of sales tax is paid?

11.　Marley is at Peoples Jewellers in New Brunswick wanting to purchase an engagement ring for his girlfriend. The price of the ring is $2,699.95. If the credit limit on his credit card is $3,000, will he be able to purchase the ring on his credit card? If not, what is the minimum amount of cash that he must put down to use his credit card?

12.　In the IKEA store in Vancouver, British Columbia, you are considering the purchase of a set of kitchen cabinets priced at $3,997.59. Calculate the amount of GST and PST you must pay for the cabinets, along with the total price including taxes.

13.　A company in Saskatchewan recorded the following GST-eligible purchases and sales throughout the year. Determine the GST remittance or refund per quarter.

Quarter	Purchases	Sales
1st	$2,164,700	$2,522,000
2nd	$1,571,300	$2,278,700
3rd	$1,816,100	$1,654,000
4th	$2,395,900	$1,911,700

14.　A manufacturer in Nova Scotia recorded the following HST-eligible purchases and sales in its first three months of its fiscal year. Determine the HST remittance or refund per month.

Month	Purchases	Sales
March	$20,209	$26,550
April	$28,861	$20,480
May	$22,649	$42,340

Challenge, Critical Thinking, & Other Applications

15. If the selling price of an item is 6% higher in Yukon than in Ontario, will the price including taxes be higher in Yukon or Ontario? What percentage more?

16. Colin just travelled across the country on a road trip. He bought some skis in Alberta for $879.95 plus tax, a boombox in British Columbia for $145.58 including taxes, a Niagara Falls souvenir in Ontario for $99.97 plus tax, and some maple syrup in Quebec for $45.14 including tax. Overall, how much GST, PST, and HST did Colin pay on his trip?

17. Cisco Enterprises in Ontario purchased the following in a single month:

- 16,000 units of network routers at $79.25 each, priced at $97.97 each
- 12,000 units of wireless LAN adapters at $129.95 each, priced at $189.55 each
- 13,500 units of computer boards at $229.15 each, priced at $369.50 each.

Assuming that all units purchased are sold during the same month and that all purchases and sales are taxable, calculate the tax remittance or refund for the month.

18. In Quebec, the PST used to be calculated on the price including GST. When the PST was calculated in this manner, what PST rate did Quebec set to arrive at the same price including taxes?

19. For each of the following situations, compute the selling price of the product before taxes in the other province/territory that would result in the same selling price including taxes as the item listed.

	Price before Tax	Sold In
a.	$363.75	British Columbia
b.	$1,795.00	Alberta
c.	$19,995.95	Saskatchewan
d.	$4,819.35	New Brunswick

20. A company made the following taxable transactions in a single month. Compute the GST remittance on its operations assuming all sales and purchases are eligible for GST.

Transaction Type	Unit Price	Quantity Involved
Purchase	$168.70	5,430
Sale	$130.00	4,000
Sale	$148.39	3,600
Purchase	$93.47	2,950
Purchase	$24.23	3,325
Purchase	$121.20	2,770
Sale	$188.88	6,250

5.2 Property Taxes

As you drive through your neighbourhood, you pass a city crew repairing the potholes in the road. Hearing sirens, you check your rear-view mirror and pull to the side of the road as a police car and fire engine race by, heading toward some emergency. Pulling back out, you drive slowly through a public school zone, where you smile as you watch children playing on the gigantic play structure. A city worker mows the lawn.

Where does the municipality get the money to pay for all you have seen? No one owns the roads, schools are free, fire crews and police do not charge for their services, play structures have no admission, and parks are open to everyone. These are just some examples of what your municipality does with the money it raises through property taxes.

Property Taxation

Property taxes are annual taxes paid by real estate owners to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services. Every

individual and every business pays property taxes. Even if you don't own property, you pay property taxes that are included in your rental and leasing rates from your landlord.

Property taxes are imposed on real estate owners by their municipal government along with any other bodies authorized to levy taxes. For example, in Manitoba each divisional school board is authorized to levy property taxes within its local school division boundaries.

The Formula

Since property taxes are administered at the municipal level and every municipality has different financial needs, there are a variety of ways to calculate a total property tax bill. Formula 5.3 (on next page) is designed to be flexible to meet the varying needs of municipal tax calculations throughout Canada.

How It Works

Follow these steps when working with calculations involving property taxes:

Step 1: Identify all known variables. This includes the market value, tax policy, assessed value, all property tax rates, and the total property taxes.

Step 2: If you know the assessed value, skip this step. Otherwise, calculate the assessed value of the property by multiplying the market value and the tax policy.

Step 3: Calculate either the tax amount for each property tax levy or the grand total of all property taxes by applying Formula 5.3.

Continuing with the Winnipeg example in which a home has a market value of $200,000, the tax policy of Winnipeg is to tax 45% of the market value. A Winnipegger receives a property tax levy from both the City of Winnipeg itself and the local school board. The mill rates are set at 14.6 and 16.724, respectively. Calculate the total property tax bill.

Step 1: The known variables are market value = $200,000, tax policy = 45%, City of Winnipeg mill rate = 14.6, and school board mill rate = 16.724.

Step 2: Calculate the assessed value by taking the market value of $200,000 and multiplying by the tax policy of 45%, or Assessed Value =$200,000 × 45% = $90,000.

Step 3: To calculate each property tax, apply Formula 5.3. The City of Winnipeg Property Tax =

$90,000 × 14.6 ÷ 1,000 =$1,314. The school board Property Tax = $90,000 × 16.724 ÷ 1,000 = $1,505.16. Add these separate taxes together to arrive at Total Property Tax = $1,314 $1,505.16 = $2,819.16.

Property Tax

Property Tax is Property Tax Amount: The amount of property taxes that are owing represents the total of all property taxes from all taxable services. For example, your property tax bill could consist of a municipal tax, a public school tax, a water tax, and a sewer/sanitation tax. Each of these taxes is levied at a specific rate. Therefore, the formula is designed to sum all applicable taxes, as represented by the summation symbol (Σ) on the right-hand side of the equation.

Formula 7.3 – Property Taxes: $$\text{Property Tax} = \Sigma(\text{AV} \times \text{PTR})$$

AV is Assessed Value of Property: Every piece of real estate has two valuations: the market value and the assessed value. Only the assessed value is relevant when computing property taxes.

1. The **market value** of a property is a snapshot of the estimated selling price of your property. It is what you might have been able to sell your property for at a certain time period. For example, the City of Winnipeg updates the market value of all property in the city every two years.
2. The **assessed value** of a property is the portion of the market value that is subjected to the property tax rate. It is calculated by taking the market value and multiplying it by a percentage determined by the municipality's **tax policy**:

    Market Value × Tax Policy = Assessed Value

In some municipalities, the tax policy is to tax 100% of the market value. In others, the tax policy can be substantially less. Continuing with the example, the tax policy for Winnipeg is to tax 45% of the market value. Therefore, a $200,000 market value home in Winnipeg has a $90,000 assessed value. This $90,000 is the base for the tax.

PTR is Property Tax Rate: Two methods commonly express how assessed value is taxed, a tax rate and a mill rate.

1. A **tax rate** is a tax per $100 of assessed value. Most municipalities in Ontario and further east use this system. The mathematical expression for the tax rate is

    $$\frac{\text{Tax rate}}{100}$$

2. A **mill rate** is a tax per $1,000 of assessed value. Most municipalities in Manitoba and further west use this system. The mathematical expression for the mill rate is

    $$\frac{\text{Mill rate}}{1,000}$$

Important Notes

Mill rates are commonly expressed with four decimals and tax rates are expressed with six decimals. Although some municipalities use other standards, this text uses these common formats in its rounding rules. In addition,

each property tax levied against the property owner is a separate tax. Therefore, you must round each property tax to two decimals before summing the grand total property tax.

Things To Watch Out For

The most common mistake is to use the wrong denominator in the tax calculation. Ensure that you read the question accurately, noting which term it uses, tax rate or mill rate. If neither appears, remember that Ontario eastward uses tax rates and Manitoba westward uses mill rates.

A second common mistake is to add multiple property tax rates together when the assessed value remains constant across all taxable elements. For example, if the assessed value of $250,000 is used for two tax rates of 2.168975 and 1.015566, you may be tempted to sum the rates, which would yield a rate of 3.184541. This does not always work and may produce a small error (a penny or two) since each tax is itemized on a tax bill. You must round each individual tax to two decimals before summing to the total property tax.

Example 5.2A Calculating Property Taxes

[table id=28 /]

Paths To Success

The collective property taxes paid by all of the property owners form either all or part of the operating budget for the municipality. Thus, if a municipality consisted of 1,000 real estate owners each paying $2,000 in property tax, the municipality's operating income from property taxes is $2,000 × 1,000 = $2,000,000. If the municipality needs a larger budget from property owners, either the assessed values, the mill/tax rate, or some combination of the two needs to increase.

Example 5.2B Setting a New Mill Rate

[table id=29 /]

Exercises

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

	Market Value	Tax Policy	Assessed Value	Rate	Type of Rate	Property Tax
1.	$320,000	55%	?	26.8145	Mill	?
2.	?	85%	$136,000	1.984561	Tax	?
3.	$500,000	?	?	9.1652	Mill	$3,666.08
4.	?	50%	$650,000	?	Tax	$4,392.91

Applications

5. A house with an assessed value of $375,000 is subject to a tax rate of 1.397645. What is the property tax?

6. If a commercial railway property has a property tax bill of $166,950 and the mill rate is 18.5500, what is the assessed value of the property?

7. A house in Calgary has a market value of $450,000. The tax policy is 100%. The property is subject to a 2.6402 mill rate from the City of Calgary and a 2.3599 mill rate from the province of Alberta. What are the total property taxes?

8. A residential property in Regina has a market value of $210,000. The Saskatchewan tax policy is 70%. The property is subject to three mill rates: 13.4420 in municipal taxes, 1.4967 in library taxes, and 10.0800 in school taxes. What amount of tax is collected for each, and what are the total property taxes?

9. A municipality needs to increase its operating budget. Currently, the assessed value of all properties in its municipality total $1.3555 billion and the tax rate is set at 0.976513. If the municipality needs an additional $1.8 million next year, what tax rate should it set assuming

the assessed values remain constant?

10. A municipality set its new mill rate to 10.2967, which increased its total operating budget by $10 million on a constant assessed value of $7.67 billion. What was last year's mill rate?

Challenge, Critical Thinking, & Other Applications

11. A school board is determining the mill rate to set for next year. The assessed property values for next year total $5.782035 billion, representing an increase of 5% over the current year. If the school board needs an additional $5.4 million in funding next year, by what amount should it change its current year mill rate of 11.9985?

12. In the current year, the market value of properties totals $6.896 billion. The current tax policy is 85% and the current mill rate is 15.6712. If the municipality requires an additional $2 million in its operating budget next year, market values increase by 4%, and the tax policy changes to 90%, what mill rate should it set for next year?

13. A $600,000 market value property is assessed with a tax policy of 75% and subject to two mill rates. If the total property taxes are $6,766.67 and the second mill rate is half of the first tax rate, calculate each mill rate.

14. Two properties in different provinces pay the same property taxes of $2,840. One province uses a mill rate of 24.6119 with a 60% tax policy, while the other province uses a tax rate of 1.977442 with an 80% tax policy. Compute the market values for each of these properties.

15. A water utility funded through property taxes requires $900 million annually to operate. It has forecasted increases in its operating costs of 7% and 3.5% over the next two years. Currently, properties in its area have a market value of $234.85 billion, with projected annual increases of 3% and 5% over the next two years. The provincial government has tabled a bill that might change the tax policy from 70% to 75% effective next year, but it is unclear if the bill will pass at this point. For planning purposes, the utility wants to forecast its new mill rates for the next two years under either tax policy. Perform the necessary calculations for the utility.

5.3 Exchange Rates and Currency Exchange

You have set aside $6,000 in Canadian funds toward hostel costs during a long backpacking trip through the United States, Mexico, and Europe. After searching on the Internet, you decide to use Hotwire.com to reserve your hostel rooms. The website quotes you the following amounts for each country: 1,980 US dollars, 21,675 Mexican pesos, and 1,400 euros. Have you allocated enough money to cover these costs?

Whether you are a consumer backpacking around the world on distant vacations, investing in international securities, or shopping online at global Internet sites, you must pay for your purchases in local currencies out of your Canadian currency accounts. Businesses are no different as they export and import products to and from other countries. With outsourcing on the rise, it is also common for business services such as call centres and advertising agencies to be located abroad along with manufacturing facilities. Large-scale companies may have operations in several countries throughout the world.

All of these transactions and operations require the conversion of Canadian currency to a foreign currency or vice versa. This section shows you the basics of currency conversion rates and then explores finer details such as charges for currency conversion and what happens when one currency gets stronger or weaker relative to another.

Exchange Rates

An **exchange rate** between two currencies is defined as the number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa. Since two currencies are involved in every transaction, two published exchange rates are available. Let's use Canada and the United States to illustrate this concept.

- The first exchange rate indicates what one dollar of Canada's currency becomes in US currency.
- The second exchange rate indicates what one dollar of US currency becomes in Canadian currency.

These two exchange rates allow Canadians to determine how many US dollars their money can buy and vice versa. These currency rates have an inverse relationship to one another: if 1 Canadian dollar equals 0.80 US dollars, then 1 US dollar equals = 1.25 Canadian dollars.

Most Canadian daily and business newspapers publish exchange rates in their financial sections. Although exchange rates are published in a variety of ways, a currency cross-rate table, like the table below, is most common. Note that exchange rates fluctuate all of the time as currencies are actively traded in exchange markets. Therefore, any published table needs to indicate the date and time at which the rates were determined. Also note that the cells where the same currency appears show no published rate as you never need to convert from Canadian dollars to Canadian dollars!

	Per C$	Per US$	Per €	Per ¥	Per MXN$	Per AU
Canadian Dollar (C$)	——	0.9787	1.4012	0.0122	0.0823	1.0
US Dollar (US$)	1.0218	——	1.4317	0.0125	0.0841	1.0
Euro (€)	0.7137	0.6985	——	0.0087	0.0588	0.7
Japanese Yen (¥)	82.0233	80.2765	114.9287	——	6.7540	84
Mexican Peso (MXN$)	12.1445	11.8859	17.0165	0.1481	——	12
Australian Dollar (AU$)	0.9652	0.9447	1.3525	0.0118	0.0795	—

In the table, all exchange rates have been rounded to four decimals. In true exchange markets, most exchange rates are expressed in 10 decimals or more such that currency exchanges in larger denominations are precisely performed. For the purposes of this textbook, we will use a four decimal standard to simplify the calculations while still illustrating the principles of currency exchange.

Technically, only half of the table is needed, since one side of the diagonal line is nothing more than the inverse of the other. For example, the euro is 0.7137 per C$ on the bottom of the diagonal. The inverse, or $\frac{1}{0.7137}$ = 1.4011 per €, is what is listed on top of the diagonal (the difference of 0.0001 is due to rounding to four decimals).

The Formula

Currency Exchange

Desired Currency is what you are converting to: In Formula 2.2, the desired currency represents the portion. This is what your money is worth in the other currency.

Formula 7.4 – Currency Exchange: **Desired Currency = Exchange Rate × Current Currency**

Exchange Rate is per unit of Current Currency: In Formula 2.2, the exchange rate represents the rate, which is the current exchange rate per currency unit in which your money is currently valued. Looking at a Cross-Rate table, find the column that represents your existing currency and then cross-tabulate this column with the row for the desired currency. For example, if you have one euro that you want to convert into yen, locate the euro column and the yen row. The exchange rate is 114.9287. It is critical that both the exchange rate and current currency always be expressed in the same unit, which is the type of currency.

Current Currency is what you are converting from: In Formula 2.2, the current currency represents the base, which is the amount of money (number of units) in your existing currency.

Currencies, exchange rates, and currency cross-tables all raise issues regarding decimals and financial fees.

1. **Decimals in Currencies**. Not all currencies in the world have decimals. Here in North America, MXN$, US$, and C$ have two decimals. Mexicans call those decimals *centavos* while Canadians and Americans call them *cents*. Australia and the European countries using the euro also have cents. However, the Japanese yen does not have any decimals in its currency. If you are unsure about the usage of decimals, perform a quick Internet search to clarify the issue.

2. **Financial Fees**. Technically, the rates in a cross-rate table are known as mid-rates. A **mid-rate** is an exchange rate that does not involve or provide for any charges for currency conversion. When you convert currencies, you need to involve a financial organization, which will charge for its services.

3. **Sell Rates**. When you go to a bank and convert your domestic money to a foreign currency, the bank charges you a **sell rate**, which is the rate at which a foreign currency is sold. When you exchange your money, think of this much like a purchase at a store—the bank's product is the foreign currency and the price it charges is marked up to its selling price. The sell rate is always higher than the mid-rate in terms of C$ per unit of foreign currency and thus is always lower than the mid-rate in terms of foreign currency per unit of C$. For example, the exchange rate of C$ per US$ is 0.9787 (you always look up the "per currency" column that you are purchasing). This means it will cost you C$0.9787 to purchase US$1.00. The bank, though, will sell you this money for a sell rate that is higher, say $0.9987. This means it will cost you an extra C$0.02 per US$ to exchange the money. That $0.02 difference is the fee from the bank for its services, and it is how the bank makes a profit on the transaction.

4. **Buy Rates**. When you go to a bank and convert your foreign currency back into your domestic money,

the bank charges you a **buy rate**, which is the rate at which a foreign currency is purchased. The buy rate is always lower than the mid-rate in terms of C$ per unit of foreign currency and thus is always higher than the mid-rate in terms of foreign currency per unit of C$. Using the same example as above, if you want to take your US$1.00 to the bank and convert it back to Canadian funds, the bank charges you a buy rate that is lower, say $0.9587. In other words, you receive C$0.02 less per US$. Again, the $0.02 difference is the bank's fee for making the currency exchange on your behalf.

How It Works

Follow these steps when performing a currency exchange:

Step 1: Identify all known variables. Specifically, identify the currency associated with any amounts. You also require the mid-rate. If buy rates and sell rates are involved, identify how these rates are calculated.

Step 2: If there are no buy or sell rates, skip this step. If buy and sell rates are involved, calculate these rates in the manner specified by the financial institution.

Step 3: Apply Formula 7.4 using the appropriate mid-rate, buy rate, or sell rate to convert currencies.

This section opened with your backpacking vacation to the United States, Mexico, and Europe, for which you were quoted prices of US$1,980, MXN$21,675, and €1,400 for hostels. Assume all purchases are made with your credit card and that your credit card company charges 2.5% on all currency exchanges. Can your C$6,000 budget cover these costs?

Step 1: There are three currency amounts: US$1,980, MXN$21,675, and €1,400. Using the cross-rate table, the Canadian exchange mid-rate per unit of each of these currencies is 0.9787 US$, 0.0823 MXN$, and 1.4012 €.

Step 2: Calculate the buy rates (since you are converting foreign currency into domestic currency) for each currency:

US$ = 0.9787(1.025) =1.0032

MXN$ = 0.0823(1.025) = 0.0844

€ = 1.4012(1.025) = 1.4362

Step 3: Apply Formula 7.4 to each of these currencies:

US$: Desired Currency = US$1,980 × 1.0032 = C$1,986.34.

MXN$: Desired Currency = MXN$21,675 × 0.0844 = C$1,829.37.

€: Desired Currency = €1,400 × 1.4362 = C$2,010.68.

Putting the three amounts together, your total hostel bill is:

$1,986.34 $1,829.37 $2,010.68 = $5,826.39

Because this is under budget by $173.61, all is well with your vacation plans.

Things To Watch Out For

When working with currency exchange, probably the trickiest element is that you have to choose one of two inverse exchange rates depending on which way the money conversion is taking place. In any currency situation, it is important that you take the time to understand the basis on which the currency rate is being expressed. Typically, exchange rates are expressed on a per-unit basis in the country's domestic currency. For example, Canadians express the US dollar exchange rate on a per C$ basis. From the cross-rate table, that exchange rate is 1.0218. In contrast, Americans express the Canadian dollar exchange rate on a per US$ basis, or 0.9787.

Paths To Success

Let the buyer beware when it comes to international transactions. If you have ever purchased and returned an item to an international seller, you may have noticed that you did not receive all of your money back. For most consumers, international purchases are made via credit cards. What most consumers do not know is that the credit card companies do in fact use buy and sell rates that typically charge 2.5% of the exchange rate when both buying and selling.

For example, if you purchase a US$2,000 item at the rates listed in the cross-rate table, your credit card is charged $2,000 × 0.9787(1.025) = $2,006.40. If you return an item you do not want, your credit card is refunded $2,000 × 0.9787(0.975) = $1,908.40. In other words, you are out $2,006.40 − $1,908.40 = $98! This amount represents your credit card company's charge for the currency conversion—a whopping 4.9% of your purchase price!

Give It Some Thought

In each of the following situations and using the cross-rate table, determine on a strictly numerical basis whether you would have more or fewer units of the target currency than of the original currency. You want to convert:

1. C$ into US$
2. AU$ into MXN$
3. MXN$ into ¥
4. C$ into €

Solutions:

1. More, since C$1 becomes US$1.0218
2. More, since AU$1 becomes MXN$12.5814
3. More, since MXN$1 becomes ¥6.7540
4. Less, since C$1 becomes €0.7137

Solutions:

1. US dollars 2. Increase 3. Mexican pesos   4. Decrease

Example 5.3 A: Straight Currency Conversion

[table id=30 /]

Example 5.3 B: Calculating Profit Using Buy and Sell Rates

[table id=31 /]

Currency Appreciation and Depreciation

An analyst on *Global News* is discussing how the Canadian dollar has strengthened against the US dollar. Your

first reaction is that a strong Canadian dollar ought to be good thing, so hearing that this change might hurt Canada's exports, you wonder how that could be.

Currencies are actively traded in the international marketplace, which means that exchange rates are changing all the time. As such, exchange rates rise and decline. A currency **appreciates** (or strengthens) relative to another currency when it is able to purchase *more* of that other currency than it could previously. A currency **depreciates** (or weakens) relative to another currency when it is able to purchase *less* of that other currency than it could previously. Take a look at two examples illustrating these concepts:

- EXAMPLE 1: If C$1 buys US$1.02 and the exchange rate rises to US$1.03, then your C$1 purchases an additional penny of the US dollars. Therefore, the Canadian currency appreciates, or strengthens, relative to the US dollar.
- EXAMPLE 2: Similarly, if the exchange rate drops to US$1.01, then your C$1 purchases one less penny of the US dollars. Therefore, the Canadian currency depreciates, or weakens, relative to the US dollar.

These concepts are particularly important to international business and global economies. Generally speaking, when a currency appreciates it has a positive effect on imports from the other country because it costs less money than it used to for domestic companies to purchase the same amount of products from the other country. However, the currency appreciation tends to also have a negative effect on exports to other countries because it costs the foreign companies more money to purchase the same amount of products from the domestic companies.

Important Notes

It is critical to observe that if currency A appreciates relative to currency B, then the opposite is true for currency B relative to currency A (currency B depreciates relative to currency A). The figure to the right illustrates this concept. Recalling Example 1 above, the Canadian currency appreciated, so the US currency depreciated. In Example 2, the Canadian currency depreciated, so the US currency appreciated.

As one currency depreciates…

…the other currency appreciates
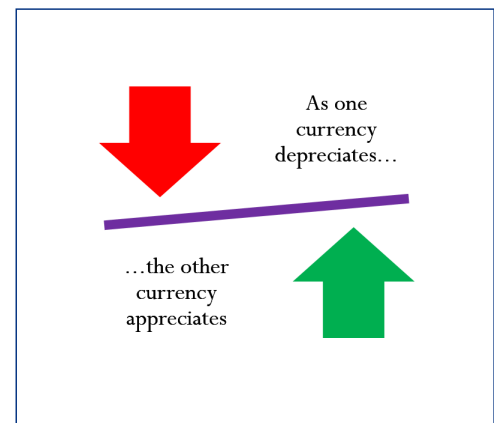
How It Works

When you work with currency appreciation or depreciation, you still use the same basic steps as before. The additional skill you require in the first step is to adjust an exchange rate appropriately based on how it has appreciated or depreciated.

Things To Watch Out For

It is very easy to confuse the two relative currencies, their values, and the concepts of appreciation and depreciation. For example, if the exchange rate increases between currency A relative to a unit of currency B, which exchange rate appreciated? If currency A has increased per unit of currency B, then it takes more money of currency A to buy one unit of currency B. As a result, currency B appreciates because a single unit of currency B can now buy more of currency A. For example, if the exchange rate is US$1.0218 relative to C$1 and it increases to US$1.0318 relative to C$1, then the Canadian dollar purchases one penny more. The figure helps you understand the relationships involved and provides a visual reminder of which direction everything is moving.

If Currency A relative to one unit of Currency B...	Currency A	Currency B
Increases ⬆	Depreciates or weakens ⬇	Appreciates or strengthens ⬆
Decreases ⬇	Appreciates or strengthens ⬆	Depreciates or weakens ⬇

Give It Some Thought

1. If the exchange rate in terms of US dollars per unit of euros increases, which currency weakened?
2. If the Australian dollar weakens against the Canadian dollar, did the exchange rate increase or decrease in terms of Australian dollars per unit of Canadian dollars?
3. If the exchange rate in terms of yen per unit of the Mexican pesos decreases, which currency weakened?
4. If the British pound (£) appreciates against the US dollar, did the exchange rate increase or decrease in terms of pounds per unit of US dollars?

Example 5.3 C: Effect of Currency Appreciation

[table id=32 /]

Exercises

For questions 1–6, use the mid-rates in the cross-rate table below to convert the current currency to the desired currency.

	Per C$	Per US$	Per €	Per ¥	Per MXN$	Per AU$
Canadian Dollar (C$)	——	0.9787	1.4012	0.0122	0.0823	1.0360
US Dollar (US$)	1.0218	——	1.4317	0.0125	0.0841	1.0585
Euro (€)	0.7137	0.6985	——	0.0087	0.0588	0.7394
Japanese Yen (¥)	82.0233	80.2765	114.9287	——	6.7540	84.9747
Mexican Peso (MXN$)	12.1445	11.8859	17.0165	0.1481	——	12.5814
Australian Dollar (AU$)	0.9652	0.9447	1.3525	0.0118	0.0795	——

Mechanics

Current Currency	Desired Currency
1. C$68,000	US$
2. ¥15,000,000	€
3. AU$3,000	MXN$
4. US$180,000	AU$
5. €230,500	C$
6. MXN$1,300,000	¥

Applications

7. If the exchange rate is ¥95.3422 per C$, what is the exchange rate for C$ per ¥?

8. Procter & Gamble just received payment for a large export of Tide in the amount of 275,000 Denmark kroner (DKK). If the exchange mid-rate is C$0.1992 per DKK, and the bank charges 3% on its buy rates, how many Canadian dollars will Procter & Gamble receive?

9. The exchange rate per US$ is C$0.9863. If the Canadian dollar depreciates by $0.0421 per US$, how many more or less US$ is C$12,500 able to purchase?

10. Jack is heading home to visit his family in Great Britain and decided to stop at the airport kiosk to convert his money. He needs to convert C$5,000 to British pounds (£). The exchange mid-rate per C$ is £0.5841. The kiosk charges a commission of 4.5% on the conversion, plus a flat fee of £5.00.

    a. How many pounds will Jack have?
    b. What is the percentage cost of his transaction?

11. Yarianni is heading on a vacation. She converts her C$4,000 into Chinese yuan renminbi (CNY) at a sell rate of CNY6.3802 per C$. While in China, she spends CNY14,000 of her money. At the airport, she converts her remaining money into Indian rupees (INR) at a sell rate of INR6.7803 per CNY. In India, she spends INR50,000. When she returns home, she converts her INR back to C$ at a buy rate of C$0.0231 per INR. How many Canadian dollars did she return with? Note that all currencies involved have two decimals.

12. Elena is an international investor. Four years ago she purchased 2,700 shares of a US firm at a price of US$23.11 per share when the exchange rate was US$0.7536 per C$. Today, she sold those shares at a price of US$19.87 per share when the exchange rate was US$1.0063 per C$. In Canadian dollars, determine how much money Elena earned or lost on her investment.

13. International Traders regularly imports products from Hong Kong. If the exchange rate of C$ per Hong Kong dollar (HKD) is 0.1378 and the Canadian dollar appreciates by HKD0.0128, by what amount would the cost of a HKD1,000,000 purchase increase or decrease in Canadian dollars for International Traders?

14. Brian needs to purchase some Brazilian reals (BRL). He takes C$7,500 to the bank and leaves the bank with BRL12,753.20. If the exchange mid-rate per C$ is 1.7621, determine the sell rate commission percentage (rounded to two decimals) charged by the bank.

Challenge, Critical Thinking, & Other Applications

15. Fernando could purchase a 55" Samsung HDTV in Winnipeg, Manitoba, for $2,999.99 plus taxes. Alternatively, he could head across the border on Black Friday and shop in Grand Forks, North Dakota, where the same product is selling for US$2,499.99 (plus 5% state sales tax and

1.75% local sales tax) at Best Buy. He estimates he would incur $65 in gasoline and vehicle wear and tear, $130 in accommodations, and $25 in food (all money in US$). He would make all purchases on his credit card, which uses the mid-rate plus 2.5%. When returning across the border, he would have to pay in Canadian dollars the 5% GST on the Canadian value of the HDTV not including taxes. Once home, he can then have the North Dakota government refund all taxes paid on the HDTV through their Canadian sales tax rebate program. For all currency exchanges, assume a mid-rate of US$0.9222 per C$. Which alternative is Fernando's better choice and by how much?

16. The current mid-rate is C$1.5832 per €. Scotiabank has a sell rate of C$1.6196 per € while an airport kiosk has a sell rate of C$1.6544 per € plus a service charge of C$4.75. You need to purchase €800.

   a. Calculate the fee percentages charged by each financial institution. Round your answers to one decimal.
   b. Rounded to two decimals, what percentage more than Scotiabank is the airport kiosk charging on your purchase?

17. Henri and Fran have retired and are considering two options for a two-month vacation in Europe. Their local Lethbridge travel agent is offering them an all-inclusive package deal at C$7,975 per person. Alternatively, they can book their own flights for C$1,425 per person, stay in Britain at a small apartment averaging £65 per night for 30 days, and then in France for €70 average per night for 30 days. Estimated groceries cost a total of £250 in Britain and €400 in France. They will need to purchase a Eurail pass for €986 each while they are there. The exchange rates are €0.6808 per C$ and £0.5062 per C$. Which alternative is their cheapest option and by how much in Canadian dollars?

18. A Canadian manufacturer imports three parts from different countries. It assembles the three parts into a finished item that is then exported to the United States. Every transaction always involves 25,000 units. Expenses average $6.25 per unit.

Component	Price per unit	Exchange rate last month per C$	Exchange rate this month per C$
Part A	¥1,500	¥107.9420	¥108.9328
Part B	AU$14.38	AU$1.1319	AU$1.0928
Part C	€10.73	€0.6808	€0.6569
Finished product export	forUS$59.45	US$1.0128	US$1.0243

19. In each of the following situations, convert the old amount to the new amount using the information provided.

	Old Amount	Old Exchange Rate per C$	Exchange Rate Change	New Amount
a.	US$625.00	US$0.9255	C$ appreciated by US$0.0213	? US$
b.	€16,232.00	€0.5839	C$ depreciated by €0.0388	? €
c.	¥156,500	¥93.4598	C$ depreciated by ¥6.2582	? ¥
d.	MXN$136,000	MXN$13.5869	C$ appreciated by MXN$0.4444	? MXN$

20. Compare the following four situations and determine which one would result in the largest sum of money expressed in Canadian funds.

Situation	Amount	Exchange Rate per C$
a.	65,204 Algerian dinars (DZD)	DZD65.5321
b.	1,807,852 Colombian pesos (COP)	COP1,781.1354
c.	3,692 Israeli new shekels (ILS)	ILS3.7672
d.	30,497 Thai baht (THB)	THB30.4208

Key Takeaways

Section 5.1: Sales Taxes (Everybody Wants a Piece of My Pie)

- The three sales taxes, current rates, and how to calculate prices including taxes
- How businesses complete GST/HST remittances

Section 5.2: Property Taxes (I Owe, I Owe)

- How municipalities levy mill rates and tax rates against

real estate owners

Section 5.3: Exchange Rates and Currency Exchange (It Is a Global World)

- Converting currencies through exchange rates
- The rise and decline of exchange rates: currency appreciation and depreciation

The Language of Business Mathematics

assessed value The portion of the market value of a property that is subject to municipal taxes.

buy rate The rate at which a foreign currency is bought; it will always be lower than the mid-rate in terms of money per unit of foreign currency.

cash discount A percentage of the balance owing on an invoice that can be deducted for payment received either in full or in part during the discount period.

credit period The number of interest-free days from the date of commencement before full payment of the invoice is required.

currency appreciation When one currency strengthens relative to another currency, resulting in an ability to purchase more of that other currency than it could previously.

currency depreciation When one currency weakens relative to another currency, resulting in an ability to purchase less of that other currency than it could previously.

date of commencement The first day from which the invoice terms extend forward in time and from which all due dates are established.

discount period The number of days from the date of commencement for which a cash discount is offered.

end-of-month invoice dating A term of payment where the date of commencement is the last day of the same month as indicated by the invoice date.

exchange rate The number of units of a foreign currency that are bought with one unit of the domestic currency, or vice versa.

GST The goods and services tax is a Canadian federal sales tax on most products purchased by businesses and consumers alike.

HST The harmonized sales tax is a tax that combines the GST and PST into a single tax.

market value A snapshot of the estimated selling price of a property at some specified point in time.

mid-rate An exchange rate that does not involve or provide for any charges for currency conversion.

mill rate A tax per $1,000 of assessed value to determine property taxes.

ordinary invoice dating A term of payment where the date of commencement is the same date as the invoice date.

property taxes Annual taxes paid by the owners of real estate to local levying authorities to pay for services such as roads, water, sewers, public schools, policing, fire departments, and other community services.

PST The provincial sales tax is a consumer sales tax administered by Canadian provinces and territories.

receipt-of-goods invoice dating A term of payment where the date of commencement is the day on which the customer physically receives the goods.

sales tax A percent fee levied by a government on the supply of products.

sell rate The rate at which a foreign currency is sold. It is always higher than the mid-rate in terms of money per unit of foreign currency.

tax policy A municipality-based percentage of the market value of a property that is used to convert the market value into an assessed value.

tax rate A tax per $100 of assessed value to determine property taxes.

tax remittance The fulfillment of a tax obligation.

The Formulas You Need to Know

Symbols Used

AV = assessed value of a property

Current Currency = currency you are converting from

Desired Currency = currency you are converting to

Exchange Rate = per-unit conversion rate for current currency

Property Tax = property tax amount

PTR = property tax rate, usually set on a per $100 (tax rate) or per $1,000 (mill rate) basis

Rate = sales tax

Remit = tax remittance

S = selling price

Stax = selling price including taxes

Tax Collected = total tax collected through sales

Tax Paid = total tax paid through purchases

Exercises

Mechanics

1. Suppose you own property with a market value of $412,000 in a municipality that has a 60% tax policy. What are your total property taxes if the mill rate is set at 15.4367?
2. Convert 140,000 Indian rupees (INR) to Belgian francs (BEF) if the exchange rate is BEF0.6590 per INR.
3. Calculate the price including taxes for a product that has a regular selling price of $995.95 in: a) Ontario b) Alberta c) Saskatchewan d)Prince Edward Island
4. An invoice for $15,000.00 dated February 16 of a leap year has terms of 3/15, 2/30. What payment in full is required if the payment is received on March 3?
5. The exchange rate is US$0.9063 per Canadian dollar. If over the course of a year the Canadian dollar appreciates by $0.0644 per C$, how many more or less US dollars are purchased with C$1,750?
6. A company recorded the following purchases and sales throughout the year. Assume all

amounts are taxable.

Quarter	Purchases	Sales
1st	$2,138,450.25	$1,444,297.93
2nd	$3,496,821.15	$4,989,201.01
3rd	$1,265,733.49	$6,422,604.19
4th	$2,725,706.28	$2,003,125.50

a) Calculate the GST remittance or refund per quarter.

b) Calculate the HST remittance or refund per quarter in Newfoundland and Labrador.

7) The assessed values in a municipality total $4.986 billion. If the city council needs an operating budget of $100 million, what mill rate should it set for its constituent property owners?

Applications

8. An invoice for $37,650 is dated June 24 with terms of 2/10, 1/30, ROG. Merchandise is received on June 29. If a payment of $20,000 is made on July 5 and another payment of $10,000 is made on July 26, what is the balance remaining on the invoice? Assume all payments are received on the dates indicated.

9. If the price including taxes for an item is $1,968.35 in Quebec, what is the dollar amount of each of the PST and GST on the item?

10. Obiwan is going on vacation to Slovakia and stops at a currency shop in his local airport. He needs to convert C$7,500 to the Slovakian koruna (SKK). The exchange mid-rate per C$ is 21.4624. The shop charges a commission of 6.5% on the conversion, plus a flat fee of C$15.75. How many koruna will Obiwan have?

11. A town in Saskatchewan has residential property values totalling $968 million in market value. The councillors have set the tax policy at 75% of market value. The following is a list of projected municipal expenditures in the coming year.

Sewers & water	$5,250,000
Public education	$13,500,000
City services (fire, police, ambulance)	$1,115,000
Roads & maintenance	$3,300,000
Other	$2,778,000

If the council expects to raise 85% of these expenditures from residential property taxes, what mill rate must it set?

12. What amount must be remitted if invoices dated January 25 for $1,700, February 1 for $1,800, and February 15 for $900, all with terms of 3/25, n/50, EOM are paid with the same cheque on March 12? Assume it is a non–leap year.

13. Many Canadians complain about the high cost of automobile gasoline. In the United States, regular grade gasoline costs approximately $3.652 per gallon (as of June 2011).

14. What is the equivalent Canadian cost per liter (rounded to three decimals) if the exchange rate is US$1.0152 per C$? A US gallon is equivalent to 3.7854 litres.

15. At the same time, Canadians were paying $1.239 per litre. What percentage more (rounded to two decimals in percent format) than Americans do Canadians pay?

16. An invoice for $13,398.25 dated August 27 with terms of 5/10, 2/25, n/45 was partially paid on September 7. If the invoice balance was reduced to $2,598.25, what was the amount of the payment? Assume the payment was received on the date indicated.

Challenge, Critical Thinking, & Other Applications

15. In Saint-Lazare, Quebec, the average market value of a residential home in 2010 rose to $308,000 from its previous 2009 value of $262,000. The residential property tax rate was lowered from 2009 to 2010 by 9.8918% to 0.583. In Saint-Lazare, the tax policy is to tax 100% of the market value. By what percentage did the residential property tax bill go up or down in 2010 for a Saint-Lazare residential homeowner?

16. What percentage of a tax-inclusive price is each of the GST and PST in the following provinces?

17. Quebec

18. Manitoba

19. It is well known that airport kiosks are much more expensive for currency exchange than visiting a bank. The current mid-rate is $2.4226 Brazilian reals (BRL) per euro. The Banco do Brasil has a sell rate of BRL$2.4710 per euro, while a kiosk at Salvador airport has a sell rate of BRL$2.5473 per euro plus a flat service charge of BRL$3.00. A traveller through the airport wants to convert BRL$2,000 to euros. Assuming the traveller has the time, would it be worth the BRL$15.00 return taxi ride to the nearest bank to convert the currency? How many euros more would the best option present?

20. An invoice for $147,477.77 dated July 26 with terms of 5/15, 3½/25, 2/40, EOM was partially paid with two equal payments on August 10 and September 4, reducing the balance owing to $76,579.81. What amount was paid on each of the two dates? Assume the payment was received on the date indicated.

19. An American tourist is travelling across Canada and makes the following purchases on her credit card (all purchases are fully taxable):

Total Purchase Amount before Taxes
$319.95
$114.75
$82.45
$183.85
$59.49
$123.99

On the first purchase, the credit card company used an exchange rate of C$1.0412 per US$. Each subsequent purchase used an exchange rate that appreciated (for the US dollar) by 0.2% each time. What total amount in US dollars was charged to the traveller's credit card?

20. A Canadian manufacturer imports 2,000 units of product worth US$262.25 each from an American supplier. These imports are subject to GST on the equivalent Canadian value of the product. All of the product is then sold for C$419.50 each to another Canadian distributor. The sale is subject to GST. Assume an exchange rate of US$0.9345 per C$. If the company needs to submit a GST remittance on this transaction, what amount is remitted or refunded?

Chapter 5 Case Study Completing Summary Financial Reports

The Situation

An accountant at Lightning Wholesale needs to summarize the GST remittances, costs of goods sold, and sales for 2013 by quarter. The table below presents the purchases (in US$) that it made from its American supplier, average monthly supplier cash discounts, total monthly sales (in C$), and the average monthly exchange rate appreciation or depreciation.

The Data

Quarter	Month	2013 Purchases in US$ Before Discounts	Average Cash Discount	2013 Sales in C$	C$ Appreciation ( ) or Depreciation (−) Against US$
I	January	$2,021	2.00%	$1,798	Start 0.9423
	February	$2,200	3.00%	$2,407	1.00%
	March	$2,531	2.50%	$2,568	−0.50%
II	April	$2,623	1.50%	$2,985	0.35%
	May	$2,676	1.00%	$3,114	−1.50%
	June	$2,805	2.00%	$3,242	1.75%
III	July	$3,309	2.25%	$3,306	1.00%
	August	$4,146	2.00%	$3,852	0.50%
	September	$6,196	3.00%	$4,815	−1.40%
IV	October	$10,981	3.00%	$7,222	−0.30%
	November	$8,332	3.25%	$12,839	0.90%
	December	$1,520	1.75%	$9,630	−0.75%

(all figures are in thousands of dollars)

Important Information

- The imported products require GST to be paid on the equivalent Canadian value of the imported goods before any discounts.
- All sales are subject to GST.
- Lightning Wholesale's suppliers offer it various cash discounts, which the accounting department always takes advantage of. These discounts have not been deducted from the purchase amounts listed in the table.

Your Tasks

1. Calculate the exchange rates for each month throughout the 2013 operating year. Round all exchange rates to four decimals before calculating the next exchange rate.
2. Compute quarterly GST remittances.
3. Convert monthly purchases into Canadian funds and determine the monthly GST paid to the CBSA (Canada Border Services Agency).
4. Assess the monthly GST collected on all sales.
5. Complete a quarterly GST remittance table summarizing GST paid, GST collected, and the net amount refunded or remitted in 2013.
6. Create a quarterly cost of goods sold and revenue table.
7. Apply the average monthly cash discount to the Canadian monthly purchases amounts. Round all answers to the nearest thousands of dollars.
8. Create a quarterly table summarizing costs of goods sold and sales with yearly totals of each.

Chapter 5: Accounting Applications (answers to section exercises)

- ### 5.1 Sales Taxes

  1. S_{tax} =$682.49
  2. S_{tax} =$750.74
  3. S_{tax} =$747.49
  4. S_{tax} =$727.99
  5. S = $796.45; HST Tax Amount = $103.54
  6. S = $818.17; GST Tax Amount = $40.91; PST Tax Amount = $40.91
  7. January Remit $5,471.40; February Refund $2,202.96
  8. Winter Remit $7,171.20; Spring Refund $1,025.25; Summer Remit $9,803.30; Fall Refund $1,191.65
  9. a.Alberta is best = $1,396.45; b. Savings = $17.48
  10. S =$529.99; GST Tax Amount = $26.50
  11. Can't purchase at S_{tax}=$3,050.94; Down Payment = $45.08
  12. S_{tax}=$4,477.30; GST Tax Amount = $199.88; PST Tax Amount = $279.83
  13. Q1 Remit $17,865; Q2 Remit $35,370; Q3 Refund $8,105; Q4 Refund $24,210

14. March Remit $951.15; April Refund; $1,257.15; May Remit $2,953.65
15. Higher in Ontario by 1.5274%
16. Total PST = $13.02; Total GST = $52.46; Total HST = $13
17. Remit $378,227.85
18. PST Rate = 9.505%
19. a. S = $352.72; b. S = $1,682.81; c. S = $19,465.09; d. S = $4,736.57
20. Remit $31,331.89

- ## 5.2 Property Taxes

1. AV = $176,000;
    Property Tax = $4,719.35
2. Market Value = 247,272.73; Property Tax = $2,699.00
3. Assessed Value = $400,000; Tax Policy = 80%
4. Market Value = $1,300,000; Tax Rate = 0.675832
5. Property Tax = $5,241.17
6. AV = $9,000,000
7. Property Tax = $2,250.05
8. Municipal Property Tax = $1,975.97; Library Property Tax = $220.01; School Property Tax = $1,481.76; Total Property Tax = $3,677.74
9. Tax Rate = 1.109305
10. Mill Rate = 8.9929
11. Increase the mill rate by 0.3626
12. Mill Rate = 14.5412
13. 10.0247 and 5.0124
14. $192,318.89 and $179,524.86
15. Mill Rates Under 70% Policy: 5.6872 then 5.6060; Mill Rates Under 75% Policy: 5.3081 then 5.2322

- ## 5.3 Exchange Rates and Currency Exchange

1. 69,482.40 US$
2. 130,500€
3. 37,744.20 AU$
4. 170,046 AU$

5. 322,976.60 C$
6. 8,780,200 ¥
7. 0.0103 ¥ per C$
8. 53,130 C$
9. Purchase $518.75 less than before
10. a. 2,784 £; b. 4.6739%
11. 649.44 C$
12. Elena lost $29,489.81 on her investment
13. The products cost $12,800 C$ less than before
14. 3.5%.
15. Buy in the U.S. to save $196.60
16. a.Scotiabank's sell rate fee is 2.3%; The kiosk's sell rate fee is 4.5%; b. The airport kiosk charges 2.52% more than Scotiabank
17. Booking it themselves saves $1,824.98
18. Currency fluctuations have decreased profits by $38,432.12
19. a. 639.38 US$; b. €15,153.19; c. ¥166,983; d. 140,447.70 MXN$
20. Columbia Pesos = $1,084.71 C$

6.

SIMPLE INTEREST

<div style="background-color:#c8591a; color:white;">

Chapter 6 Topics

</div>

6.1 Principal, Rate, Time

6.2 Moving Money Involving Simple Interest

6.3 Savings Accounts and Short-Term GICs

If you have a lot of surplus cash, it would hardly make sense to hide it under your mattress or bury it in the backyard of your townhouse. As any investment specialist will tell you, you want your money to work for you instead of just sitting idle. So you can invest and be rewarded by earning interest.

Do you know if your account at the Royal Bank of Canada (RBC) has paid you the right amount of interest? You might assume that because everything at a bank is fully automated it is guaranteed to be correct. However, ultimately all automation at the Royal Bank relies on input by an employee. What if your interest rate is keyed in wrong? Maybe the computer programmer erred in a line of computer code and the amount of interest is miscalculated. What if RBC had a computer glitch? These pitfalls highlight the importance of always checking your account statements.

In business, it would be rare for an organization to borrow or lend money for free. That is not to say that businesses are ruthless with each other; in the previous chapter, you saw that most business transactions are completed through interest-free credit transactions involving invoicing. However, this generosity does not extend indefinitely. After the credit period elapses, the business essentially treats an unpaid invoice as a loan and starts charging an interest penalty.

When businesses need to borrow money, promissory notes, demand loans, and commercial papers are just three of the possibilities discussed later in this chapter.

Even governments need to borrow money. Have you ever thought about how the government of Canada or your provincial government goes about doing that? While a government has many methods at its disposal, a common method is to use treasury bills—in essence, borrowing the money from investors.

On a personal level, almost all people invest and borrow money at some time. This chapter examines consumer lines of credit and student loans, both of which carry simple interest rates. On the earnings side, it is common for people to have savings accounts that earn low rates of interest. Other short-term investments include treasury bills, commercial papers, and short-term guaranteed investment certificates (GICs).

The bottom line is that money does not come free. In this chapter, you will explore the concept of simple interest, learn how to calculate it, and then apply simple interest to various financial tools.

6.1 Principal, Rate, Time

Your investments may be at risk if stock and bond markets slump, as a story in the *Globe and Mail* predicts. You wonder if you should shift your money into relatively secure short-term investments until the market booms again. You consider your high-interest savings account, but realize that only the first $60,000 of your savings account is insured. Perhaps you should put some of that money into treasury bills instead.

Looking ahead, what income will you live on once you are no longer working? As your career develops, you need to save money to fund your lifestyle in retirement. Some day you will have $100,000 or more that you must invest and reinvest to reach your financial retirement goals.

To make such decisions, you must first understand how to calculate simple interest. Second, you need to understand the characteristics of the various financial options that use simple interest. Armed with this knowledge, you can make smart financial decisions!

The world of finance calculates interest in two different ways:

1. **Simple Interest.** A **simple interest** system primarily applies to short-term financial transactions, with a time frame of less than one year. In this system, which is explored in this chapter, interest accrues but does not compound.
2. **Compound Interest.** A **compound interest** system primarily applies to long-term financial transactions, with a time frame of one year or more. In this system, which the next chapter explores, interest accrues and compounds upon previously earned interest.

Simple Interest

In a simple interest environment, you calculate interest solely on the amount of money at the beginning of the transaction. When the term of the transaction ends, you add the amount of the simple interest to the initial amount. Therefore, throughout the entire transaction the amount of money placed into the account remains unchanged until the term expires. It is only on this date that the amount of money increases. Thus, an investor has more money or a borrower owes more money at the end.

How Simple Interest Works
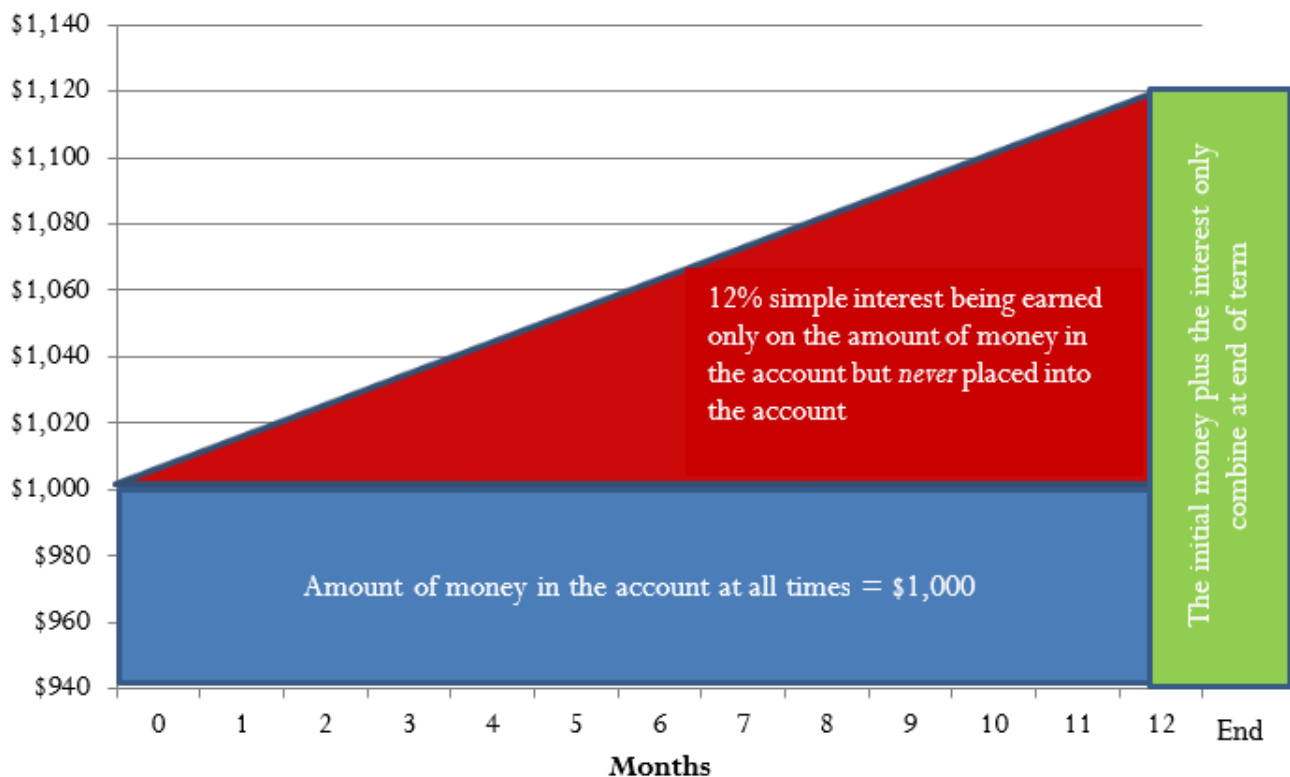
The figure illustrates the concept of simple interest. In this example, assume $1,000 is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals $1,000. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the $120 of interest and the initial $1,000 are then combined to total $1,120.

A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

- If you take out a personal loan from a bank, the bank gives you the money and you receive the money. In

this situation, the bank earns the simple interest and you are being charged simple interest on your loan. In the figure, this means you must pay back not only the $1,000 you borrowed initially but an additional $120 in interest.

• If you place your money into an investment account at the bank, you have given the money and the bank has received the money. In this situation, you earn the simple interest on your money and the bank pays simple interest to your investment account. In the figure, this means the bank must give you back your initial $1,000 at the end plus an additional $120 of interest earned.

The Formula

The best way to understand how simple interest is calculated is to think of the following relationship:

Amount of simple interest = *How much* at *What simple interest rate* for *How long*

Notice that the key variables are the amount, the simple interest rate, and time. Formula 6.1 combines these elements into a formula for simple interest.

Simple Interest

I is Interest Amount: The interest amount is the dollar amount of interest that is paid or earned. To interpret the interest amount properly, remember who you are in the transaction. If you are borrowing the money, this is the interest amount charged to you. If you are investing the money, this is the interest amount you earn.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed is known as the present value, or P. If this is in fact the amount at the start of the financial transaction, it is also called the principal, which is the original amount of money that is borrowed or invested. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 8.1 – Simple Interest: $I = Prt$

r is Interest Rate: The interest rate is the rate of interest that is charged or earned during a specified time period. It is usually expressed in percent format. Unless noted otherwise, interest rates are expressed on an annual basis. An interest rate is the result of Formula 2.2, where

$$Rate = \frac{Portion}{Base}$$

In this case, you calculate an annual interest rate in its decimal format as follows:

$$Annual\ Interest\ Rate = \frac{Annual\ Interest\ Amount}{Principal}$$

Thus, if you are earning $3 of interest annually on a $100 principal, then your annual interest rate is 3/100 = 0.03 or 3%.

t is Time: The time period or term is the length of the financial transaction for which interest is charged or earned.

How It Works

Follow these steps when you calculate the amount of simple interest:

Step 1: Formula 6.1 has four variables, and you need to identify three for any calculation involving simple interest. If necessary, draw a timeline to illustrate how the money is being moved over time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If they are not already, you need to convert one of the two variables to the same units as the other.

Step 3: Apply Formula 6.1 and solve for the unknown variable. Use algebra to manipulate the formula if necessary.

Assume you have $500 earning 3% simple interest for a period of nine months. How much interest do you earn?

Step 1: Note that your principal is $500, or P = $500. The interest rate is assumed to be annual, so *r* = 3% per year. The time period is nine months.

Step 2: Convert the time period from months to years: $t = \frac{9}{12}$

Step 3: According to Formula 6.1, *I* = $500 × 3% × $\frac{9}{12}$

= $11.25. Therefore, the amount of interest you earn on the $500 investment over the course of nine months is $11.25.

Important Notes

Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for Formula 6.1, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual interest rate for nine months, then either

- The time needs to be expressed annually as 9/12 of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as 3%/12 = 0.25% per month to match the number of months.
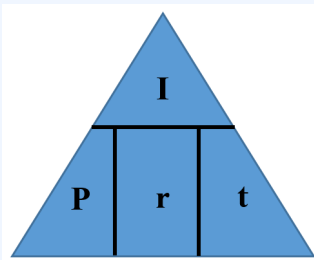
Paths To Success

Four variables are involved in the simple interest formula, which means that any three can be known, requiring you to solve for the fourth missing variable. To reduce formula clutter, the triangle technique illustrated here helps you remember how to rearrange the formula as needed.

Give It Some Thought

1. If you have money in your savings account (or any other investment) and it earns simple interest over a period of time, would you have more or less money in your account in the future?
2. If you have a debt for which you haven't made any payments, yet it is being charged simple interest on the principal, would you owe more or less money in the future?

Solutions:

1. More, because the interest is earned and therefore is added to your savings account.

2. More, because you owe the principal and you owe the interest, which increases your total amount owing.

Example 6.1A: How Much Interest Is Owed?

Start   r = 5% annually   t = 5 months

P = $1,100   I = $?
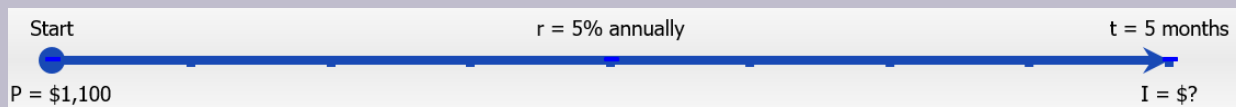
[table id=46 /]

Example 6.1B: Do You Know Your Interest Rate?

Start   r = ?% annually   t = 6 months

P = $3,500   I = $70
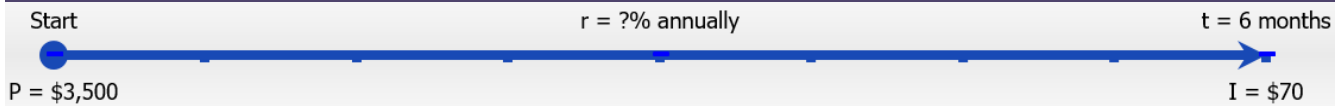
[table id=48 /]

Example 6.1C: What Did You Start with?

Start   r = 6% annually   t = 11 months

P = $?   I = $2,035
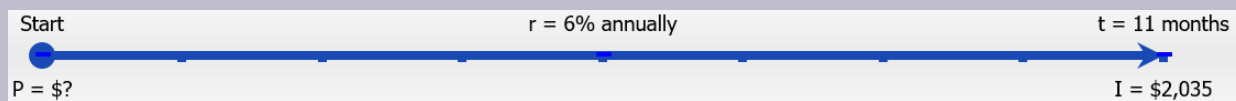
[table id=49 /]

Example 6.1D: How Long?

Start   r = 5% annually   t = ? months

P = $95,000   I = $1,187.50
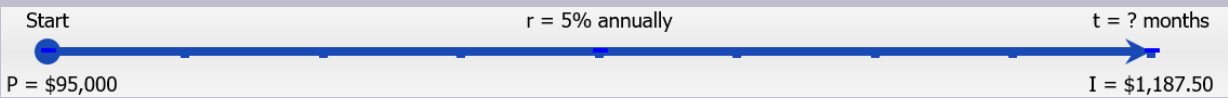
[table id=52 /]

Time and Dates

In the examples of simple interest so far, the time period equalled an exact number of months. While this is convenient in many situations, financial institutions and organizations calculate interest based on the exact number of days in the transaction, which changes the interest amount.

To illustrate this, assume you had money saved for the entire months of July and August, where $t = \frac{2}{12}$ or $t = 0.1\overline{6}$ of a year. However, if you use the exact number of days, the 31 days in July and 31 days in August total 62 days. In a 365-day year that is $t = \frac{62}{365}\%$

or $t = 0.169863$ of a year. Notice a difference of 0.003196 occurs. Therefore, to be precise in performing simple interest calculations, you must calculate the exact number of days involved in the transaction.

Figuring Out the Days

In practice, when counting the number of days in a transaction you include the first day but not the last day.

This is because interest is calculated on the daily closing balance, not on interim balances throughout a day. For example, if you borrowed $500 on June 3 and paid it back with interest on June 5, the closing balance on June 3 and June 4 is $500. However, on June 5 a zero balance is restored. Therefore, you count June 3 and 4 as days of interest but not June 5, so you will owe two days of interest, not three.

There are three common ways to calculate the number of days involved in a transaction.

Method 1: Use Your Knuckles

You can use your knuckles as an aid to remembering which months of the year have more or fewer days.
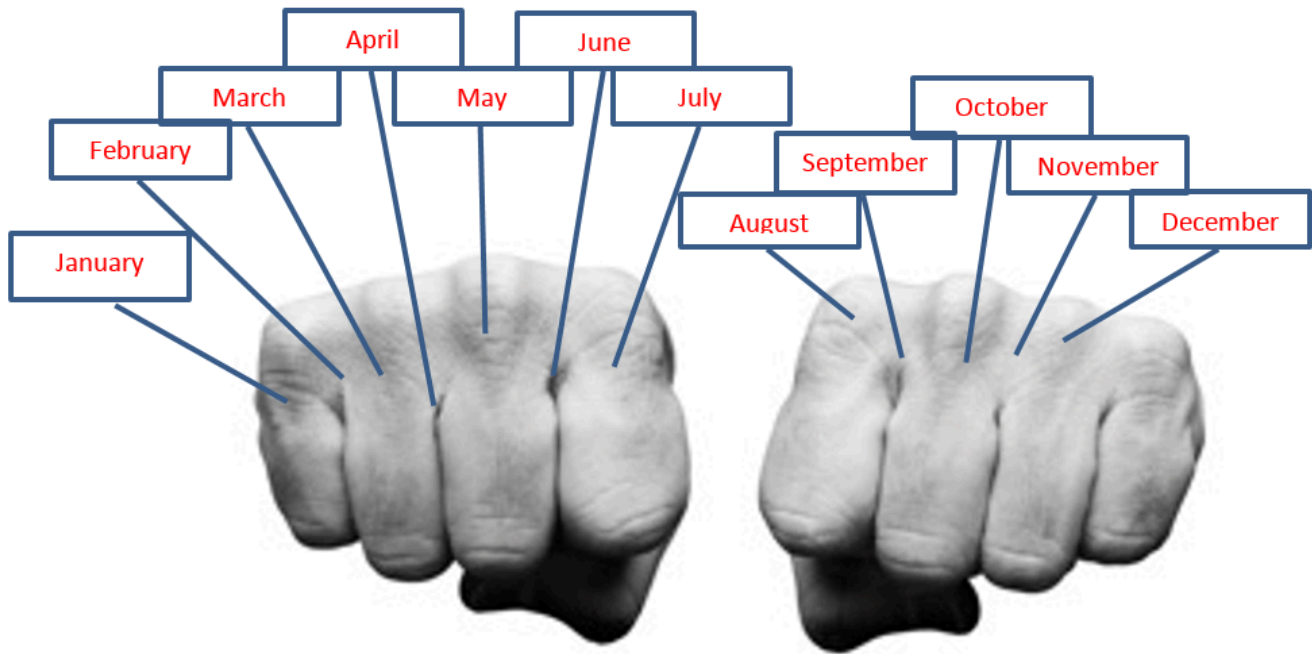
Put your knuckles side by side and start on your outermost knuckle. Every knuckle represents a month with 31 days. Every valley between the knuckles represents a month that does not have 31 days (30 in all except February, which has 28 or 29). Now start counting months. Your first knuckle (pinkie finger) is January, which has 31 days. Next is your first valley, which is February and it does not have 31 days. Your next knuckle (ring finger) is March, which has 31 days … and so on. When you get to your last knuckle (your index finger), move to the other hand's first knuckle (an index finger again). Notice that July and August are the two months back to back with 31 days.

If you are trying to calculate the number of days between March 20 and May 4, use your knuckles to recall that March has 31 days and April has 30 days. The calculations are illustrated in the table here.

Start Date	End Date	Days between Dates
March 20	March 31	$31 - 20 = 11$ days
March 31 (which precedes April 1, so it is day "0" for April)	April 30	$30 - 0 = 30$ days
April 30 (which precedes May 1, so it is day "0" for May)	May 4	$4 - 0 = 4$ days
TOTAL		$11 + 30 + 4 = 45$ days

Why does the end date of one line become the start date of the next line? Recall that you count the first day, but not the last day. Therefore, on the first line March 31 has not been counted yet and must be counted on the second line. Ultimately a transaction extending from March 20 to May 4 involves 45 days as the time period.

Method 2: Use A Table of Serial Numbers

In the table on the next page, the days of the year are assigned a serial number. The number of days between any two dates in the same calendar year is simply the difference between the serial numbers for the dates.

Using the example in Method 1, when the dates are within the same calendar year, find the serial number for the later date, May 4 (Day 124) and the serial number for the earlier date, March 20 (Day 79). Then, find the difference between the two serial numbers (124 – 79 = 45 days).

Day of Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29	*	88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

*For Leap years, February 29 becomes Day 60 and the serial number for each subsequent day is increased by 1.

Method 3: Use The BAII Plus Date Function

Your BAII Plus calculator is programmed with the ability to calculate the number of days involved in a transaction according to the principle of counting the first day but not the last day. The function "Date" is located on the second shelf above the number one. To use this function, open the window by pressing 2nd 1.

- Scroll amongst the four display lines by using your ↑ and ↓ arrows. There are four variables:

1. DT1 is the starting date of the transaction.
2. DT2 is the ending date of the transaction.
3. DBD is the days between dates, which is your *t* in Formula 6.1.
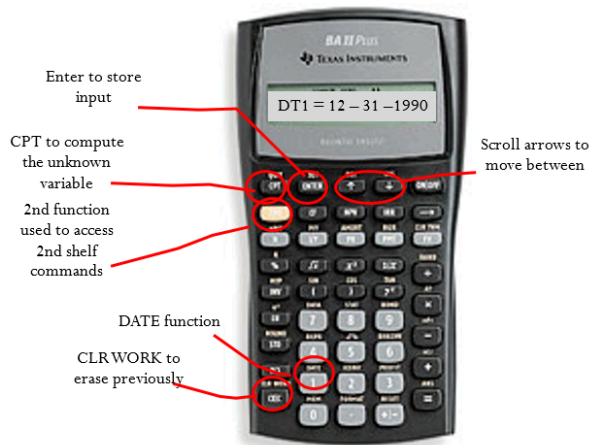
4. ACT or 360 is the method of calculating the days between dates. In ACT (for "actual"), the calculator calculates the DBD based on the correct number of days in the month. In 360, the DBD is based on treating every month as having 30 days (30 × 12 = 360 days). For all calculations, set the calculator to the ACT mode. Toggle this setting by pressing 2nd SET.

- To enter a date into the calculator, the calculator uses the format of MM.DDYY where:
- MM is the numerical value of the month; you do not need to enter the leading zero on months with only one digit.
- DD is the day of the month; you must always enter this value with two digits using a leading zero for days 1 through 9 in a month.
- YY is the last two digits of the year; you must always enter this with two digits. If your question does not involve a year, select an appropriate year of your choice.
- As long as you know any two of the DT1, DT2, or DBD variables, you can calculate the third. Enter any two of the three variables, pressing ENTER after each and ensuring the window shows the output variable you are seeking, and then press CPT.

Using the example in Method 1, to calculate the time from March 20 to May 4 you first need to choose a year arbitrarily, perhaps 2011. Therefore, perform the following sequence:

2nd DATE

2nd CLR Work (to erase previous calculations)

3.2011 ENTER ↓ (this is March 20, 2011)

5.0411 ENTER ↓ (this is May 4, 2011)

CPT

Answer: 45

Note that the answer is the same as the one you arrived at manually using your knuckles.

Important Notes

When solving for t, decimals may appear in your solution. For example, if calculating t in days, the answer may show up as 45.9978 or 46.0023 days; however, interest is calculated only on complete days. This occurs because the interest amount (I) used in the calculation has been rounded off to two decimals. Since the interest amount is imprecise, the calculation of t is imprecise. When this occurs, round t off to the nearest integer.

Things To Watch Out For

Miscalculating the number of days in February has to be one of the most common errors. To help you remember how many days are in February, recall your knowledge of leap years.

- February has 28 days except for leap years, in which it has 29 days.
- Leap years are those years that are evenly divisible by four or 400. However, those years that are evenly divisible by 100 (excepting years that are evenly divisible by 400) are not leap years. Here are some examples to illustrate this rule:

1. The year 1900 was not a leap year, since it is evenly divisible by 100 and not by 400.
2. The year 1996 was a leap year, since it is evenly divisible by four and not by 100.
3. The year 2000 was one of those exception years, and it was a leap year since it is evenly divisible by 400.
4. The year 2004 was once again a leap year since it is evenly divisible by four and not by 100. Thus, in this century 2004, 2008, 2012, 2016, 2020, 2024, and so on by fours are all leap years with 29 days.
5. The year 2100 will not be a leap year, since it is evenly divisible by 100.

Note that when it comes to the total number of days in a year, simple interest calculations ignore the 366th day in a leap year. Therefore, assume a year has 365 days in all calculations.

Paths To Success

When entering two dates either on the calculator or in Excel, the order in which you key in the dates is not important. If you happen to put the last day of the transaction into the first field (DT1 or Start Date) and the start day of the transaction into the last field (DT2 or End Date), then the number of days will compute as a negative number since the dates are reversed. Ignore the negative sign in these instances. Using the example, on your calculator if you had keyed May 4 into DT1 and March 20 into DT2, the DBD computes as −45, in which you ignore the negative sign to determine the answer is 45 days.

Example 6.1E: Time Using Dates

Start Date Unknown   r = 9% annually   September 13, 2011

P = $15,000   I = $1,283.42
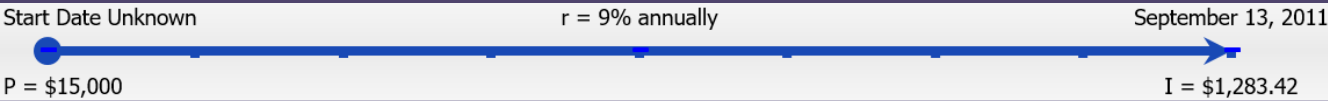
[table id=53 /]

Calculator Instructions

To calculate the start date, use the DATE function:

DT1	DT2	DBD	Mode
Answer: 10-01-2010	9.1311	347	ACT

Example 6.1F: Figuring Out Dates Using Time

[table id=54 /]

Variable Interest Rates

Not all interest rates remain constant. There are two types of interest rates:

1. **Fixed.** A **fixed interest rate** is an interest rate that is unchanged for the duration of the transaction.
2. **Variable.** A **variable interest rate** is an interest rate that is open to fluctuations over the duration of a transaction. This variable interest rate is usually tied to the **prime rate**, which is an interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans. Most people and companies are usually charged the prime rate plus a percentage (typically 0.5% to 5%) to arrive at the variable interest rate. The Bank of Canada periodically adjusts this rate because of economic and financial considerations in Canada.

In all of the previous examples in this section, the interest rate (r) was fixed for the duration of the transaction. When r fluctuates, you must break the question into time periods or time fragments for each value of r. In each of these time fragments, you calculate the amount of simple interest earned or charged and then sum the interest amounts to form the total interest earned or charged.
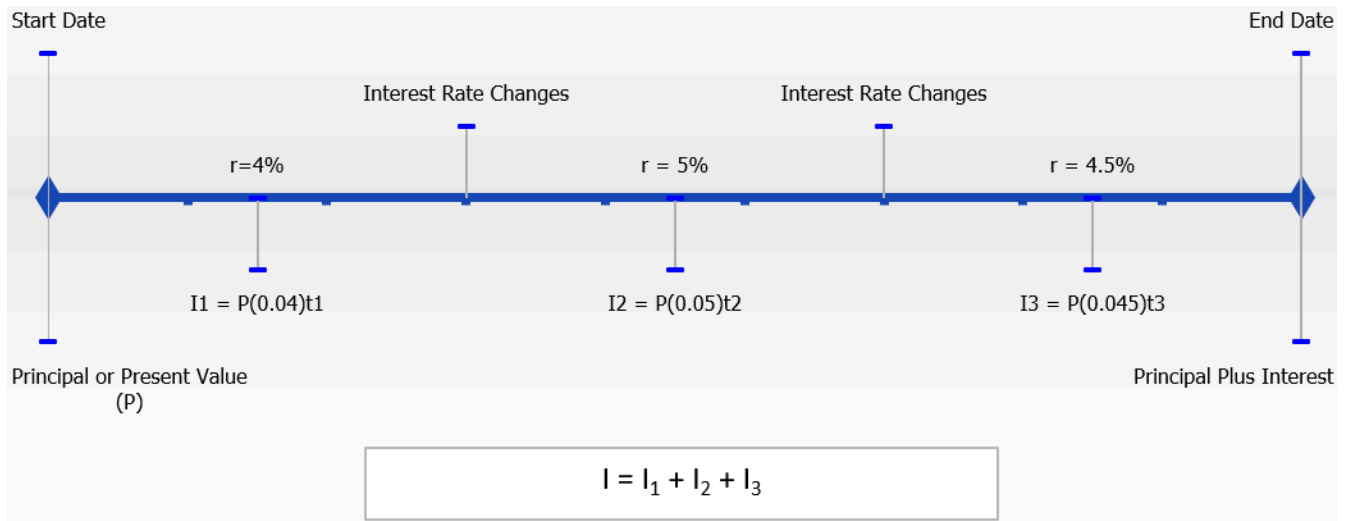
This figure illustrates this process with a variable rate that changes twice in the course of the transaction. To calculate the total interest amount, you need to execute Formula 6.1 for each of the time fragments. Then total the interest amounts from each of the three time fragments to calculate the interest amount (I) for the entire transaction.

Example 6.1G: A Floating Rate

[table id=55 /]

Calculator Instructions
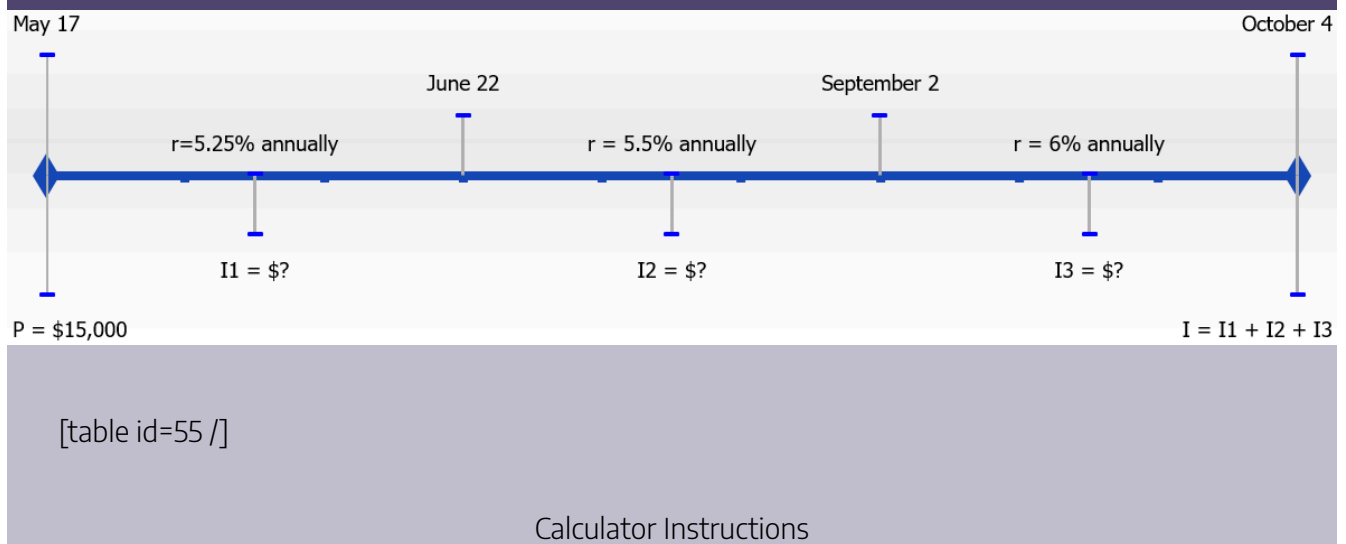
Arbitrarily pick the year 2011 and use the DATE function:

Time Fragment	DT1	DT2	DBD	Mode
1	5.1711	6.2211	Answer: 36	ACT
2	6.2211	9.0211	Answer: 72	ACT
3	9.0211	10.0411	Answer: 32	ACT

Exercises

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

Interest Amount	Principal or Present Value	Interest Rate	Time
1. ?	$130,000.00	8%	9 months
2. $4,000.00	?	2% per month	8 months
3. $1,437.50	$57,500.00	? per year	4 months
4. $103.13	$1,250.00	9%	? months
5. ?	$42,250.00	1½% per month	½ year
6. $1,350.00	?	6¾%	3 months

Applications

7. Brynn borrowed $25,000 at 1% per month from a family friend to start her entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?

8. Alex took out a variable rate $20,000 loan on July 3 at prime + 4% when prime was set at 3%. The prime rate increased ½% on August 15. How much simple interest does Alex owe if the loan is paid back on September 29?

9. How much simple interest is earned on $50,000 over 320 days if the interest rate is:

10. 3%

11. 6%

12. 9%

13. What relationship is evident between simple interest amounts and rates in your three answers above?

14. If you placed $2,000 into an investment account earning 3% simple interest, how many months does it take for you to have $2,025 in your account?

15. Cuthbert put $15,000 into a nine-month term deposit earning simple interest. After six months he decided to cash the investment in early, taking a penalty of 1% on his interest rate. If he received $393.75 of interest, what was the original interest rate before the penalty on his

term deposit?

16. If you want to earn $1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?

17. On March 3, when the posted prime rate was 4%, an investment of $10,000 was placed into an account earning prime + 2½%. Effective June 26, the prime rate rose by ¾%, and on October 4 it rose another ½%. On November 2, it declined by ¼%. If the money was withdrawn on December 15, how much simple interest did it earn?

18. If $6,000 principal plus $132.90 of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?

19. Jessica decided to invest her $11,000 in two back-to-back three-month term deposits. On the first three-month term, she earned $110 of interest. If she placed both the principal and the interest into the second three-month term deposit and earned $145.82 of interest, how much higher or lower was the interest rate on the second term deposit?

Challenge, Critical Thinking, & Other Applications

16. The simple interest formula applies to any instance of constant unit growth. Assume that 200 units of product are sold today and you forecast a constant 20-unit sales increase every four weeks. What will be the sales in one year's time (assume a year has exactly 52 weeks)?

17. Cade received an invoice for $5,200 dated August 13 with terms of 2/10, net 30 with a stated penalty of 2% simple interest per month. If Cade pays this invoice on December 13, what penalty amount is added to the invoice?

18. Marrina is searching for the best way to invest her $10,000. One financial institution offers 4.25% on three-month term deposits and 4.5% on six-month term deposits. Marrina is considering either doing two back-to-back three-month term deposits or just taking the six-month deposit. It is almost certain that interest rates will rise by 0.5% before her first three-month term is up. She will place the simple interest and principal from the first three-month term deposit into the second three-month deposit. Which option should Marrina pursue? How much better is your recommended option?

19. Goliath borrowed $20,000 on January 3 at prime + 4.5%. The prime rate changed throughout his term (of a non–leap year) as follows:

Date Effective	Prime Rate
January 3	2.25%
February 17	2.5%
April 30	3%
June 15	2.75%
September 15	3.25%

1. If he repaid his loan on December 24, what amount of simple interest should Goliath be charged?
2. What equivalent fixed rate of simple interest did he get charged on his loan?
3. Evaluate each of the following $10,000 investment alternatives and recommend the best alternative for investing any principal regardless of the actual amount. Assume in all cases that the principal and simple interest earned in prior terms are placed into subsequent investments.

Alternative 1	Alternative 2	Alternative 3
6% for 1 year	5% for 6 months, then 7% for 6 months	5.25% for 3 months, then 5.75% for 3 months, then 6.25% for 3 months, then 6.75% for 3 months

What percentage more interest is earned in the best alternative versus the worst alternative?

6.2 Moving Money Involving Simple Interest

Can you calculate the amount of money required to meet a future goal? Assume you will graduate college with your business administration diploma in a few months and have already registered at your local university to continue with your studies toward a bachelor of commerce degree. You estimate your total tuition, fees, and textbooks at $8,000. After investigating some short-term investments, you find the best simple rate of interest obtainable is 4.5%. How much money must you invest today for it to grow with interest to the needed tuition money?

In the previous section, you calculated the amount of interest earned or charged on an investment or loan. While this number is good to know, most of the time investors are interested solely in how much in total, including both principal and interest, is either owed or saved. Also, to calculate the interest amount in Formula 6.1, you must know the principal. When people plan for the future, they know the future amount of money that they need but do not know how much money they must invest today to arrive at that goal. This is the case in the opening example above, so you need a further technique for handling simple interest.

This section explores how to calculate the principal and interest together in a single calculation. It adds the flexibility of figuring out how much money there is at the beginning of the time period so long as you know the value at the end, or vice versa. In this way, you can solve almost any simple interest situation.

Maturity Value (or Future Value)

The **maturity value** of a transaction is the amount of money resulting at the end of a transaction, an amount that includes both the interest and the principal together. It is called a maturity value because in the financial world the termination of a financial transaction is known as the "maturing" of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the **future value**.

Simple Interest for Single Payments

S is Future Value or Maturity Value: This is the combined value of the principal and the simple interest together at a future point in time.

P is Present Value or Principal: The amount of money at the beginning of the time period being analyzed. If this is in fact the amount at the start of the financial transaction, it is the principal. Or it can simply be the amount at some earlier time before the future value was known. In any case, the amount excludes the interest.

Formula 8.2 – Simple Interest for Single Payments: $S = P(1 + rt)$

r is Interest Rate: The interest rate in Formula 8.2 operates in the same manner as in Formula 8.1. It is the simple rate of interest charged or earned during the specified time period. It is usually expressed annually and must share a common unit with the time period.

t is Time: This is the length of time for the financial transaction in which interest is charged or earned. It must be expressed in a common unit with the interest rate.

Depending on the financial scenario, what information you know, and what variable you need to calculate, you may need a second formula offering an alternative method of calculating the simple interest dollar amount.

Interest Amount

I is Interest Amount: The interest amount is the dollar amount of interest earned or charged over the span of a financial transaction involving a single payment.

S is Future Value or Maturity Value: The amount of money in the future (or at maturity) including both the principal (or present value) and the interest together.

Formula 8.3 – Interest Amount for Single Payments: $I = S - P$

P is Present Value or Principal: The amount of money at the start of the transaction or at some other time before maturity that excludes the interest involved in the transaction.

How It Works

Follow these steps when working with single payments involving simple interest:

Step 1: Formula 6.2 has four variables, any three of which you must identify to work with a single payment involving simple interest.

- If either the P or S is missing but the *I* is known, apply Formula 6.3 to determine the unknown value.
- If only the interest amount *I* is known and neither P nor S is identified, recall that the interest amount is a portion and that the interest rate is a product of rate and time (e.g., 6% annually for nine months is an earned interest rate of 4.5%). Apply Formula 2.2 on rate, portion, and base to solve for base, which is the present value or P.
- If necessary, draw a timeline to illustrate how the money is being moved through time.

Step 2: Ensure that the simple interest rate and the time period are expressed with a common unit. If not, you need to convert one of the two variables to achieve the commonality.

Step 3: Apply Formula 6.2 and solve for the unknown variable, manipulating the formula as required.

Step 4: If you need to calculate the amount of interest, apply Formula 6.3.

Assume that today you have $10,000 that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

Step 1: The principal is P = $10,000, the simple interest rate is 7% per year, or $r = 0.07$, and the time is $t = 11$ months.

Step 2: The time is in months, but to match the rate it needs to be expressed annually as $t = \frac{11}{12}$

Step 3: Applying Formula 6.2 to calculate the future value including interest,

$S = \$10{,}000 \times (1 + 0.07 \times \frac{11}{12}$

$) = \$10{,}641.67$. This is the total amount after 11 months.

Step 4: Applying Formula 6.3 to calculate the interest earned, $I = \$10{,}641.67 - \$10{,}000.00 = \$641.67$. You could also apply Formula 6.1 to calculate the interest amount; however, Formula 6.3 is a lot easier to use. The $10,000 earns $641.67 in simple interest over the next 11 months, resulting in $10,641.67 altogether.

Things To Watch Out For

As with Formula 6.1, the most common mistake in the application of Formula 6.2 is failing to ensure that the rate and time are expressed in the same units. Before you proceed, always check these two variables!

Paths To Success

When solving Formula 6.2 for either rate or time, it is generally easier to use Formulas 6.3 and 6.1 instead. Follow these steps to solve for rate or time:

- If you have been provided with both the present value and future value, apply Formula 6.3 to calculate the amount of interest (I).
- Apply and rearrange Formula 6.1 to solve for either rate or time as required.

Give It Some Thought

In each of the following situations, determine which statement is correct.

- If you have a future value of $5,000 and interest is involved, then your present value is
- More than $5,000
- The same
- Less than $5,000
- If you had a principal of $10,000 earning 10% simple interest for any period less than a year, you earn
- More than $1,000 interest
- Exactly $1,000 interest
- Less than $1,000 interest
- If you took out a loan for $2,500 at a 5% simple interest rate for six months and you made no payments during that time frame, you owe
- More than $2,500
- Exactly $2,500

Solutions: Less than $5,000. When being charged or earning interest, the present value is always less than the future value. Less than $1,000 interest. The $10,000 earns exactly $1,000 interest per year, so any time frame less than a year produces a smaller interest amount.More than $2,500. When interest is being charged, the future value is always larger than the present value.

Less than $2,500

Example 6.2A: Saving for a Down Payment on a Home

Today   r = 4.25% annually   4 Months

P = $35,000   S= $?
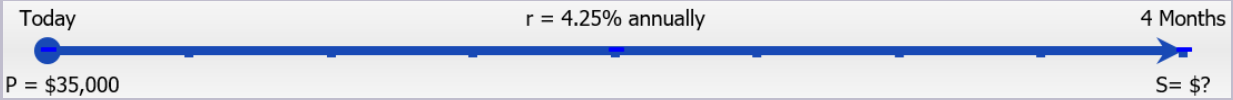
[table id=56 /]

Example 6.2B: Saving for Tuition

Today   r = 4.5% annually   8 Months

P = $?   S= $8,000
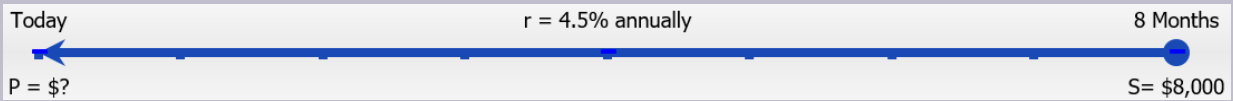
[table id=57 /]

Example 6.2C: What Exactly Are You Being Offered?

August 4   r = ?% annually   September 1

P = $20,000   S = $20,168.77
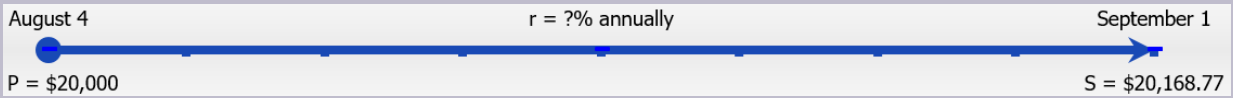
[table id=58 /]

Assume the year 2011 and use the DATE function:			
DT1	DT2	DBD	Mode
08.0411	09.0111	Answer: 28	ACT

Equivalent Payments

Life happens. Sometimes the best laid financial plans go unfulfilled. Perhaps you have lost your job. Maybe a reckless driver totalled your vehicle, which you now have to replace at an expense you must struggle to fit into your budget. No matter the reason, you find yourself unable to make your debt payment as promised.

On the positive side, maybe you just received a large inheritance unexpectedly. What if you bought a scratch ticket and just won $25,000? Now that you have the money, you might want to pay off that debt early. Can it be done?

Whether paying late or paying early, any amount paid must be equivalent to the original financial obligation. As you have learned, when you move money into the future it accumulates simple interest. When you move money into the past, simple interest must be removed from the money. This principle applies both to early and late payments:

- **Late Payments.** If a debt is paid late, then a financial penalty that is fair to both parties involved should be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe $100 to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of $10 should be added, making your debt payment $110. This is no different from your friend receiving the $100 today and investing it himself at 10% interest so that it accumulates to $110 in one year.
- **Early Payments.** If a debt is paid early, there should be some financial incentive (otherwise, why bother?). Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend $110 one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of $10 should be deducted, making your debt payment today $100. If your friend then invests this money at 10% simple interest, one year from now he will have the $110, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means $100 today is the same thing as having $110 one year from now. This illustrates the concept that two payments are **equivalent payments** if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.
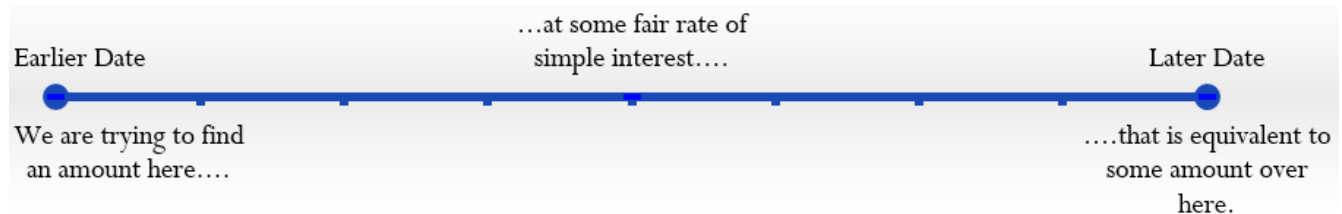
How It Works

The steps required to calculate an equivalent payment are no different from those for single payments. If an early payment is being made, then you know the future value, so you solve for the present value (which removes the interest). If a late payment is being made, then you know the present value, so you solve for the future value (which adds the interest penalty).

Paths To Success

Being financially smart means paying attention to when you make your debt payments. If you receive no financial benefit for making an early payment, then why make it? The prudent choice is to keep the money yourself, invest it at the best interest rate possible, and pay the debt off when it comes due. Whatever interest is earned is yours to keep and you still fulfill your debt obligations in a timely manner!

Example 6.2D: Making a Late Payment
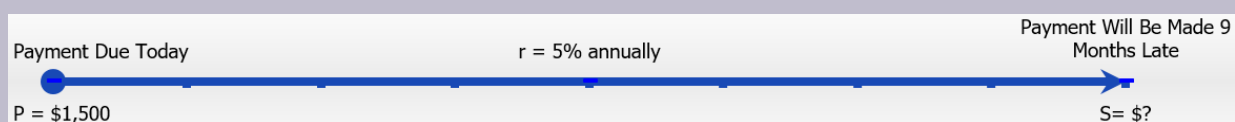
[table id=59 /]

Example 6.2E: Making an Early Payment
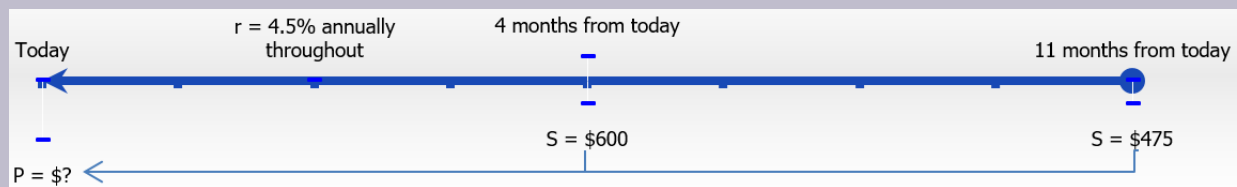
[table id=60 /]

Exercises

For each of the following questions, round all money to two decimals and percentages to four decimals.

Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided.

	Interest Amount	Principal / Present Value	Interest Rate	Time	Maturity / Future Value
1.	?	$16,775.00	0.5% per month	6 months	?
2.	?	?	9%	2 months	$61,915.00
3.	$1,171.44	?	5¾%	? months	$23,394.44
4.	$2,073.67	$41,679.00	?	227 days	?

Questions 5–8 involve either early or late payments. Calculate the following for each:

- The dollar amount of the early or late payment
- The amount of interest that forms the late penalty or early benefit

	Original Payments	Proposed Payments	Interest Rate
5.	$1,700 today	Seven months from today	8%
6.	$3,800 two years from now	Today	6.35%
7.	$950 two months from now; $1,000 seven months from now	A single payment one year from now	4.95%
8.	$2,150 three months from now; $1,875 thirteen months from now	A single payment today	11%

Applications

- On January 23 of a non–leap year, a loan is taken out for $15,230 at 8.8% simple interest. What is the maturity value of the loan on October 23?
- An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was $10,267.21 for a loan that spanned April 14 to

July 31 at 9.1%, how much was the principal and how much was the interest?

- Suppose Robin borrowed $3,600 on October 21 and repaid the loan on February 21 of the following year. What simple interest rate was charged if Robin repaid $3,694.63?
- How many weeks will it take $5,250 to grow to $5,586 at a simple interest rate of 10.4%? Assume 52 weeks in a year.
- Jayne needs to make three payments to Jade requiring $2,000 each 5 months, 10 months, and 15 months from today. She proposes instead making a single payment eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?
- Markus failed to make three payments of $2,500 scheduled one year ago, nine months ago, and six months ago. As his creditor has successfully sued Markus in small claims court, the judge orders him to pay his debts. If the court uses a simple interest rate of 1.5% per month, what amount should the judge order Markus to pay today?

Challenge, Critical Thinking, & Other Applications

- Over a period of nine months, $40.85 of simple interest is earned at 1.75% per quarter. Calculate both the principal and maturity value for the investment.
- The Home Depot is clearing out lawn mowers for $399.75 in an end-of-season sale that ends on September 30. Alternatively, you could wait until April 1 (assume February is in a non–leap year) and pay $419.95 on the same lawn mower. Assume your money can earn simple interest of 4.72% on a short-term investment.
- Which option should you choose?
- How much money will you have saved on September 30 by making that choice?
- Tula lent $15,000 to a business associate on August 12 at 6% simple interest. The loan was to be repaid on November 27; however, the business associate was unable to make the payment. As a consequence, they agreed that the associate could repay the debt on December 29 and that Tula would add $200 to the balance that was due on November 27. What rate of simple interest is Tula charging on the late payment?
- Merina is scheduled to make two loan payments to Bradford in the amount of $1,000 each, two months and nine months from now. Merina doesn't think she can make those payments and offers Bradford an alternative plan where she will pay $775 seven months from now and another payment seven months later. Bradford determines that 8.5% is a fair interest rate. What is the amount of the second payment?

- Alia took out three back-to-back short-term simple interest investments. On November 10, Alia had saved up a total of $12,986.75. Her interest rate for each investment is listed below:

Dates	Interest Rate
March 2 to June 19	4.13%
June 19 to October 7	4.63%
October 7 to November 10	4.43%

How much did Alia originally invest on March 2? Assume that the principal and interest from a prior investment are both placed into the next investment.

- An accountant just received a payment from a client on January 26 in the amount of $14,500. The cheque stub states the payment is to be applied toward the client's three outstanding loans. The three loans were for $4,500 on May 12, $6,750 on July 11, and $8,000 on August 23. All loans are being charged simple interest of 6.85%. What is the remaining balance on the client's account?

6.3 Application: Savings Accounts and Short-Term GICs

When the stock market offers potentially huge returns, why would anyone put money into an investment that earns just a few percent annually? The answer of course is that sometimes stocks decrease in value. For example, after reaching a historic high in June 2008, the Toronto Stock Exchange plunged 40% over the course of the next six months. So if you are not willing to take the risk in the stock market, there are many options for investing your money that are safe and secure, particularly savings accounts and short-term guaranteed investment certificates (GICs). Some of the benefits and drawbacks associated with safe and secure investments are listed in the table below.

Benefits	Drawbacks
• Investments are virtually 100% safe, meaning that the chances of losing your investment are almost zero. • They always have a positive rate of interest. • It is easy to access the money when the investor needs it. • Any interest earned is better than zero interest. • It is suitable for small amounts.	• Interest rates are extremely low and offer little growth for your money. • Many come with stiff fees and interest rate penalties if certain conditions are not met. • Minimum balances are a common requirement, otherwise the interest rate is zero. • Balances are only insured up to $100,000.

Savings Accounts

A **savings account** is a deposit account that bears interest and has no stated maturity date. These accounts are found at most financial institutions, such as commercial banks (Royal Bank of Canada, TD Canada Trust, etc.), trusts (Royal Trust, Laurentian Trust, etc.), and credit unions (FirstOntario, Steinbach, Assiniboine, Servus, etc.). Owners of such accounts make deposits to and withdrawals from these accounts at any time, usually accessing the account at an automatic teller machine (ATM), at a bank teller, or through online banking.

A wide variety of types of savings accounts are available. This textbook focuses on the most common features of most savings accounts, including how interest is calculated, when interest is deposited, insurance against loss, and the interest rate amounts available.

1. **How Interest Is Calculated.** There are two common methods for calculating simple interest:
2. Accounts earn simple interest that is calculated based on the daily closing balance of the account. The closing balance is the amount of money in the account at the end of the day. Therefore, any balances in the account throughout a single day do not matter. For example, if you start the day with $500 in the account and deposit $3,000 at 9:00 a.m., then withdraw the $3,000 at 4:00 p.m., your closing balance is $500. That is the principal on which interest is calculated, not the $3,500 in the account throughout the day.
3. Accounts earn simple interest based on a minimum monthly balance in the account. For example, if in a

single month you had a balance in the account of $900 except for one day, when the balance was $500, then only the $500 is used in calculating the entire month's worth of interest.

4. **When Interest Is Deposited.** Interest is accumulated and deposited (paid) to the account once monthly, usually on the first day of the month. Thus, the interest earned on your account for the month of January appears as a deposit on February 1.

5. **Insurance against Loss.** Canadian savings accounts at commercial banks are insured by the national Canada Deposit Insurance Corporation (CDIC), which guarantees up to $100,000 in savings. At credit unions, this insurance is usually provided provincially by institutions such as the Deposit Insurance Corporation of Ontario (DICO), which also guarantees up to $100,000. This means that if your bank were to fold, you could not lose your money (so long as your deposit was within the maximum limit). Therefore, savings accounts carry almost no risk.

6. **Interest Rate Amounts.** Interest rates are higher for investments that are riskier. Savings accounts carry virtually no risk, which means the interest rates on savings accounts tend to be among the lowest you can earn. At the time of writing, interest rates on savings accounts ranged from a low of 0.05% to a high of 1.95%. Though this is not much, it is better than nothing and certainly better than losing money!

While a wide range of savings accounts are available, these accounts generally follow one of two common structures when it comes to calculating interest. These structures are flat rate savings accounts and tiered savings accounts. Each of these is discussed separately.

How It Works

Flat-Rate Savings Accounts. A flat-rate savings account has a single interest rate that applies to the entire balance. The interest rate may fluctuate in synch with short-term interest rates in the financial markets.

Follow these steps to calculate the monthly interest for a flat-rate savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a flat-rate table as illustrated here. Create a number of rows equalling the number of monthly transactions (deposits or withdrawals) in the account plus one.

Dates	Closing Balance in Account	# of Days	Simple Interest Earned
			I $= Prt$
Total Monthly Interest Earned			

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: Calculate the number of days that the closing balance is maintained for each row.

Step 5: Apply Formula 6.1, I = Prt, to each row in the table. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: Sum the Simple Interest Earned column and round off to two decimals.

When you are calculating interest on any type of savings account, pay careful attention to the details on how interest is calculated and any restrictions or conditions on the balance that is eligible to earn the interest.

Example 6.3A: Savings at the Royal Bank

[table id=61 /]

How It Works

Tiered Savings Accounts. A tiered savings account pays higher rates of interest on higher balances in the account. This is very much like a graduated commission on gross earnings, as discussed in Section 4.1. For example, you might earn 0.25% interest on the first $1,000 in

your account and 0.35% for balances over $1,000. Most of these tiered savings accounts use a portioning system. This means that if the account has $2,500, the first $1,000 earns the 0.25% interest rate and it is only the portion above the first $1,000 (hence, $1,500) that earns the higher interest rate.

Follow these steps to calculate the monthly interest for a tiered savings account:

Step 1: Identify the interest rate, opening balance, and the monthly transactions in the savings account.

Step 2: Set up a tiered interest rate table as illustrated below. Create a number of rows equalling the number of monthly transactions (deposits or withdrawals) in the account plus one. Adjust the number of columns to suit the number of tiered rates. Fill in the headers for each tiered rate with the balance requirements and interest rate for which the balance is eligible.

Dates	Closing Balance in Account	# of Days	Tier Rate #1 Balance Requirements and Interest Rate	Tier Rate #2 Balance Requirements and Interest Rate	Tier Rate #3 Balance Requirements and Interest Rate
			Eligible P = I = Prt	Eligible P = I = Prt	Eligible P = I = Prt
Total Monthly Interest Earned					

Step 3: For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

Step 4: For each row, calculate the number of days that the closing balance is maintained.

Step 5: Assign the closing balance to the different tiers, paying attention to whether portioning is being used. In each cell with a balance, apply Formula 6.1, where $I = Prt$. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

Step 6: To calculate the Total Monthly Interest Earned, sum all interest earned amounts from all tier columns and round off to two decimals.

Example 6.3B: A Rate Builder Tiered Account

The Rate Builder savings account at your local credit union pays simple interest on the daily closing balance as indicated in the table below:

Balance	Interest Rate
$0.00 to $500.00	0% on entire balance
$500.01 to $2,500.00	0.5% on entire balance
$2,500.01 to $5,000.00	0.95% on this portion of balance only
$5,000.01 and up	1.35% on this portion of balance only

[table id=62 /]

Short-Term Guaranteed Investment Certificates (GICs)

A **guaranteed investment certificate** (**GIC**) is an investment that offers a guaranteed rate of interest over a fixed period of time. In many countries around the world, a GIC is often called a time or term deposit. GICs are found mostly at commercial banks, trust companies, and credit unions. Just like savings accounts, they have a very low risk profile and tend to have much lower rates than are available through other investments such as the stock market or bonds. They are also unconditionally guaranteed, much like savings accounts. In this section, you will deal only with short-term GICs, defined as those that have a time frame of less than one year.

The table below summarizes three factors that determine the interest rate on a short-term GIC: principal, time, and redemption privileges.

	Higher Interest Rates	Lower Interest Rates
Principal Amount	**Large**	**Small**
Time	**Longer**	**Shorter**
Redemption Privileges	**Nonredeemable**	**Redeemable**

Amount of Principal. Typically, a larger principal is able to realize a higher interest rate than a smaller principal.

1. **Time.** The length of time that the principal is invested affects the interest rate. Short-term GICs range from 30 days to 364 days in length. A longer term usually realizes higher interest rates.
2. **Redemption Privileges.** The two types of GICs are known as redeemable and nonredeemable. A redeemable GIC can be cashed in at any point before the maturity date, meaning that you can access your money any time you want it. A nonredeemable GIC "locks in" your money for the agreed-upon term. Accessing that money before the end of the term usually incurs a stiff financial penalty, either on the interest rate or in the form of a financial fee. Nonredeemable GICs carry a higher interest rate.

Example 6.3C: GIC Choices

[table id=63 /]

Exercises

Mechanics

1. A savings account at your local credit union holds a balance of $5,894 for the entire month of September. The posted simple interest rate is 1.35%. Calculate the amount of interest earned for the month (30 days).
2. If you place $25,500 into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?
3. In November, the opening balance on a savings account was $12,345. A deposit of $3,000 was made to the account on November 19, and a withdrawal of $3,345 was made from the account

on November 8. If simple interest is paid at 0.95% based on the daily closing balance, how much interest for the month of November is deposited to the account on December 1?

4. A 320-day short-term GIC earns 0.78% simple interest. If $4,500 is invested into the GIC, what is the maturity value and how much interest is earned?

Applications

5. In February of a leap year, the opening balance on your savings account was $3,553. You made two deposits of $2,000 each on February 5 and February 21. You made a withdrawal of $3,500 on February 10, and another withdrawal of $750 on February 17. If simple interest is calculated on the daily closing balance at a rate of 1.45%, how much interest do you earn for the month of February?

6. An investor places $30,500 into a short-term 120-day GIC at the Bank of Montreal earning 0.5% simple interest. The maturity value is then rolled into another short-term 181-day GIC earning 0.57% simple interest. Calculate the final maturity value.

7. The Mennonite Savings and Credit Union posts the following rate structure for its Daily Interest Savings Account. The entire balance is subject to the highest posted rate, and interest is calculated daily based on the closing balance.

Balance	Interest Rate
$0–$10,000.00	0.1%
$10,000.01–$25,000.00	0.21%
$25,000.01–$50,000.00	0.34%
$50,000.01–$100,000.00	0.85%
$100,000.01 and up	1.02%

The opening balance in February of a non–leap year was $47,335. The transactions on this account were a deposit of $60,000 on February 8, a withdrawal of $86,000 on February 15, and a deposit of $34,000 on February 24. What interest for the month of February will be deposited to the account on March 1?

1. An investor with $75,000 is weighing options between a 200-day GIC or two back-to-back 100-day GICs. The 200-day GIC has a posted simple interest rate of 1.4%. The 100-day GICs have a posted simple interest rate of 1.35%. The maturity value of the first 100-day GIC would be reinvested in the second 100-day GIC (assume the same interest rate upon renewal). Which alternative is best and by how much?

2. Canadian Western Bank offers a Summit Savings Account with posted interest rates as indicated in the table below. Only each tier is subject to the posted rate, and interest is calculated daily based on the closing balance.

Balance	Interest Rate
$0–$5,000.00	0%
$5,000.01–$1,000,000.00	1.05%
$1,000,000.01 and up	0.80%

December's opening balance was $550,000. Two deposits in the amount of $600,000 each were made on December 3 and December 21. Two withdrawals in the amount of $400,000 and $300,000 were made on December 13 and December 24, respectively. What interest for the month of December will be deposited to the account on January 1?

1. Sun Life Financial Trust offers a 360-day short-term GIC at 0.65%. It also offers a 120-day short-term GIC at 0.58%. You are considering either the 360-day GIC or three consecutive 120-day GICs. For the 120-day GICs, the entire maturity value would be "rolled over" into the next GIC. Assume that the posted rate increases by 0.1% upon each renewal. If you have $115,000 to invest, which option should you pursue and how much more interest will it earn?

Challenge, Critical Thinking, & Other Applications

2. The Regular Savings Account at Steinbach Credit Union posts the following tiered rate structure. Interest is calculated on the entire balance for any given tier.

Balance	Interest Rate
$0–$100,000.00	1.75%
$100,000.01–$250,000.00	1.85%
$250,000.01 and up	1.95%

For any given month, simple interest is calculated on the lowest monthly closing balance, but the interest is not deposited until January 1 of the following year. On January 1, 2014, the opening balance on an account was $75,000. The following transactions took place throughout the year:

Date	Activity	Amount
February 14	Deposit	$133,000
March 30	Deposit	$42,000
June 15	Withdrawal	$110,000
August 1	Deposit	$212,000
October 29	Withdrawal	$300,000

Calculate the annual amount of interest earned on this account that will be deposited on January 1, 2015.

Interest rates can fluctuate throughout the year. Working with the information from question 11, recalculate the annual amount of interest if all of the posted interest rates were adjusted as follows throughout the year:

12.

Effective Date	Change	Percent
March 1	Increase	0.1%
July 1	Decrease	0.15%
September 1	Increase	0.25%
November 1	Decrease	0.05%

13. Interest rates in the GIC markets are always fluctuating because of changes in the short-term financial markets. If you have $50,000 to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.

14. A Citibank Savings Account posts the following simple interest rate structure. All interest is paid on entire balances, and interest is deposited to the account every quarter.

Balance	Interest Rate
$0–$2,500.00	0.05%
$2,500.01–$5,000.00	0.1%
$5,000.01 and up	0.75%

On January 1, 2013, the opening balance was $3,300. The following transactions took place during the first quarter (January 1 to March 31):

Deposits		Withdrawals	
January 4	$350	January 15	$500
January 29	$1,800	January 19	$1,000
February 7	$2,300	February 20	$1,300
March 2	$2,400	March 7	$4,000
March 22	$1,100	March 25	$2,000

Calculate the total amount of interest that will be deposited to the account on April 1, 2013.

1. You are considering the following short-term GIC investment options for an amount of $90,000.

	Option 1	Option 2	Option 3	Option 4
Institution	Coast Capital Savings	MCAN Mortgage Corp.	DUCA Financial Services	Sun Life Financial Trust
Term	360 day	2 × 180 day	3 × 120 day	4 × 90 day
Posted Rate	0.8%	0.75%	0.72%	0.715%

In all cases, assume that the posted rates remain unchanged and that the entire maturity value will be reinvested in the next short-term GIC. Calculate the total maturity value for each option at the end of 360 days.

The Language of Business Mathematics

accrued interest Any interest amount that has been calculated but not yet placed (charged or earned) into an account.

commercial paper A short-term financial instrument with maturity no longer than one year that is issued by large corporations.

compound interest A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

current balance The balance in an account plus any accrued interest.

demand loan A short-term loan that generally has no specific maturity date, may be paid at any time without any interest penalty, and where the lender may demand repayment at any time.

discount rate An interest rate used to remove interest from a future value.

equivalent payments Two payments that have the same value on the same day factoring in a fair interest rate.

face value of a T-bill The maturity value of a T-bill, which ispayable at the end of the term. It includes both the principal and interest together.

fixed interest rate An interest rate that is unchanged for the duration of the transaction.

Key Concepts Summary

Section 6.1: Principal, Rate, Time

- Calculating the amount of simple interest either earned or charged in a simple interest environment
- Calculating the time period when specific dates or numbers of days are involved
- Calculating the simple interest amount when the interest rate is variable throughout the transaction

Section 6.2: Moving Money Involving Simple Interest

- Putting the principal and interest together into a single calculation known as maturity value
- Altering a financial agreement and establishing equivalent payments

Section 6.3: Application: Savings Accounts and Short-Term GICs

- How to calculate simple interest for flat-rate and tiered savings accounts
- How to calculate simple interest on a short-term GIC

future value The amount of principal with interest at a future point of time for a financial transaction. If this future point is the same as the end date of the financial transaction, it is also called the *maturity value*.

guaranteed investment certificate (GIC) An investment that offers a guaranteed rate of interest over a fixed period of time.

interest amount The dollar amount of interest that is paid or earned.

interest rate The rate of interest that is charged or earned during a specified time period.

legal due date of a note Three days after the term specified in an interest-bearing promissory note is the date when a promissory note becomes legally due. This grace period allows the borrower to repay the note without penalty in the event that the due date falls on a statutory holiday or weekend.

maturity date The date upon which a transaction, such as a promissory note, comes to an end and needs to be repaid.

maturity value The amount of money at the end of a transaction, which includes both the interest and the principal together.

present value The amount of money at the beginning of a time period in a transaction. If this is in fact the amount at the start of the financial transaction, it is also called the *principal*. Or it can simply be the amount at some time earlier before the future value was known. In any case, the amount excludes the interest.

prime rate An interest rate set by the Bank of Canada that usually forms the lowest lending rate for the most secure loans.

principal The original amount of money that is borrowed or invested in a financial transaction.

proceeds The amount of money received from a sale.

promissory note An unconditional promise in writing made by one person to another person to pay a sum of money on demand or at a fixed or determinable future time.

repayment schedule A table that details the financial transactions in an account, including the balance, interest amounts, and payments.

savings account A deposit account that bears interest and has no stated maturity date.

secured loan Those loans that are guaranteed by an asset such as a building or a vehicle that can be seized to pay the debt in case of default.

simple interest A system for calculating interest that primarily applies to short-term financial transactions with a time frame of less than one year.

student loan A special type of loan designed to help students pay for the costs of tuition, books, and living expenses while pursuing postsecondary education.

time period The length of the financial transaction for which interest is charged or earned. It may also be called the term.

treasury bills Short-term financial instruments with maturities no longer than one year that are issued by both federal and provincial governments.

unsecured loan Those loans backed up by the general goodwill and nature of the borrower.

variable interest rate An interest rate that is open to fluctuations over the duration of a transaction.

yield The percentage increase between the sale price and redemption price on an investment such as a T-bill or commercial paper.

The Formulas You Need to Know

Symbols Used

S = maturity value or future value in dollars

I = interest amount in dollars

P = principal or present value in dollars

r = interest rate (in decimal format)

t = time or term

Formulas Introduced

Formula 6.1 Simple Interest: $I = Prt$ (Section 6.1)

Formula 6.2 Simple Interest for Single Payments: $S = P(1 + rt)$ (Section 6.2)

Formula 6.3 Interest Amount for Single Payments: $I = S - P$ (Section 6.2)

Technology

Calculator

The following calculator functions were introduced in this chapter:

Date Function
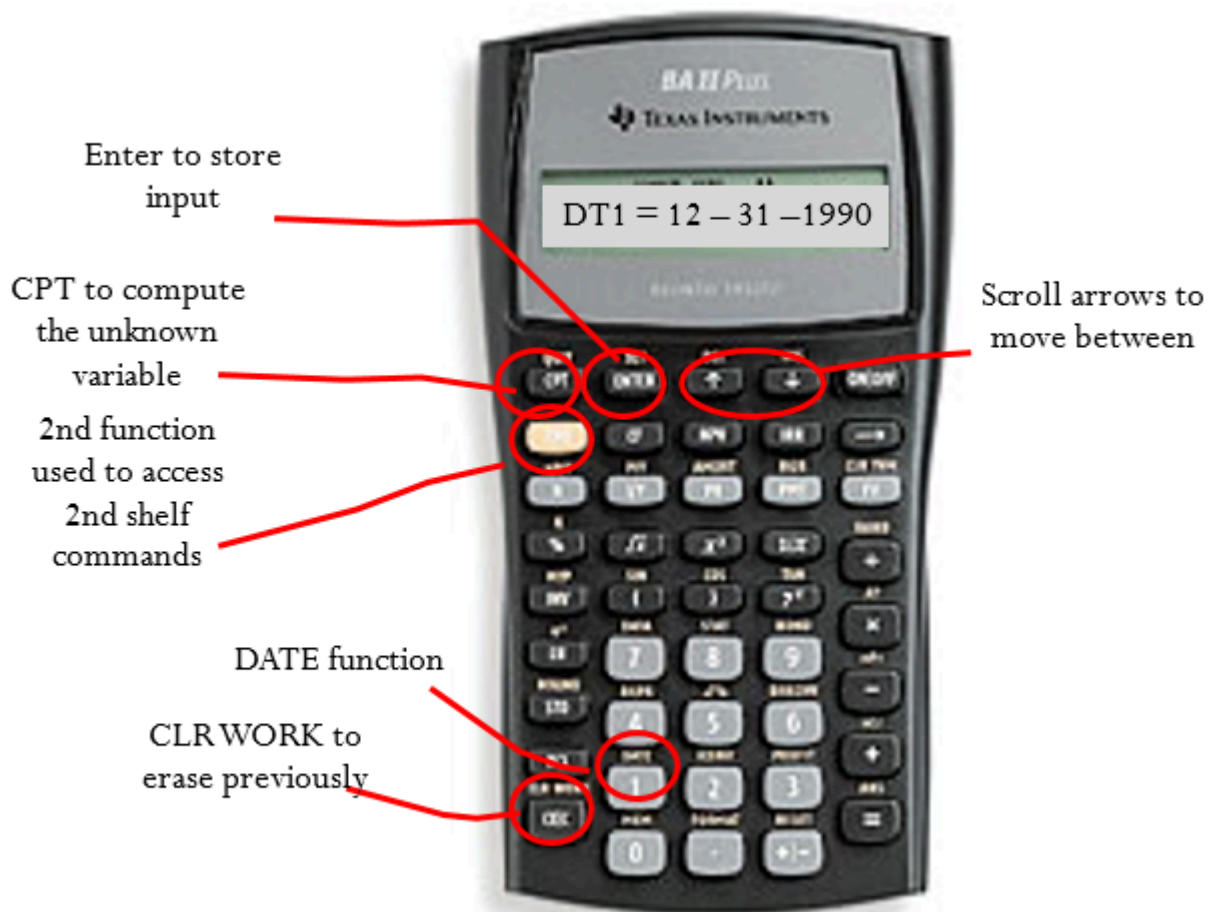
- 2nd Profit to access.
- Enter two of the three variables (DT1, DT2, DBD) by pressing Enter after each input and using and to scroll through the display. The variables are:
- DT1 = The starting date of the transaction
- DT2 = The ending date of the transaction
- DBD = The days between the dates, counting the first day but not the last, which is the time period of the transaction.
- ACT / 360 = A setting for determining how the calculator determines the DBD. In Canada, you should

maintain this setting on ACT, which is the actual number of days. In other countries, such as the United States, they treat each year as having 360 days (the 360 setting) and each month as having 30 days. If you need to toggle this setting, press 2nd SET.

- Enter all dates in the format of MM.DDYY, where MM is the numerical month, DD is the day, and YY is the last two digits of the year. DD and YY must always be entered with both digits.
- Press CPT on the unknown (when it is on the screen display) to compute the answer.

Chapter 8 Review Exercises

Mechanics

- If $4,000 is borrowed from April 3 to June 22 at a simple interest rate of 3.8%, how much interest is paid on the loan?
- A savings account pays flat-rate interest of 1.45%. If a balance of $3,285.40 is maintained for the entire month of August, how much interest does the savings account earn?
- What is the legal due date and maturity value on that date for a 125-day $51,000 promissory note issued on May 14 at
- A full-time student graduated from college on December 16, 2013, with $26,500 in outstanding student loans. Calculate the interest accrued during his grace period if the prime rate is 4.7%.
- A 182-day $1,000,000 Government of British Columbia T-bill was issued when the market rate of return was 4.21%. Calculate the purchase price of the T-bill on its issue date.
- If you place $8,000 into a 300-day short-term GIC at Scotiabank earning 0.95% simple interest, how much will you receive when the investment matures?

Applications

- Proper accounting procedures require accountants to separate principal and interest components on any loan. Allocate the principal and interest portions of a $24,159.18 payment clearing a 147-day loan at 8.88%.
- Cadillac Fairview withdrew $115,000 from its operating loan account on September 4 to perform some needed maintenance on one of its properties. The operating loan requires interest at prime + 3% and fixed $25,000 monthly payments starting October 1. The company thinks it can make an additional payment of $35,000 on November 18. The prime rate on the date of withdrawal was 3.6%, and it increased by 0.5% on October 27. Construct a full repayment schedule for this loan.
- As part of your financial plan for retirement, you purchased a 270-day $25,000 commercial paper on its date of issue, July 14, when market yields were 2.94%. 234 days later, you sold the note when market yields were 2.76%. What rate of return did you realize on your investment?

- Alterna Savings and Bank offers a Daily Interest Savings Account with posted interest rates as indicated in the table below. The entire balance qualifies for the posted rate, and interest is calculated daily based on the closing balance.

Balance	Interest Rate
$0–$5,000.00	0.10%
$5,000.01–$10,000.00	0.15%
$10,000.01–$25,000.00	0.20%
$25,000.01 and up	0.25%

  BOS Designer Candles Inc. has an opening balance in July of $17,500. Three deposits in the amounts of $6,000, $4,000, and $1,500 were made on July 4, July 18, and July 22, respectively. Two withdrawals in the amount of $20,000 and $8,000 were made on July 20 and July 27, respectively. What interest for the month of July will Alterna deposit to the account on August 1?

- Shannon has a $68,000 student loan when she graduates on August 28, 2014, and the prime rate is set at 4.9%. She will convert the interest to principal at the end of her grace period, and she elects to take the variable rate on her student loan. She sets her fixed monthly payment at $1,400. The prime rate decreases by 0.5% on January 15 and rises by 0.75% on April 25. Compute the first six repayments of her repayment schedule. Calculate the total interest charged for both the grace period and the six payments combined.
- Sturm put $48,700 into a 10-month term deposit, but needed to withdraw the funds after five months to deal with a family emergency. The credit union penalized him 2.35% off of his interest rate for the early withdrawal and deposited $49,602.98 into his account.
- What was Sturm's original interest rate?
- How much interest, in dollars, was he penalized for the early withdrawal?
- Jerry's Concrete allows his customers to create six-month promissory notes on any stamped concrete driveway project with interest at 8.9%. Three months after completing a $17,300 job on March 2, 2014, Jerry had some liquidity problems and sold the promissory note to a financial institution at an interest rate of 10.35%. Calculate Jerry's proceeds on the sale.
- Three hundred days from now you will be departing on a backpacking trip through Europe. You need $4,000 in spending money to take with you. Today, you currently have saved $3,960.
- If you place your money in consecutive 100-day short-term GICs earning 1.02%, will you meet your

goal? Assume the full maturity values are reinvested into the next GIC.
- What would the interest rate need to be if you had placed your money into a single 300-day short-term GIC to reach your goal?

Challenge, Critical Thinking, & Other Applications

- A credit union posts the following tiered interest rate structure for its savings accounts. Only each tier is subject to the posted rate and is computed using the daily *opening* balance in the account. Interest is deposited to the account on the last day of every month.

Balance	Interest Rate
$0–$4,000.00	0.15%
$4,000.01–$8,000.00	0.25%
$8,000.01 and up	0.45%

On August 1, the opening balance was $6,400. Three deposits of $2,000, $3,500, and $1,500 were made on August 3, August 10, and August 27, respectively. Two withdrawals of $7,000 and $1,900 were made on August 6 and August 21, respectively. Compute the total interest earned for the month of August.

- Island Lakes Dental (ILD) found itself with an urgent need to replace its X-ray machine when its existing machine suddenly became inoperable. It borrowed $43,000 on May 12 from its operating loan with interest set at prime + 2.4%. The prime rate was 2.75% and increased by 0.5% on June 5. The loan requires payment of the balance in full when the balance is less than $5,000, or 10% of the current balance on the first of every month. ILD made payments of $15,000 on May 29 and $21,000 on June 21. Create a full repayment schedule for its operating loan and calculate the total interest charged on the purchase.
- Pendragon Inc. has an operating loan with a balance of $52,000 on September 1 with interest set at prime + 4.75%. On the first of every month the operating loan requires repayment of the accrued interest only. The current prime rate is 4.75% and will decrease to 4.25% on October 7. On September 15, it has a 270-day $20,000 GIC with 3.8% interest maturing. On October 15, it has a 320-day $18,000 GIC with 3.68% interest maturing. It will use these maturing investments to pay down its operating loan. Construct a repayment schedule from September 1 to November 1 only.
- A 364-day, $50,000 face value T-bill is issued when market yields are 2.85%. The T-bill is sold to another

investor every 91 days until maturity, with yields of 3.1%, 2.98%, and 3.15% on each of the dates of sale, respectively. Compute the purchase price for each investor, including the date of issue. For each investor, calculate the actual rate of return realized on their investment.

- Brant and Sylvia have a prime + 2.35% HELOC with a $30,000 balance on March 1 with accrued interest payable on the first of every month. They also have a savings account with a $10,000 balance on March 1 earning 0.85% payable on the first of every month. On March 18 they deposited $4,000 into the savings account. On both March 5 and March 30 they transferred $3,000 from their savings to their HELOC. Prime was initially at 3.75% but increased to 4.25% on March 22. Calculate the balance on April 1 in both the savings account and the HELOC.
- Norbert is looking to make a short-term investment for 120 days with the $25,000 he just inherited from his father's estate. His options are as follows. Compute the maturity values of each of the options and rank his choices.
- Two back-to-back 60-day GICs earning 2.25% today, with a forecasted rate of 2.5% 60 days from now. Assume the full maturity value is rolled over into the next investment.
- One 120-day GIC earning 2.35%.
- A 90-day $25,000 face value T-bill earning 2.28% followed by a 30-day $25,000 face value T-bill earning a projected 2.38%. Assume any leftover funds are invested to earn 0.85% interest.

Chapter 6 Case Study

Managing A Company's Investments

The Situation

As with most mid-size to large companies, Lightning Wholesale has a finance department to manage its money. The structure of Lightning Wholesale's financial plan allows each department to have its own bank account from which all expenses, purchases, and charges are deducted. This same account has all revenues and interest deposited into it.

The manager of the sporting goods department wants a summary of all interest amounts earned or charged to her department for the year 2013. This will allow her to better understand and assess the financial policies of the company and make any necessary changes for 2014.

Select Dates	T-bill Yields	Prime rate
December 31, 2012	2.85%	4.25%
January 31, 2012	2.90%	4.25%
February 28, 2013	2.87%	4.25%
March 31, 2013	3.04%	4.75%
April 30, 2013	3.09%	4.75%
May 31, 2013	3.11%	4.75%
June 30, 2013	3.07%	4.75%
July 31, 2013	2.95%	4.50%
August 31, 2013	2.92%	4.50%
September 30, 2013	3.40%	5.00%
October 31, 2013	3.45%	5.00%
November 30 , 2013	3.47%	5.00%

The Data

Month	Total Sales Revenue	Cost of Goods Sold	Operating Expenses
January	$1,798	$2,102	$156
February	$2,407	$2,242	$100
March	$2,568	$2,606	$222
April	$2,985	$2,719	$258
May	$3,114	$2,831	$269
June	$3,242	$2,887	$280
July	$3,306	$3,363	$286
August	$3,852	$4,203	$333
September	$4,815	$6,306	$525
October	$7,222	$11,210	$625
November	$12,839	$8,408	$1,111
December	$9,630	$1,569	$833

(All numbers in thousands of dollars)

Important Information

- The balance in the sporting goods bank account on December 31, 2012, was $2,245,636.45.
- The bank account permits a negative balance, which the bank treats as an operating loan. The interest rate on any operating loan is prime + 0.5%. Accrued interest is placed into the account on the last day of each month.
- When a positive balance exists, the bank pays interest at 0.85% on the first $50,000 in the account and 1.2% only on the portion above $50,000. Interest is deposited on the last day of each month.
- On the last day of each month, the finance department purchases T-bills in the market that will mature

by the last day of the next month. The face value of T-bills are bought in denominations of exactly $100,000 in a quantity as permitted by the current balance in the bank account. If the bank account has a negative balance (i.e., it is using its operating loan), no T-bills are purchased that month. For example, if the bank account has a balance of $350,000 on March 31, three $100,000 T-bills will be purchased with 30 days left to maturity.

- Assume all revenues are deposited to the account at the end of the corresponding month.
- Assume all cost of goods sold and operating expenses are deducted at the end of the corresponding month.
- For simplicity, assume the balance in the bank account remains unchanged throughout each month.

Your Tasks

The manager wants a report that summarizes the following information from December 31, 2012, to December 31, 2013:

- The total interest earned through T-bill investments.
- The total interest earned from the bank account.
- The total interest charged by any operating loans.
- The final balance in the bank account as of December 31, 2013, when no purchases of T-bills for January 2014 have been made.

In order to meet the manager's requirements, work through 2013 month by month starting from December 31, 2012, by following the steps below. Once arriving at December 31, 2013, use the answers to provide the four pieces of information requested by the manager.

- Using the opening balance, determine the face value of T-bills that can be purchased. If the balance is negative, no T-bills are purchased, so skip to step 4.
- Calculate the purchase price of the T-bills using the current market yield and the number of days until the end of the next month. The difference between the purchase price and the face value is the total interest earned for the month by T-bills.
- Deduct the purchase price of the T-bills from the balance in the account.
- Examine the account balance.
- If the balance is positive, calculate the interest earned for the month based on the tiered interest rate structure. This is the total interest earned by the investment for the month.
- If the balance is negative, charge interest to the account for the month using the interest rate charged by the bank. This is the total interest charged by the operating loan for the month.

- To figure out the balance at the end of the next month, take the balance from step 4, add the face value of the T-bills that are maturing at the end of the month, add any interest earned from the bank account (step 4a), deduct any interest charged on the operating loan (step 4b), add the revenues for the month, and deduct the expenses and cost of goods sold for the month.
- Go back to step 1 and repeat for the next month.
- When all months are complete, calculate the required totals and present the requested summary information to the manager.

8-

7.

FUNCTIONS AND LINES

7.1: Functions and Function Notation

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function

A **function** is a rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 7.1 A: Functions

In the height and age example above, is height a function of age? Is age a function of height?

Solution:

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18th birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 7.1 B: Functions in Everyday Life

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

Solution:

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 7.1 C: Percentages, Decimals, and Functions

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

Solution:

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

You try...

Let's consider bank account information.

1. Is your balance a function of your bank account number?
(If you input a bank account number does it make sense that the output is your balance?)

2. Is your bank account number a function of your balance?
(If you input a balance does it make sense that the output is your bank account number?)

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

"height is a function of age"

if we name the function f we write

"h is f of a"

or more simply

$$h = f(a)$$

we could instead name the function h and write

$$h(a)$$

which is read "h of a."

Remember we can use any variable to name the function; the notation $h(a)$ shows us that h depends on a.

The value "a" must be put into the function "h" to get a result. Be careful – the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

Function Notation

The notation output $= f(input)$ defines a function named f. This would be read "output is f of input."

Example 7.1 D: Function Notation

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

Solution:

The number of days in a month is a function of the name of the month, so if we name the function f, we could write "$days = f(month)$" or $d = f(m)$. If we simply name the function d, we could write $d(m)$.

For example, $d(March) = 31$, since March has 31 days. The notation $d(m)$ reminds us that the number of days, d (the output) is dependent on the name of the month, m (the input)

Example 7.1 E: Functions in Everyday Life

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ tell us?

Solution:

When we read $f(2005) = 300$, we see the input quantity is 2005, which is a value for the input

quantity of the function, the year (y). The output value is 300, the number of police officers (N), a value for the output quantity. Remember $N = f(y)$. So this tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.

In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, m	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month D	31	28	31	30	31	30	31	31	30	31	30	31

Table 2: The table below defines a function $Q = g(n)$. Remember this notation tells us g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) a, age in years	5	5	6	7	8	9	10
(output) h, height inches	40	42	44	47	50	52	54

Example 7.1 F: Tables as Functions

Which of these tables define a function (if any)?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

Solution:

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

You try...

3. If each percentage earned translated to one letter grade, would this be a function?

Solving and Evaluating Functions

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Example 7.1 G: Calculating Input and Output

Using the table shown, where $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

a) Evaluate $g(3)$

Solution:

Evaluating $g(3)$ (read: "g of 3") means that we need to determine the output value, Q, of the function g given the input value of $n = 3$. Looking at the table, we see the output corresponding to $n = 3$ is $Q = 7$, allowing us to conclude $g(3) = 7$.

b) Solve $g(n) = 6$

Solution:

Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$. Specifically:

- When we input 2 into the function g, our output is $Q = 6$. In other words, $g(2) = 6$.
- When we input 4 into the function g, our output is also $Q = 6$. In other words, $g(4) = 6$.

You try...

4. Using the function in Example 7.1G, evaluate $g(4)$.

Graphs as Functions

Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

The most common graph has y on the vertical axis and x on the horizontal axis, and we say y is a function of x, or $y = f(x)$ when the function is named f.
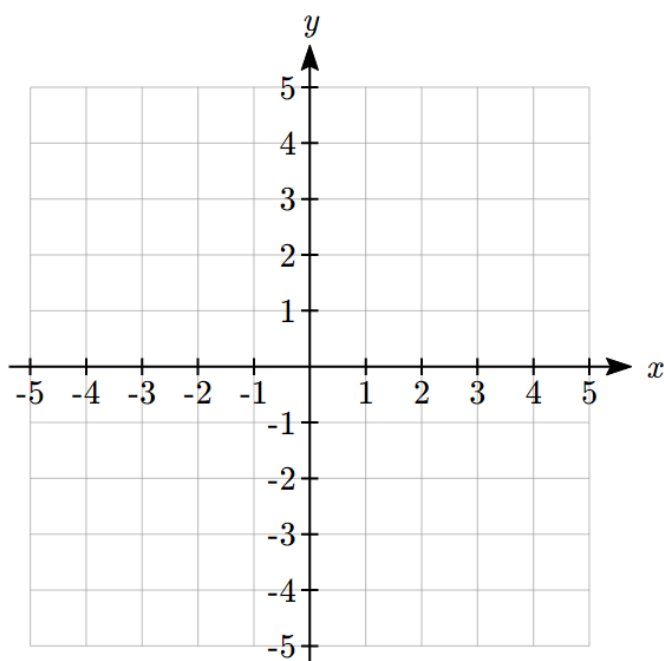
Example 7.1 H: Graphs of Functions

Which of these graphs defines a function $y = f(x)$?
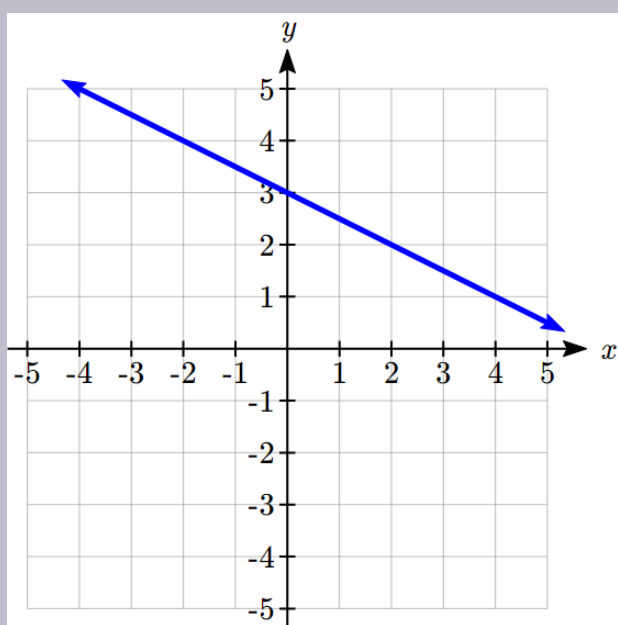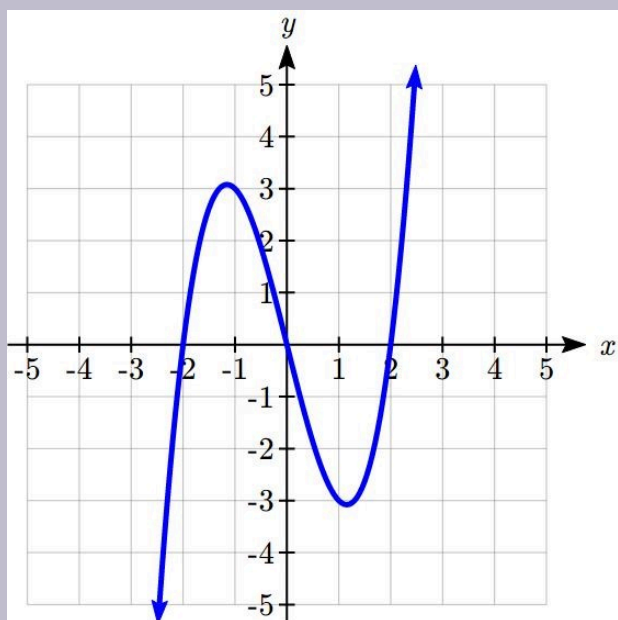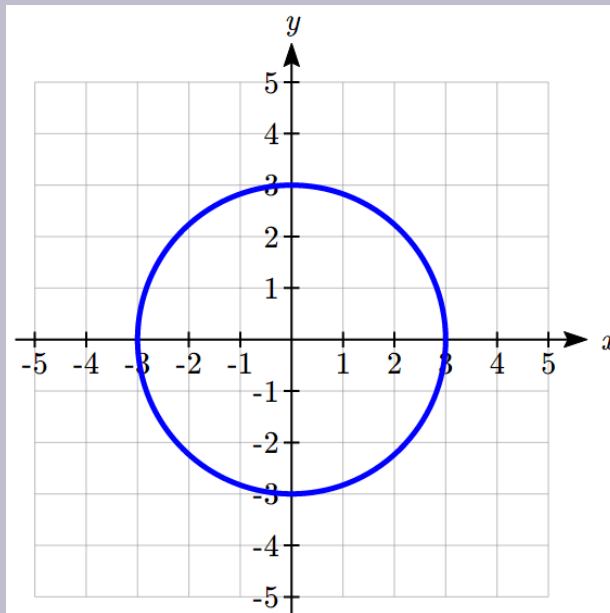
Solution:

Looking at the three graphs above, the first two define a function $y = f(x)$, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the y-value of the graph. The 3rd graph does not define a function $y = f(x)$ since some input values, such as $x = 2$, correspond with more than one output value.

Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 7.1 I: Solving Functions from Graphs

Given the graph shown below,

a) Evaluate $f(2)$

b) Solve $f(x) = 4$
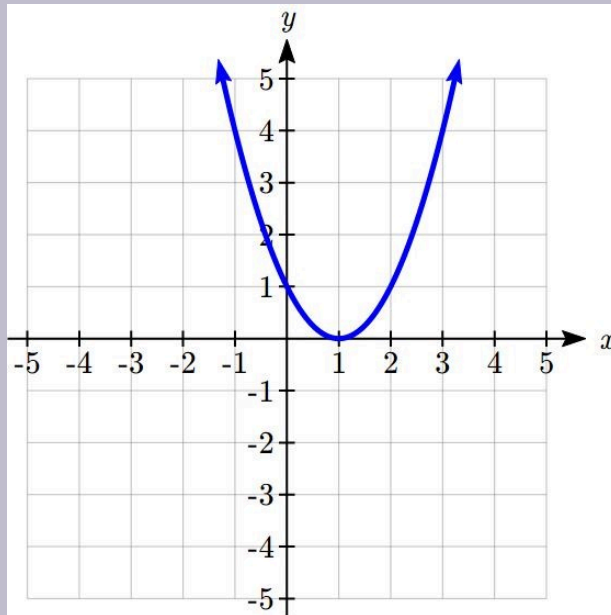
Solution:

a) To evaluate $f(2)$, we find the input of $x = 2$ on the horizontal axis. Moving up to the graph gives the point $(2, 1)$, giving an output of $y = 1$. So $f(2) = 1$

b) To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: $(-1, 4)$ and $(3, 4)$. These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$.

This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4.

You try...

5. Using the graph from Example 7.1I, solve f(x)=1.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 7.1 J: Relationships as Functions

Express the relationship $2n + 6p = 12$ **as a function** $p = f(n)$ **if possible.**

Solution:

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as p = [something involving n].

$$2n + 6p = 12$$

subtract $2n$ from both sides

$$6p = 12 - 2n$$

divide both sides by 6 and simplify

$$p = \tfrac{12-2n}{6} = \tfrac{12}{6} - \tfrac{2n}{6} = 2 - \tfrac{1}{3}n$$

Having rewritten the formula as $p =$, we can now express p as a function:

$$p = f(n) = 2 - \tfrac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations – no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 7.1 K: Evaluating Functions

Given the function $k(t) = t^3 + 2$

a) Evaluate $k(2)$
b) Solve $k(t) = 1$

Solution:

a) To evaluate $k(2)$, we plug in the input value 2 into the formula wherever we see the input variable t, then simplify

$$k(2) = 2^3 + 2$$
$$k(2) = 8 + 2$$
$$\text{So } k(2) = 10$$

b) To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output

$$k(t) = 1$$

substitute the original formula $k(t) = t^3 + 2$

$$t^3 + 2 = 1$$

subtract 2 from each side

$$t^3 = -1$$

take the cube root of each side

$$t = -1$$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t) = 1$ is true when $t = -1$.

$$k(-1) = (-1)^3 + 2$$
$$= -1 + 2$$
$$= 1$$

which was the desired result.

Example 7.1 L: Evaluating Functions

Given the function $h(p) = p^2 + 2p$
a) Evaluate $h(4)$
b) Solve $h(p) = 3$

Solution:

a) To evaluate $h(4)$ we substitute the value 4 for the input variable p in the given function.

$$h(4) = (4)2 + 2(4)$$
$$= 16 + 8$$
$$= 24$$

b)

$$h(p) = 3$$

Substitute the original function $h(p) = p^2 + 2p$:

$$p^2 + 2p = 3$$

This is quadratic, so we can rearrange the equation to get it = 0:

$$p^2 + 2p - 3 = 0$$

subtract 3 from each side:

$$p^2 + 2p - 3 = 0$$

this is factorable, so we factor it:

$$(p + 3)(p - 1) = 0$$

By the zero factor theorem since $(p + 3)(p - 1) = 0$, either $(p + 3) = 0$ or $(p - 1) = 0$ (or both of them equal 0) and so we solve both equations for p, finding $p = -3$ from the first equation and $p = 1$ from the second equation.

This gives us the solution: $h(p) = 3$ when $p = 1$ or $p = -3$

You try...

6. Given the function $g(m) = \sqrt{m - 4}$
a. Evaluate $g(5)$
b. Solve $g(m) = 2$

You try... Answers

1. Yes
2. No
3. Yes, it's a function
4. $Q = g(4) = 6$

5. $x = 0$ or $x = 2$
6. a. $g(5) = 1$
   b. $m = 8$

Section 7.1 Exercises

1. Given a specific person, is their height a function of their age? Is the person's age a function of their height? Justify your answers.
2. At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price? Justify your answers.
3. At Ontario Tech University, the final grade a student earns in the course corresponds to a letter grade (see Ontario Tech – Grading). Is the letter grade a function of the percentage a student can obtain in the course? Is the percentage a student can obtain in the course a function of the letter grade? Justify.
4. Consider bank account information.
   a. Is the account balance a function of the bank account number? Justify.
   b. Is the bank account number a function of the account balance? Justify.
5. Introduce function notation of your choice using variable names to represent a function that takes as input the name of a month, and gives as output the number of days in that month. State clearly what each variable represents.
6. A function $N(y)$ gives the number of police officers, N, in a town in year y. What does $N(2005) = 300$ tell us?
7. Which of these tables define a function (if any)? Justify.

a.

Input	Output
2	1
5	3
8	6

b.

Input	Output
-3	5
0	1
4	5

c.

Input	Output
1	0
5	2
5	4

8. Using the table shown for $Q(n)$:

n	1	2	3	4	5
Q	8	6	7	6	8

   a. Evaluate $Q(3)$ and interpret your result (describe in words what the answer means).
   b. Solve $Q(n) = 6$ and interpret your result.
   c. Evaluate $Q(7)$. Interpret your result.
   d. Solve $Q(n) = 1$. Interpret your result.

9. Which of these graphs defines a function? Justify your answers.
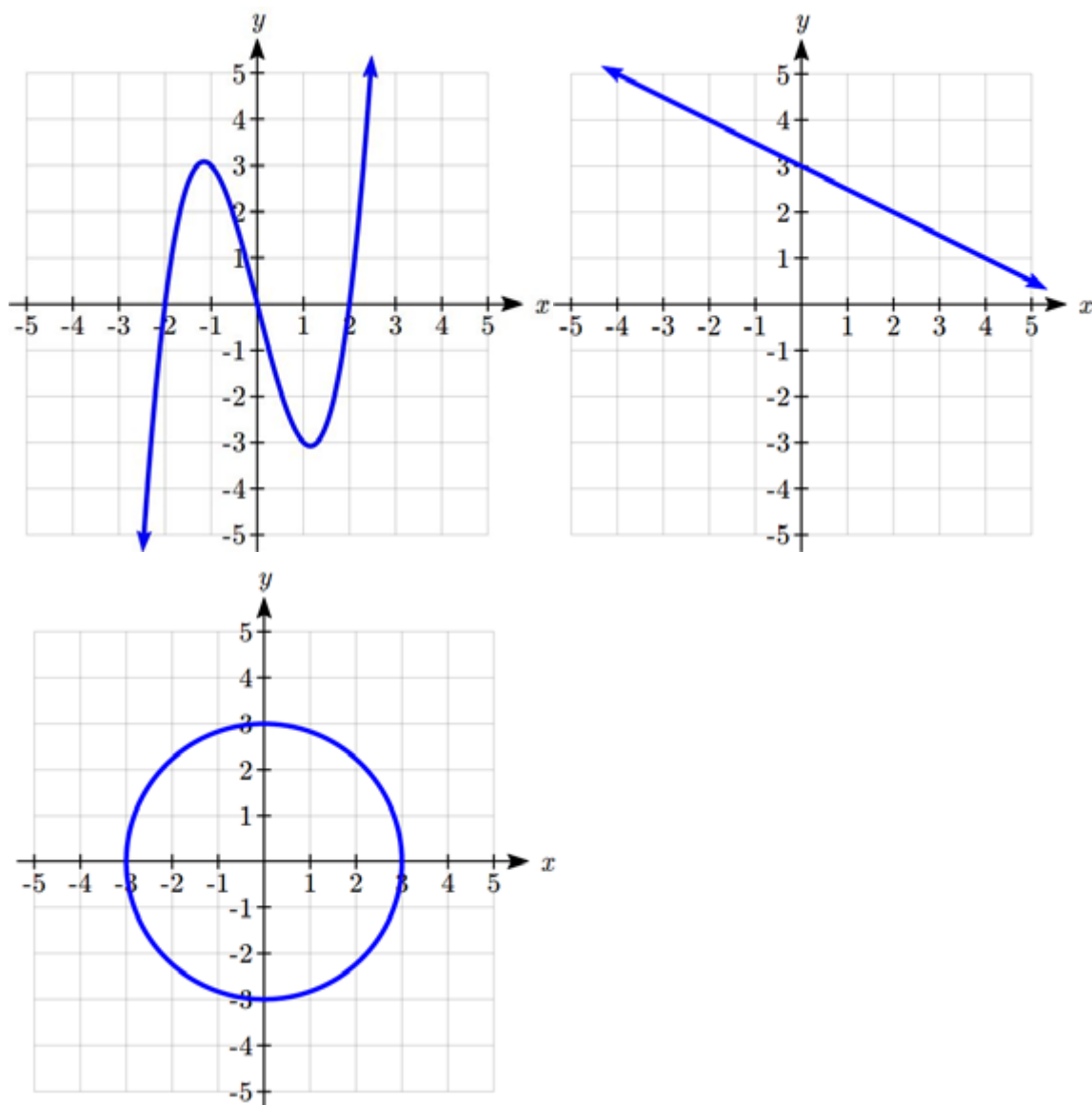
10. Given the graph shown,
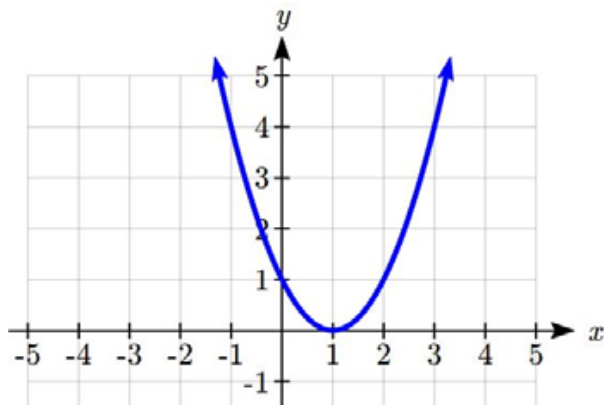
a. Evaluate $f(2)$.

b. Solve $f(x) = 4$.

    c. Evaluate $f(-1)$.

    d. Solve $f(x) = 1$.

11. If possible, express the relationship $2n + 6p = 12$ as a function $p(n)$.

12. Consider the relationship $3m^2 - 15n = 7$. Use this relationship to express the following functions, if possible. If this is not possible, justify why not.

    a. Express n as a function of m.

    b. Express m as a function of n.

13. Given the function $k(t) = t^3 + 2$.

    a. Evaluate $k(2)$.

    b. Solve $k(t) = 1$.

14. Given the function $h(p) = p^2 + 2p$,

    a. Evaluate $h(4)$.

    b. Solve $h(p) = 3$.

15. Given the function $g(m) = \sqrt{m - 4}$

    a. Evaluate $g(5)$

    b. Solve $g(m) = 2$

7.2: Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, c	1.7	2.5	5.5	8.2	13.7
Height, h	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Domain and Range

The **domain** refers to the set of possible input values to a function.

The **range** refers to the set of possible output values of a function

Example 7.2 A: Domain and Range

Using the tree table above, determine a reasonable domain and range.

Solution:

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$. For the domain, possible values for the input circumference c, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet[1]. With this information we would say a reasonable domain is $0 < c \leq 119$ feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is $0 < h \leq 379$ feet.

Example 7.2 B: Calculating Postage

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter[2], as shown in the table below. Determine the domain and range.

Letters

---

1. http://en.wikipedia.org/wiki/Tree, retrieved July 19, 2010
2. http://www.usps.com/prices/first-class-mail-prices.htm, retrieved July 19, 2010

Weight not Over	Price
1 ounce	$0.44
2 ounces	$0.61
3 ounces	$0.78
4 ounces	$0.95

Solution:

Suppose we denote weight of a letter by w and price by p, we can think of price p is a function of weight, w. We then have that $p(w)$ represents the price of mailing the letter of weight w.

Since acceptable weights are 3.5 ounces or less, and negative weights and weight of zero don't make sense, the domain would be $0 < w \le 3.5$ or, in interval form, $(0, 3.55]$.

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. We use the {...} brackets to denote the type of set made a from a list of values. Thus the range is p = {$0.44, $0.61, $0.78, $0.95}.

You try...

1. The population of a small town in the year 1960 was 100 people. Since then, the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

Notation

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one

way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163, 0 < w \leq 3.5$, and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, $p =$ \$0.44, \$0.61, \$0.78, or \$0.95. These numbers represent a set of specific values: {0.44, 0.61, 0.78, 0.95}

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable x that are 10 or greater, but less than 30. In inequalities, we would write $10 \leq x < 30$.

When describing domains and ranges, we sometimes extend this into **set notation**, which would look like this: $\{x|10 \leq x < 30\}$. The curly brackets {} are read as "the set of", and the vertical bar | is read as "such that", so altogether we would read $\{x|10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x and x is less than 30."

When describing ranges in set notation, we could similarly write something like $\{f(x)|0 < f(x) < 100\}$, or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h|0 < h \leq 379\}$. In set notation, if a domain or range is not limited, we could write $\{t|t$ is a real number$\}$, or $\{t|t \in \Re\}$, read as "the set of t-values such that t is an element of the set of real numbers."

A more compact alternative to set notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for "strictly less than," and square brackets are used for "less than or equal to." Since infinity is not a number, we can't include it in the interval, so we always use curved parentheses with ∞ and $-\infty$. The table below will help you see how inequalities correspond to set notation and interval notation:

Inequality	Set Notation	Interval Notation
$5 < h \leq 10$	$\{h\|5 < h \leq 10$	$(5, 10]$
$5 \leq h < 10$	$\{h\|5 \leq h < 10\}$	$[5, 10)$
$5 < h < 10$	$\{h\|5 < h < 10\}$	$(5, 10)$
$h < 10$	$\{h\| < 10\}$	$(-\infty, 10)$
$h \geq 10$	$\{h\|h \geq 10\}$	$[10, \infty)$
all real numbers	$\{h\|h \in \Re\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set notation we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Example 7.2 C: Intervals

Describe the intervals of values shown on the line graph below using set builder and interval notations.
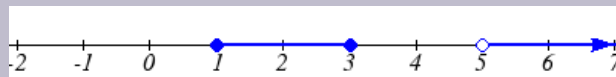
Solution:

To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

As an inequality it is:

$$1 \leq x \leq 3 \text{ or } x > 5$$

In set notation:

$$\{x|1 \leq x \leq 3 \text{ or } x > 5\}$$

In interval notation:

$$[1, 3] \cup (5, \infty)$$

Remember when writing or reading interval notation:

- Using a square bracket [means the start value is included in the set.
- Using a parenthesis (means the start value is not included in the set.

You try...

2. Given the following interval, write its meaning in words, set builder notation, and interval notation.

Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 7.2 D: Domain and Range of a Graph

Determine the domain and range of the graph below.
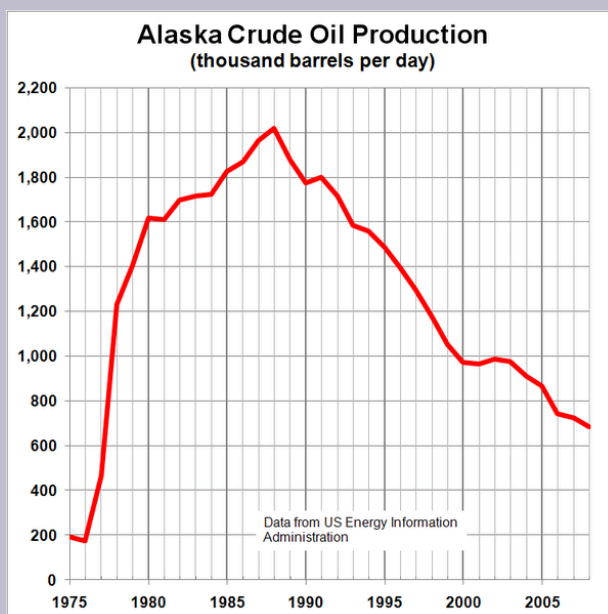
Solution:

In the graph above[3], the input quantity along the horizontal axis appears to be "year", which we could denote by the variable y. The output is "thousands of barrels of oil per day", which we might denote by the variable b, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \leq y < 2008$, and the range is approximately $180 \leq b \leq 2010$.

In interval notation, the domain would be $[1975, 2008]$ and the range would be about $[180, 2010]$. For the range, we have to approximate the smallest and largest outputs since they don't fall exactly on the grid lines.

3. http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG, CC-BY-SA, July 19, 2010

Remember that, as in the previous example, x and y are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

You try...

3. Given the graph below, write the domain and range in interval notation.

World Population Increase

source: U.S. Bureau of the Census, International Data Base (via GeoHive)
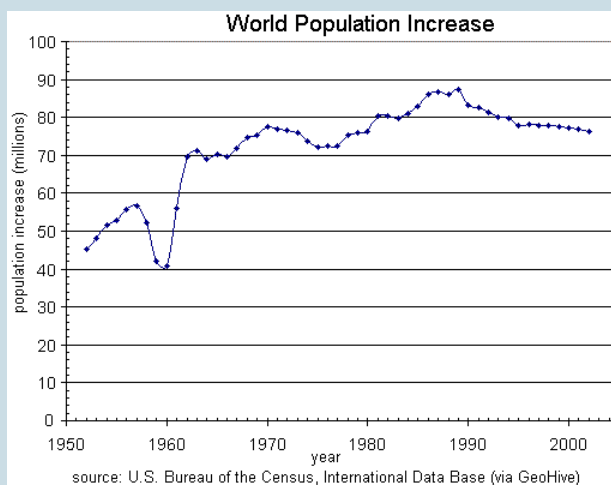
Domain and Range from Formulas

Most basic formulas can be evaluated at an input. Two common restrictions are:

- The square root of negative values is non-real.
- We cannot divide by zero.

Example 7.2 E: Finding Domain and Range

Find the domain of each function:

a) $f(x) = 2\sqrt{x+4}$

b) $g(x) = \frac{3}{6-3x}$

Solution:

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.

$$x + 4 \geq 0 \text{ when } x \geq -4.$$

The domain of $f(x)$ is $[-4, \infty)$.

b) We cannot divide by zero, so we need the denominator to be non-zero.

$$6 - 3x = 0 \text{ when } x = 2,$$

so we must exclude 2 from the domain.

The domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$.

Piecewise Functions

Some functions cannot be described by a single formula.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1} & \text{if} \quad \text{domain to use formula 1} \\ \text{formula 2} & \text{if} \quad \text{domain to use formula 2} \\ \text{formula 3} & \text{if} \quad \text{domain to use formula 3} \end{cases}$$

Example 7.2 F: Piecewise Functions

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people, n, to the cost, C.

Solution:

To set up this function, two different formulas would be needed. $C = 5n$ would work for n values under 10, and $C = 50$ would work for values of n ten or greater. We denote this as follows:

$$C(n) = \begin{cases} 5n & if & 0 < n < 10 \\ 50 & if & n \geq 10 \end{cases}$$

Example 7.2 G: Applications of Piecewise Functions

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & if & 0 < g < 2 \\ 25 + 10(g - 2) & if & g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

Solution:

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving $C(1.5) = 25$.

To find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we'll use the second formula. $C(4) = 25 + 10(4 - 2) = 45$.

Therefore, the cost of using 1.5 gigabytes of data is $25, and the cost of using 4 gigabytes of data is $45.

Example 7.2 H: Graphing a Piecewise Function

Sketch a graph of the function

$$f(x) = \begin{cases} 0 & if & x \leq 1 \\ x - 1 & if & 1 < x \leq 4 \\ 2 & if & x > 4 \end{cases}$$

Solution:

We can imagine graphing each function, then limiting the graph to the indicated domain. At the endpoints of the domain, we put open circles to indicate where the endpoint is not included, due to a strictly-less-than inequality, and a closed circle where the endpoint is included, due to a less-than-or-equal-to inequality. The first and last parts are constant functions, where the output is the same for all inputs. The middle part we might recognize as a line, and could graph by evaluating the function at a couple inputs and connecting the points with a line.
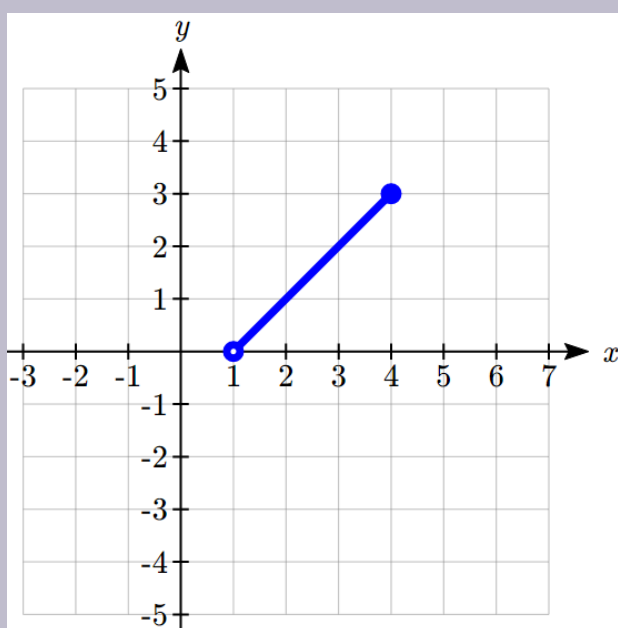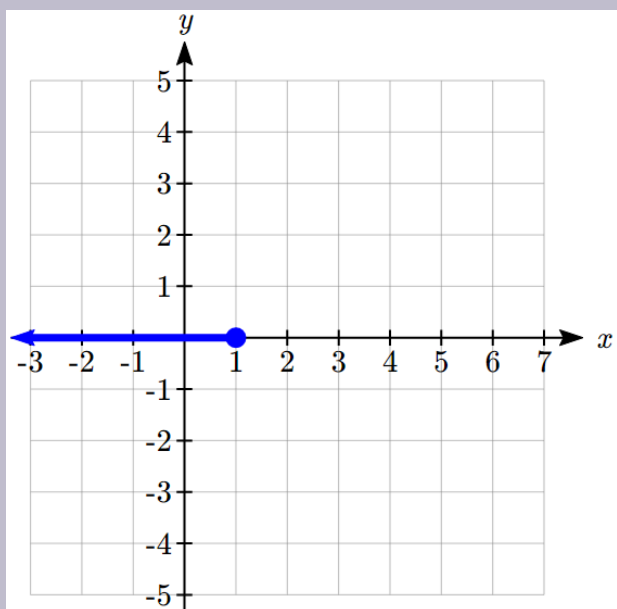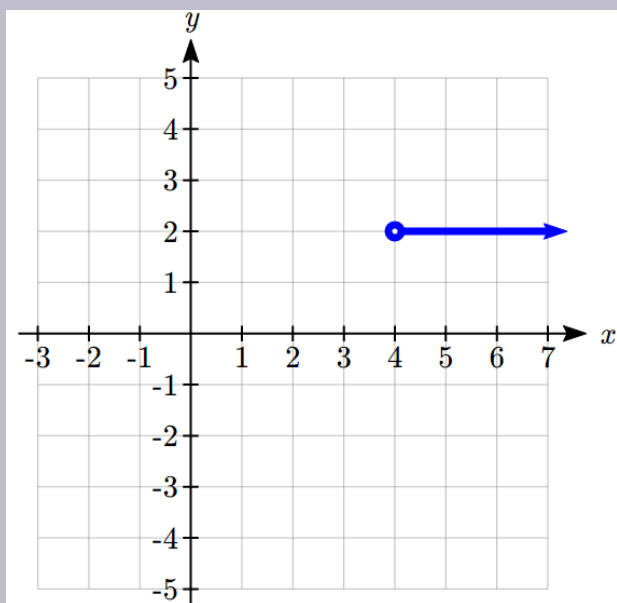
Now that we have each piece individually, we combine them onto the same graph. When the first and second parts meet at $x = 1$, we can imagine the closed dot filling in the open dot. Since there is no break in the graph, there is no need to show the dot.
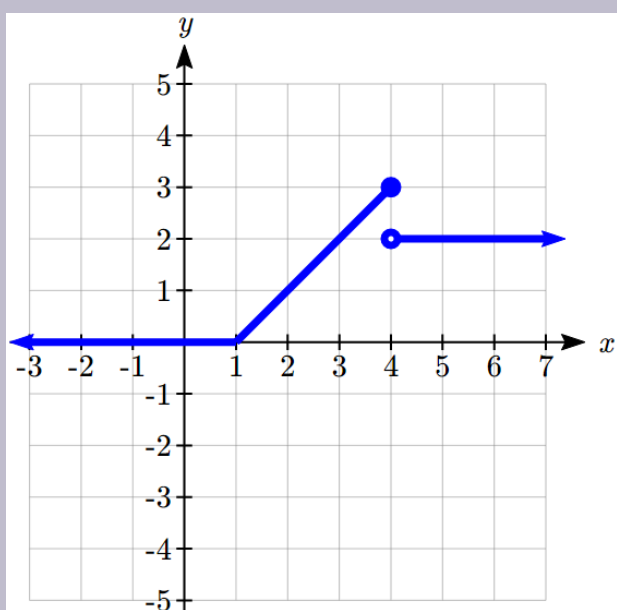
You try...

4. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were $89.50 per credit for the first 10 credits, $33 per credit for credits 11-18, and for over 18 credits the rate is $73 per credit[4].

Write a piecewise defined function for the total tuition, T, at Pierce College during 2009-2010 as a function of the number of credits taken, c. Be sure to consider a reasonable domain and range.

You try... Answers

1. Domain: y = years $[1960, 2010]$
Range: p = population, $[100, 1400]$

2. a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3
b. $\{x | x \leq -2 \ or \ -1 \leq x < 3\}$
c. $(-\infty, -2] \cup [-1, 3)$

3. Domain: y=years $[1952, 2002]$
Range: p=population in millions, $[40, 88]$

4. Tuition, T, as a function of credits, c.

$$T(c) = \begin{cases} 89.5c & if & c \leq 10 \\ 895 + 33(c - 10) & if & 10 < c \leq 18 \\ 1159 + 73(c - 18) & if & c > 18 \end{cases}$$

4. https://www.pierce.ctc.edu/dist/tuition/ref/files/0910_tuition_rate.pdf, retrieved August 6, 2010

Reasonable domain should be whole numbers 0 to (answers may vary), e.g. $[0, 23]$

Reasonable range should be $0 – (answers may vary), e.g. $[0, 1524]$

Section 7.2 Exercises

1. A data storage company rents server space for a flat annual fee of $150 and storage charge of $0.39 per GB for less than 1.8 petabytes per annual contract (one petabyte is 1000 terabytes and one terabyte is 1000 GB).
   a. Write the formula for the total cost C to rent server space as a function of the s gigabytes of data storage requirement.
   b. What is the domain of this function?
   c. What is the range of this function?
   d. Find the total cost to rent server space for 3.5 terabytes of storage space.
   e. Determine how much server space was rented under the annual contract if the bill was $1203.

2. A rental car company rents cars for a flat fee of $20 and an hourly charge of $10.25. Reservations made but cancelled are charged the flat fee. The company policy states that the rentals must be less than 5 days in duration.
   a. Write the formula for the total cost C to rent a car as a function of the hours h the car is rented.
   b. What is the domain of this function?
   c. What is the range of this function?
   d. Find the total cost to rent a car for 2 days and 7 hours.

3. In making strategic decisions in manufacturing we study the relationship between market supply of an item and the price of the item. Suppose the number of units available x at the price of the unit p of a particular product satisfy the following relationship:
   $$2x - 35p = 7000$$
   a. If possible, express the supply as a function of the price, explain what that means and why it is useful. If it is not possible, explain why.
   b. If possible, express the price as a function of the supply, explain what that means and why it is useful. If it is not possible, explain why.
   c. What is the domain of the supply function in terms of price?

4. In making strategic decisions about product pricing, we study the relationship between market demand for an item and the price of the item. Suppose the market demand for x number of units and the price of the unit p of a particular product satisfy the following relationship:
   $$9x + 150p = 6500$$
   a. If possible, express the demand as a function of price, explain what that means and why it is useful. If it is not possible, explain why.
   b. If possible, express the price as a function of demand, explain what that means and why it is useful.

If it is not possible, explain why.

   c. What is the domain of the demand function in terms of price?

   d. What is the domain of the price function in terms of demand?

5. If the revenues from sales of units of a certain product and the cost C of manufacturing x units of the same product can be expressed as $R(x) = 18x$ and $C(x) = 9.37x + 25000$,

   a. Determine the profit function in terms of the number of units .

   b. Calculate and interpret what that means.

   c. Calculate and interpret what that means.

   d. How many units would they have to produce and sell to break even?

6. The population of Oshawa in the year 1960 was 77,000 people. Since then the population has grown to 379,848 people reported during the 2016 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

7. Describe the domains of

   a. $f(x) = \frac{3x+5}{x^2+4x-5}$

   b. $f(x) = \frac{\sqrt{2x-3}}{\sqrt{(5-x)}}$

   c. $f(x) = \frac{x}{\sqrt[3]{x+3}}$

8. A phone data plan has a basic charge of $30 a month. The plan includes first 2GB free and charges $10 for each additional GB, up to 8 GB total usage, after which it charges $15 for each additional GB. If d is the amount of data used (in GB) and is the total monthly cost:

   a. Express $C(d)$ as a (piece-wise) formula.

   b. Identify the independent and the dependent variables of C.

   c. Identify the domain and the range of C.

   d. Graph C as a function of d for $0 \le d \le 10$.

   e. Calculate the cost if 9GB were used.

9. Describe the relationship shown in the graph below as a function and describe its domain and range in

Estimated monthly light crude oil production (Alberta)

Source: open.canada.ca
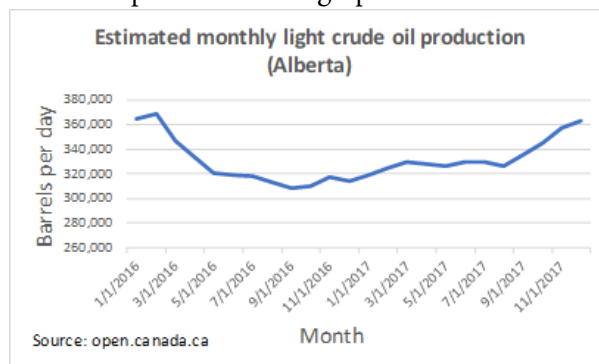
interval form.

10. **Population Growth:** The table below lists the Canada population by census years (source: Statistics Canada).

Census Year	Canadian Population (millions)
1991	27.3
1996	28.8
2001	30.0
2006	31.5
2011	33.5
2016	35.2

a. Using Microsoft Excel, find a linear regression model that represents the size of Canadian population in millions in terms of number of years since 1991 and graph the model.
b. Draw a scatter plot of the data and graph the model.
c. Interpret the slope of the model.
d. Estimate the size of Canadian population in 2003.
e. Predict the population of Canada in 2030.

7.3: Rates of Change and Behaviour of Graphs

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing.

For example, the function $C(t)$ below gives the average cost, in dollars, of a gallon of gasoline t years after 2000.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from $1.47 to $2.14, an increase of $0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by $0.67 from 2002 to 2009, over 7 years, for an average of ₋ dollars per year.

On average, the price of gas increased by about 9.6 cents each year.

Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are "<u>output units</u> per <u>input units</u>"

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns $9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by $4,000 per quarter

Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 7.3 A: Cost Functions

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009.

From the table, in 2007 the cost of gas was $2.64. In 2009 the cost was $2.14.

The input (years) has changed by 2. The output has changed by $2.14 – $2.64 = -0.50.

The average rate of change is then $\frac{-0.50}{2 years} = -0.25$ dollars per year

Give It Some Thought

1. Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Notice that in the last example the change of output was negative since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example 7.3 B: Average Rate of Change

Given the function $g(t)$ shown here, find the average rate of change on the interval $[0, 3]$.
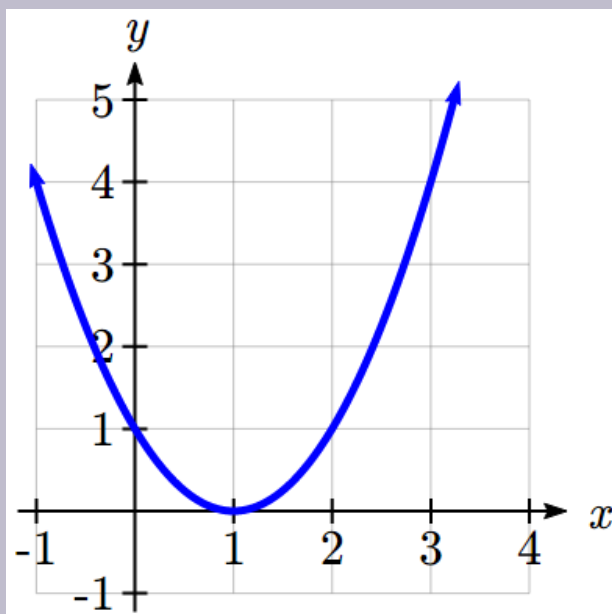
At $t = 0$, the graph shows $g(0) = 1$

At $t = 3$, the graph shows $g(3) = 4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$

Example 7.3 C: Average Speed

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t(hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Here, your average speed is the average rate of change.

You traveled 282 miles in 6 hours, for an average speed of $\frac{292-10}{6-0} = \frac{282}{6} = 47$ miles per hour

We can more formally state the average rate of change calculation using function notation.

Average Rate of Change using Function Notation

Given a function $f(x)$, the average rate of change on the interval $[a, b]$ is

Average rate of change $= \dfrac{\text{Change of Output}}{\text{Change of Input}} = \dfrac{f(b)-f(a)}{b-a}$

Example 7.3 D: Average Rate of Change

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2, 4]$.

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$
$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

Average rate of change $= \frac{f(4) - f(2)}{4 - 2} = \frac{\frac{63}{4} - \frac{7}{2}}{4 - 2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$

Give It Some Thought

2. Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval $[1, 9]$.

Example 7.3 E: Magnetic Force

The magnetic force F, measured in Newtons, between two magnets is related to the distance between the magnets d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval $[2, 6]$.

Average rate of change = $\frac{F(6)-F(2)}{6-2}$

Evaluating the function

$$\frac{F(6)-F(2)}{6-2}$$

$$\frac{\frac{2}{6^2} - \frac{2}{2^2}}{6-2}$$

Simplifying

$$\frac{\frac{2}{36} - \frac{2}{4}}{4}$$

Combining the numerator terms

$$\frac{\frac{-16}{36}}{4}$$

Simplifying further

$$\frac{-1}{9} \text{ Newtons per centimeter}$$

This tells us the magnetic force decreases, on average, by $\frac{1}{9}$ Newtons per centimeter over this interval.

Example 7.3 F: Average Rate of Change

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. Your answer will be an expression involving a.

Using the average rate of change formula

$$\frac{g(a)-g(0)}{a-0}$$

Evaluating the function

$$\frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0}$$

Simplifying

$$\frac{a^2+3a+1-1}{a}$$

Simplifying further, and factoring

$$\frac{a(a+3)}{a}$$

Cancelling the common factor a

$$a+3$$

This result tells us the average rate of change between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be 5+3 = 8.

Give It Some Thought

3. Find the average rate of change of $f(x) = x^3 + 2$ on the interval $[a, a + h]$.

Application to Business

In business, average rate of change is related to the idea of marginal cost, marginal revenue, and marginal profit.

Marginal Cost/Revenue/Profit

Marginal Cost is typically defined as the change in total cost if the quantity of items produced increases by one. Likewise, **marginal revenue** and **marginal profit** are the change in revenue and profit if the quantity of items increases by one.

In practice, marginal cost is usually calculated another way, using calculus techniques you will learn in future courses. In this course, though, we will stick with calculating marginal cost by looking at the actual change of total cost when production increases by one.

Example 7.3 G: Marginal Cost

A company has determined the total cost of producing q items is $TC(q) = 2000 + 50\sqrt{q}$. Find the marginal cost, when production is currently 5000 items.

We want to find the increase in total cost when increasing production from 5000 items to 5001 items. This is equivalent to finding the average rate of change on the interval $[5000, 5001]$.

The total cost at 5000 items is:

$$TC(5000) = 2000 + 50\sqrt{5000} \approx 5535.53$$

The total cost at 5001 items is:

$$TC(5001) = 2000 + 50\sqrt{5001} \approx 5535.89$$

The marginal cost is

5535.89 – 5535.53 = $0.36.

In other words, after producing 5000 items, the additional cost of producing the 5001st item is $0.36

Graphical Behaviour of Functions

As part of exploring how functions change, it is interesting to explore the graphical behaviour of functions.

Increasing/Decreasing

A function is **increasing** on an interval if the function values increase as the inputs increase. More formally, a function is increasing if $f(b) > f(a)$ for any two input values a and b in the interval with $b > a$. The average rate of change of an increasing function is **positive**.

A function is **decreasing** on an interval if the function values decrease as the inputs increase. More formally, a function is decreasing if $f(b) < f(a)$ for any two input values a and b in the interval with $b > a$. The average rate of change of a decreasing function is **negative**.

Example 7.3 H: Graphs of Functions

Given the function $p(t)$ graphed here, on what intervals does the function appear to be increasing?
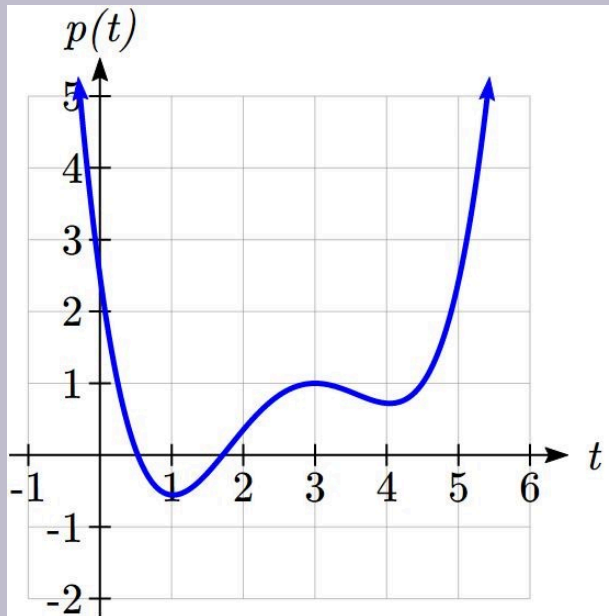
The function appears to be increasing from $t = 1$ to $t = 3$, and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Notice in the last example that we used open intervals (intervals that don't include the endpoints) since the function is neither increasing nor decreasing at $t = 1, 3$, or 4.

Local Extrema

A point where a function changes from increasing to decreasing is called a **local maximum**.

A point where a function changes from decreasing to increasing is called a **local minimum**.

Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

Example 7.3 I: Invervals

Using the cost of gasoline function from the beginning of the section, find an interval on which the function appears to be decreasing. Estimate any local extrema using the table.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

It appears that the cost of gas increased from $t = 2$ to $t = 8$. It appears the cost of gas decreased from $t = 8$ to $t = 9$, so the function appears to be decreasing on the interval $(8, 9)$.

Since the function appears to change from increasing to decreasing at $t = 8$, there is local maximum at $t = 8$.

Example 7.3 J: Estimating Extrema

Use a graph to estimate the local extrema of the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Use these to determine the intervals on which the function is increasing.
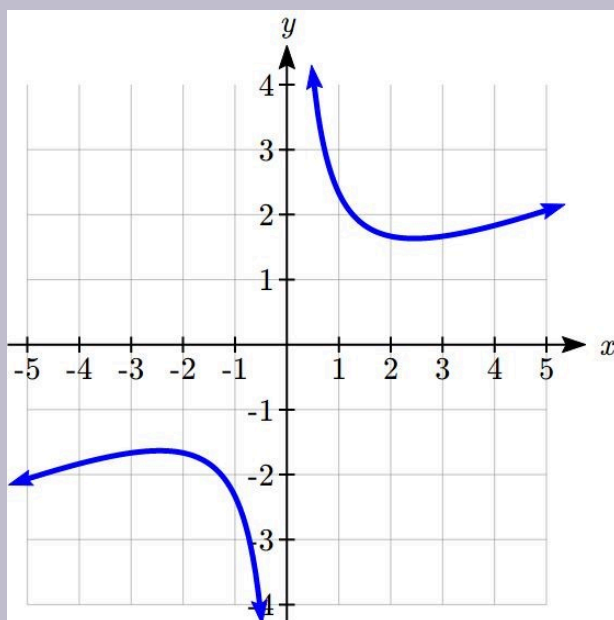
Using technology to graph the function, it appears there is a local minimum somewhere between $x = 2$ and $x = 3$, and a symmetric local maximum somewhere between $x = -3$ and $x = -2$.

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Below are screen images from two different technologies, showing the estimate for the local maximum and minimum.

2.4494898, 1.6329932
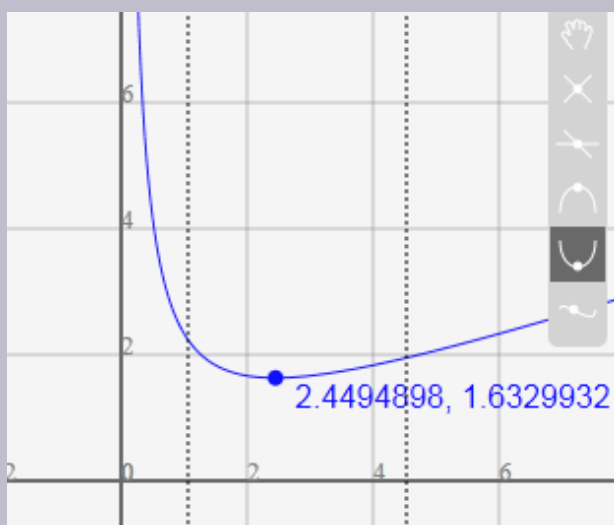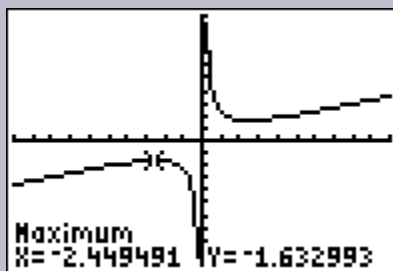
Based on these estimates, the function is increasing on the intervals $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that while we expect the extrema to be symmetric, the two different technologies agree only up to 4 decimals due to the differing approximation algorithms used by each.

Give It Some Thought

4. Use a graph of the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Give It Some Thought Answers

1. $= 0.264$ dollars per year.

2. Average rate of change $= \frac{f(9)-f(1)}{9-1} = \frac{\left(9-2\sqrt{9}\right)-\left(1-2\sqrt{1}\right)}{9-1} = \frac{(3)-(-1)}{9-1} = \frac{4}{8} = \frac{1}{2}$

3. $\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{\left((a+h)^3+2\right)-\left(a^3+2\right)}{h} = \frac{a^3+3a^2h+3ah^2+h^3+2-a^3-2}{h} =$

$\frac{3a^2h+3ah^2+h^3}{h} = \frac{h\left(3a^2+3ah+h^2\right)}{h} = 3a^2 + 3ah + h^2$

4. Based on the graph, the local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.
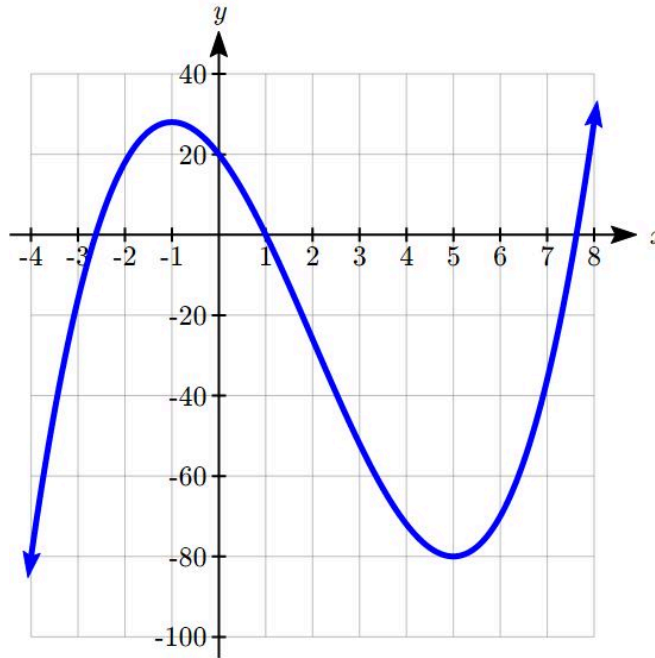
Section 7.3 Exercises

Insert review exercises here

7.4: Linear Functions

As you hop into a taxicab in Las Vegas, the meter will immediately read \$3.30; this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add \$2.60 for each mile the taxi drives[5]. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether

---

5. There is also a waiting fee assessed when the taxi is waiting at red lights, but we'll ignore that in this discussion.

it is possible to model this type of scenario with a function. Using descriptive variables, we choose m for miles and C for Cost in dollars as a function of miles: $C(m)$.

We know for certain that $C(0) = 3.30$, since the $3.30 drop charge is assessed regardless of how many miles are driven. Since $2.60 is added for each mile driven, then:

$$C(1) = 3.30 + 2.60 = 5.90$$

If we then drove a second mile, another $2.60 would be added to the cost:

$$C(2) = 3.30 + 2.60 + 2.60 = 3.30 + 2.60(2) = 8.50$$

If we drove a third mile, another $2.40 would be added to the cost:

$$C(3) = 3.30 + 2.60 + 2.60 + 2.60 = 3.30 + 2.60(3) = 11.10$$

From this we might observe the pattern, and conclude that if m miles are driven, $C(m) = 3.30 + 2.60m$ because we start with a $3.30 drop fee and then for each mile increase we add $2.60.

It is good to verify that the units make sense in this equation. The $3.30 drop charge is measured in dollars; the $2.60 charge is measured in dollars per mile. So

$$C(m) = 3.30 dollars + \left(2.60 \frac{dollars}{mile}\right)(m\ miles)$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.30, and matching the desired units for the C function.

Notice this equation $C(m) = 3.30 + 2.60m$ consisted of two quantities. The first is the fixed $3.30 charge which does not change based on the value of the input. The second is the $2.60 dollars per mile value, which is a **rate of change**. In the equation this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by $2.60 for every 1 mile increase.

m	0	1	2	3
$C(m)$	3.30	5.90	8.50	11.10

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

Graphing this equation, $C(m) = 3.30 + 2.60m$ we see the shape is a line, which is how these functions get their name: **linear functions**.
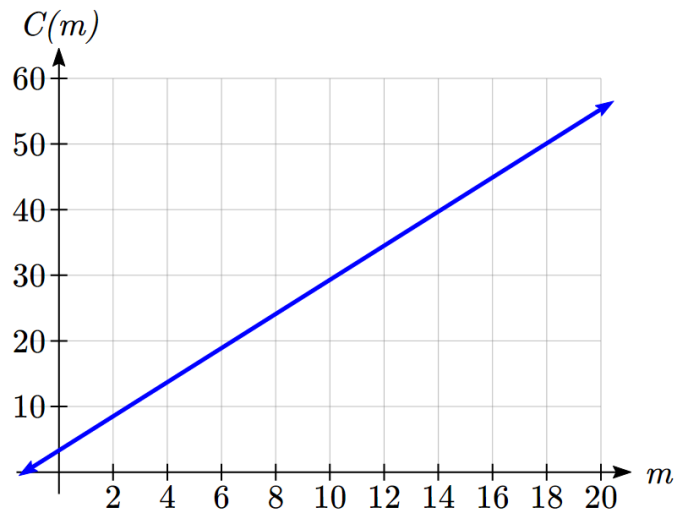
When the number of miles is zero the cost is \$3.30, giving the point (0, 3.30) on the graph. This is the vertical or $C(m)$ intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by \$2.40; this rate remains consistent.

In this example you have seen the taxicab cost modeled in words, an equation, a table and in graphical form. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.

Linear Function

A **linear function** is a function whose graph produces a line. Linear functions can always be written in the form

$$f(x) = b + mx$$

or

$$f(x) = mx + b$$

They're equivalent where:

b is the initial or starting value of the function (when input, $x = 0$), and
m is the constant rate of change of the function

Many people like to write linear functions in the form $f(x) = b + mx$ because it corresponds to the way we tend to speak: "The output starts at b and increases at a rate of m."

For this reason alone we will use the $f(x) = b + mx$ form for many of the examples, but remember they are equivalent and can be written correctly both ways.

Slope and Increasing/Decreasing

m is the constant rate of change of the function (also called slope). The slope determines if the function is an increasing function or a decreasing function.

$$f(x) = b + mx \text{ is an } \textbf{increasing} \text{ function if } m > 0$$

$$f(x) = b + mx \text{ is a } \textbf{decreasing} \text{ function if } m < 0$$

If $m = 0$, the rate of change zero, and the function $f(x) = b + 0x = b$ is just a horizontal line passing through the point $(0, b)$, neither increasing nor decreasing.

Example 7.4A

To start producing a new model of custom wheels, a company will have to purchase $30,000 in new equipment. Each wheel made costs them $40 in supplies and labor. Write a formula for the total cost, TC, of producing q wheels. What will be their total costs in the first year if they expect to sell 240 wheels?

The initial value for this function is 30,000, since that is the startup costs, so $TC(0) = 30,000$. The cost increases by $40 for each wheel made, so the rate of change is $40 per wheel. With this information, we can write the formula:

$$.TC(q) = 30,000 + 40q$$

$TC(q)$ is an increasing linear function.

With this formula we can predict the total cost of 240 wheels:

$$TC(240) = 30,000 + 40(240) = 30,000 + 9,600 = 39,600.$$

The total cost will be $39,600.

Give It Some Thought

1. If you earn $30,000 per year and you spend $29,000 per year write an equation for the amount of money you save after y years, if you start with nothing.
"The most important thing, spend less than you earn!"[6]

Calculating Rate of Change

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, (x_1, y_1) and (x_2, y_2), if we wish to find a linear function that contains both points we can calculate the rate of change, m:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Rate of change of a linear function is also called the **slope** of the line.

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 7.4B

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

The rate of change will relate the change in population to the change in time. The population

---

6. http://www.thesimpledollar.com/2009/06/19/rule-1-spend-less-than-you-earn/

increased by $27800 - 23400 = 4400$ people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400 people}{4 years} = 1100\frac{people}{year} = 1100 \text{ people per year}$$

Notice that we knew the population was increasing, so we would expect our value for m to be positive. This is a quick way to check to see if your value is reasonable.

Example 7.4C

The pressure, P, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, d, in feet, following the equation $P(d) = 14.696 + 0.434d$. Interpret the components of this function.

The rate of change, or slope, 0.434 would have units

$$\frac{output}{input} = \frac{pressure}{depth} = \frac{PSI}{ft}.$$

This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

Example 7.4D

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find the rate of change.

$f(3) = -2$ tells us that the input 3 corresponds with the output -2, and $f(8) = 1$ tells us that the input 8 corresponds with the output 1. To find the rate of change, we divide the change in output by the change in input:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1-(-2)}{8-3} = \frac{3}{5}.$$

If desired we could also write this as $m = 0.6$.

Note that it is not important which pair of values comes first in the subtractions so long as the first output value used corresponds with the first input value used.

Give It Some Thought

2. Given the two points $(2, 3)$ and $(0, 4)$, find the rate of change. Is this function increasing or decreasing?

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

Example 7.4E

Write an equation for the linear function graphed below.
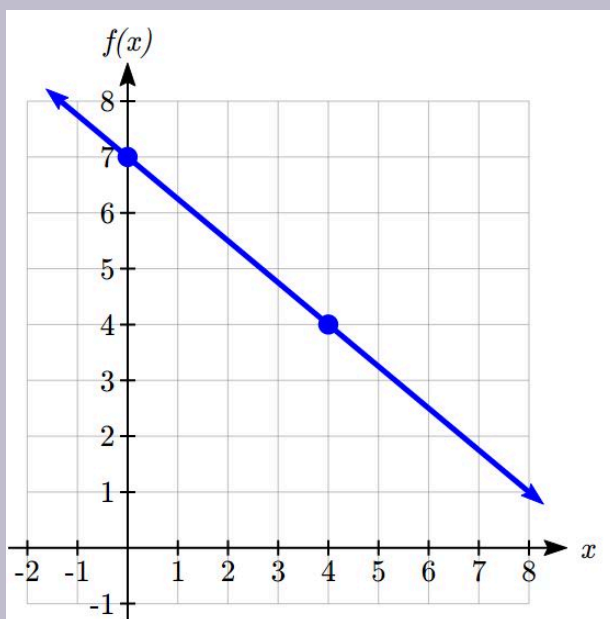
Looking at the graph, we might notice that it passes through the points $(0, 7)$ and $(4, 4)$. From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4-7}{4-0} = \frac{-3}{4}$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x$$

Example 7.4F

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.

In Example 7.4C, we computed the rate of change to be $m = \frac{3}{5}$. In this case, we do not know the initial value $f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3.

$$f(3) = b + \frac{3}{5}(3)$$

Since we know that $f(3) = -2$, we can substitute on the left side

$$-2 = b + \frac{3}{5}(3)$$

This leaves us with an equation we can solve for the initial value

$$b = -2 - \frac{9}{5} = \frac{-19}{5}$$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = \frac{-19}{5} + \frac{3}{5}x$$

Example 7.4G

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies, and earned $920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy; Ilya earns a commission of $80 for each policy sold during the week.

We can then solve for the initial value

$$I(n) = b + 80n$$

then when $n = 3, I(3) = 760$, giving

$$760 = b + 80(3)$$

this allows us to solve for b

$$b = 760 - 80(3) = 520$$

This value is the starting value for the function. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Writing the final equation:

$$I(n) = 520 + 80n$$

Our final interpretation is: Ilya's base salary is $520 per week and he earns an additional $80 commission for each policy sold each week.

Give It Some Thought

3. Looking at Example 7.4G:

Determine the independent and dependent variables.

What is a reasonable domain and range?

4. The balance in your college payment account, C, is a function of the number of quarters, q, you

attend. Interpret the function $C(a) = 20000 - 4000q$ in words. How many quarters of college can you pay for until this account is empty?

Example 7.4H

Given the table below write a linear equation that represents the table values

w, number of weeks	0	2	3	6
$P(w)$, number of rats	1000	1080	1160	1240

We can see from the table that the initial value of rats is 1000 so in the linear format $P(w) = b + mw, b = 1000$.

Rather than solving for m, we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6. The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write

$$P(w) = b + mw$$

as

$$P(w) = 1000 + 40w$$

If you didn't notice this from the table you could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$,

$$m = \frac{1240 - 1080}{6 - 2} = \frac{160}{4} = 40 \text{ rats per week}$$

Give It Some Thought Answers

1. $S(y) = 30,000y - 29,000y = 1000y$ $1000 is saved each year.

2. $m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; Decreasing because $m < 0$

3. n (number of policies sold) is the independent variable

$I(n)$ (weekly income as a function of policies sold) is the dependent variable.

A reasonable domain is $(0, 15)^*$

A reasonable range is $(\$540, \$1740)^*$

*answers may vary given reasoning is stated; 15 is an arbitrary upper limit based on selling 3 policies per day in a 5 day work week and $1740 corresponds with the domain.

4. Your College account starts with $20,000 in it and you withdraw $4,000 each quarter (or your account contains $20,000 and decreases by $4000 each quarter.) You can pay for 5 quarters before the money in this account is gone.

Section 7.4 Exercises

1. Sarah wants to go skating at Super Skate ice rink. She has to pay a $7 entrance fee and $1.25 for every minute she is on the rink.
   a. Write an equation to determine the cost (C) in terms of the number of minutes (t) that she is on the rink.
   b. If she only has $43.25, find the number of minutes she can be on the rink.

2. If you earn $30,000 per year and you spend $29,000 per year, write amount of money you save A after t years, assuming you start with no money.

3. The balance in your post-secondary studies savings account $C(t) = 30000 - 6000t$, is a function of the number of terms t you attend. Interpret the function in words and explain the meaning of each number and symbol in this equation. For how many terms of post-secondary education can you pay until this account is empty?

4. A manager for a country market will spend a total of $80 on apples at $0.25 each and pears at $0.50 each. Write the number of apples she can buy as a linear function of the number of pears. Find the slope and interpret your answer. Graph the function.

5. At $10 per ticket, Willie Williams and the Wranglers will fill all 8,000 seats in the Assembly Center. The manager knows that for every $1 increase in the price, 500 tickets will go unsold.
   1. Write the number of tickets sold n as a function of the ticket price p.
   2. What are the limits of the independent variable, if any?

6. At a price of $2.28 per bushel, the supply of barley is 7,500 million bushels and the demand is 7,900

million bushels. At a price of $2.37 per bushel, the supply is 7,900 million bushels and the demand is 7,800 bushels.

    a. Assuming that price and supply are linearly related, determine the price in terms of supply (the *price-supply equation*).

    b. Assuming that price and demand are linearly related, determine the price in terms of demand (the *price-demand equation*).

    c. Find the equilibrium point (price and the number of units for which supply and demand are equal).

    d. Graph the price-supply equation, price-demand equation and the equilibrium point in the same coordinate system.

7.5: Graphs of Linear Functions

When we are working with a new function, it is useful to know as much as we can about the function: its graph, where the function is zero, and any other special behaviours of the function.

When graphing a linear function, there are two basic ways to graph it:

- By plotting points (at least 2) and drawing a line through the points
- Using the initial value (output when x = 0) and rate of change (slope)

Example 7.5A

Graph $f(x) = 5 - \frac{2}{3}x$ by plotting points

In general, we evaluate the function at two or more inputs to find at least two points on the graph. Usually it is best to pick input values that will "work nicely" in the equation. In this equation, multiples of 3 will work nicely due to the $\frac{2}{3}$ in the equation, and of course using $x = 0$ to get the vertical intercept. Evaluating $f(x)$ at x = 0, 3 and 6:

$$f(0) = 5 - \tfrac{2}{3}(0) = 5$$
$$f(3) = 5 - \tfrac{2}{3}(3) = 3$$
$$f(6) = 5 - \tfrac{2}{3}(6) = 1$$

These evaluations tell us that the points $(0, 5)$, $(3, 3)$, and $(6, 1)$ lie on the graph of the line. Plotting these points and drawing a line through them gives us the graph.
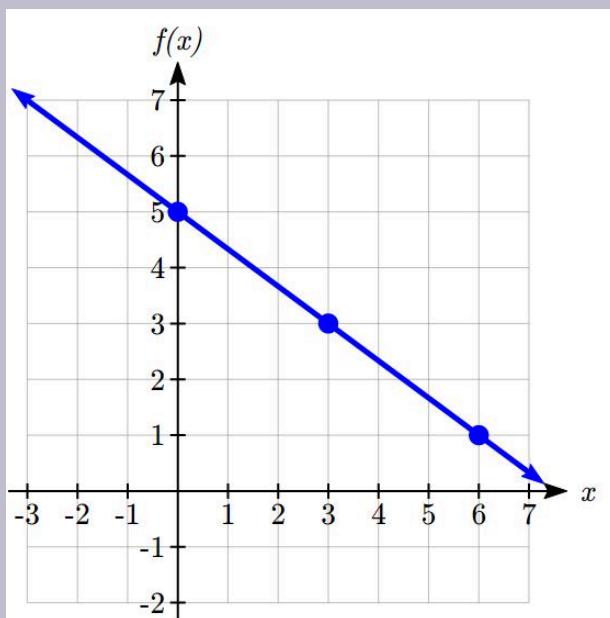
When using the initial value and rate of change to graph, we need to consider the graphical interpretation of these values. Remember the initial value of the function is the output when the input is zero, so in the equation $f(x) = b + mx$, the graph includes the point $(0, b)$. On the graph, this is the vertical intercept – the point where the graph crosses the vertical axis.

For the rate of change, it is helpful to recall that we calculated this value as

$$m = \frac{\text{change of output}}{\text{change of input}}$$

From a graph of a line, this tells us that if we divide the vertical difference, or rise, of the function outputs by the horizontal difference, or run, of the inputs, we will obtain the rate of change, also called slope of the line.

$$m = \frac{\text{change of output}}{\text{change of input}} = \frac{rise}{run}$$

Notice that this ratio is the same regardless of which two points we use
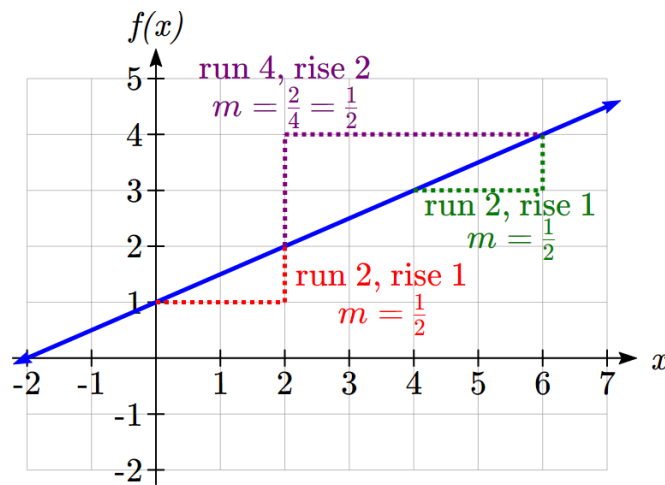
Graphical Interpretation of a Linear Equation

Graphically, in the equation $f(x) = b + mx$

b is the **vertical intercept** of the graph and tells us we can start our graph at $(0, b)$

m is the **slope of the line** and tells us how far to rise & run to get to the next point

Once we have at least 2 points, we can extend the graph of the line to the left and right.

Example 7.5B

Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope.

The vertical intercept of the function is $(0, 5)$, giving us a point on the graph of the line.

The slope is $-\frac{2}{3}$. This tells us that for every 3 units the graph "runs" in the horizontal, the vertical "rise" decreases by 2 units. In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value $(0, 5)$ the slope tells us that if we move to the right 3, we will move down 2, moving us to the point $(3, 3)$. We can continue this again to find a third point at $(6, 1)$. Finally, extend the line to the left and right, containing these points.
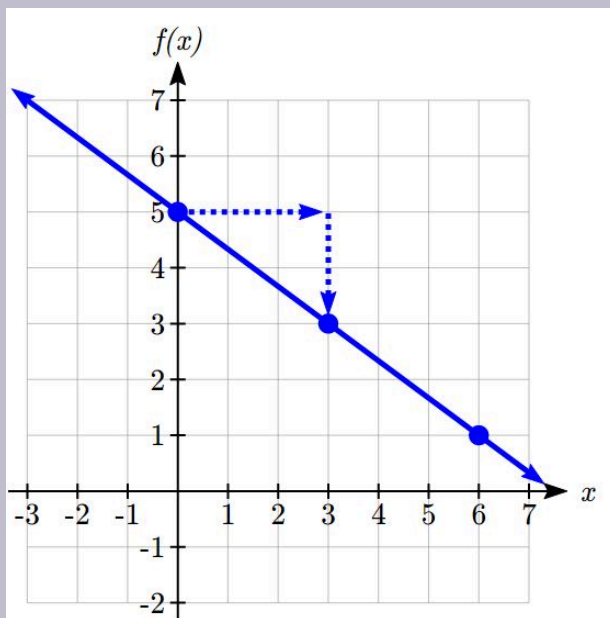
Give It Some Thought

1. Consider that the slope -2/3 could also be written as $\frac{2}{-3}$. Using $\frac{2}{-3}$, find another point on the graph that has a negative x value.

In $f(x) = mx + b$, the value of m determines the steepness of the line. When m is positive the graph will be increasing, and when m is negative the graph will be decreasing. Looking at some examples:
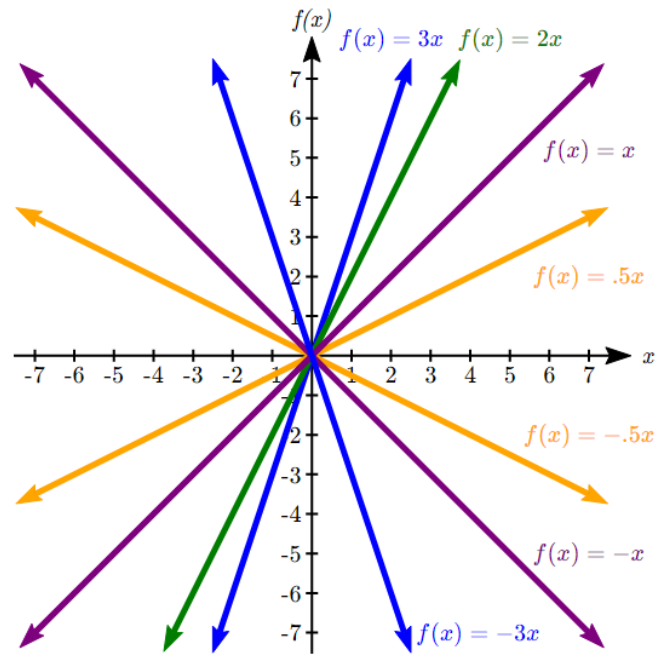
In $f(x) = mx + b$, the b acts as a vertical shift, moving the graph up and down without affecting the slope of the line. Some examples:
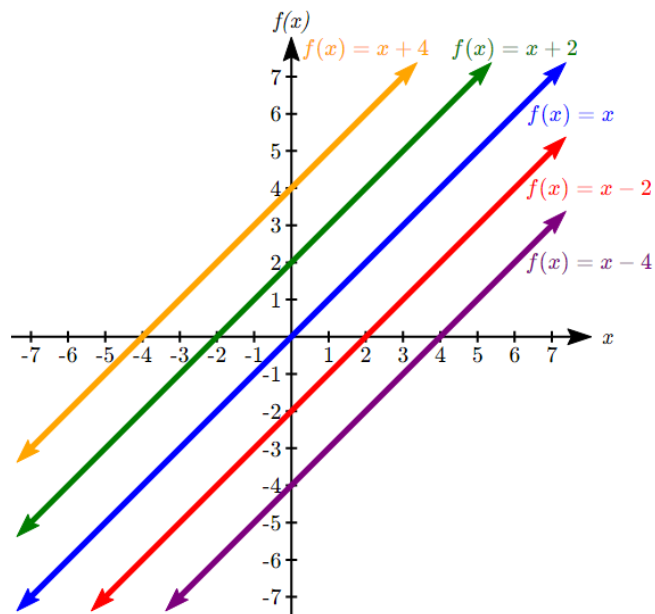
Example 7.5C

Match each equation with one of the lines in the graph below

$$f(x) = 2x + 3$$
$$g(x) = 2x - 3$$
$$h(x) = -2x + 3$$
$$j(x) = \tfrac{1}{2}x + 3$$
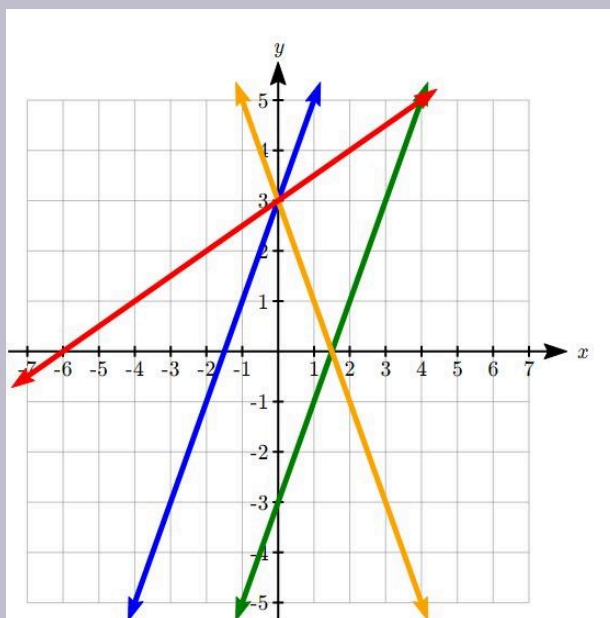
Only one graph has a vertical intercept of -3, so we can immediately match that graph with $g(x)$. For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with $h(x)$. Of the other two, the steeper line would have a larger slope, so we can match that graph with equation $f(x)$, and the flatter line with the equation $j(x)$.
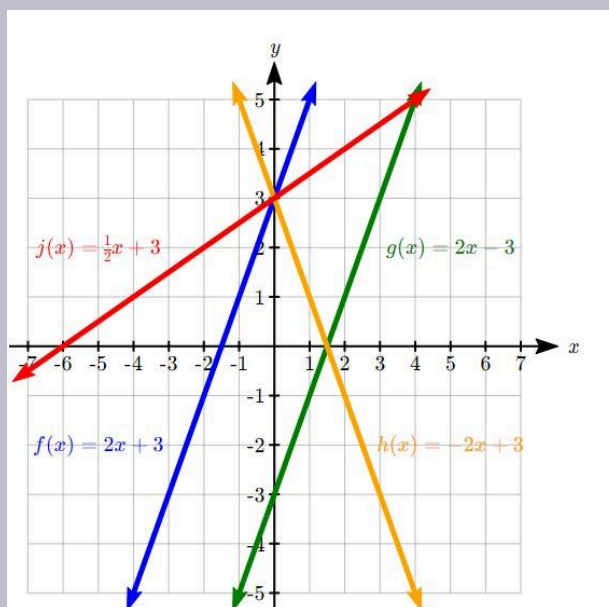
In addition to understanding the basic behaviour of a linear function (increasing or decreasing, recognizing the slope and vertical intercept), it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

Finding Horizontal Intercept

The **horizontal intercept** of the function is where the graph crosses the horizontal axis. If a function has a horizontal intercept, you can always find it by solving $f(x) = 0$.

Example 7.5D

Find the horizontal intercept of $f(x) = -3 + \frac{1}{2}x$.

Setting the function equal to zero to find what input will put us on the horizontal axis:

$$0 = -3 + \tfrac{1}{2}x$$
$$3 = \tfrac{1}{2}x$$
$$x = 6$$

The graph crosses the horizontal axis at (6,0)

There are two special cases of lines: a horizontal line and a vertical line. In a horizontal line like the one graphed below, notice that between any two points, the change in the outputs is 0. In the slope equation, the numerator will be 0, resulting in a slope of 0. Using a slope of 0 in the $f(x) = b + mx$, the equation simplifies to $f(x) = b$
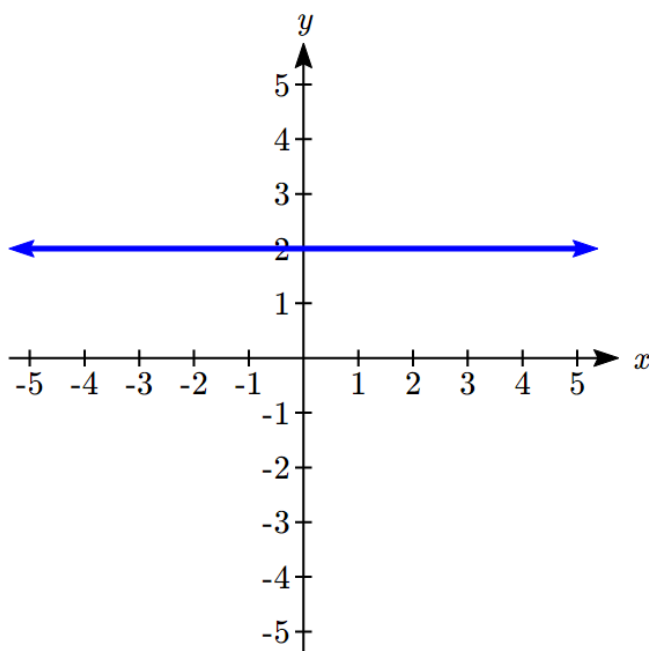
Notice a horizontal line has a vertical intercept, but no horizontal intercept (unless it's the line $f(x) = 0$).

In the case of a vertical line, notice that between any two points, the change in the inputs is 0. In the slope equation, the denominator will be 0, and you may recall that we cannot divide by the 0; the slope of a vertical line is undefined. You might also notice that a vertical line is not a function. To write the equation of vertical line, we simply write input=value, like $x = b$.
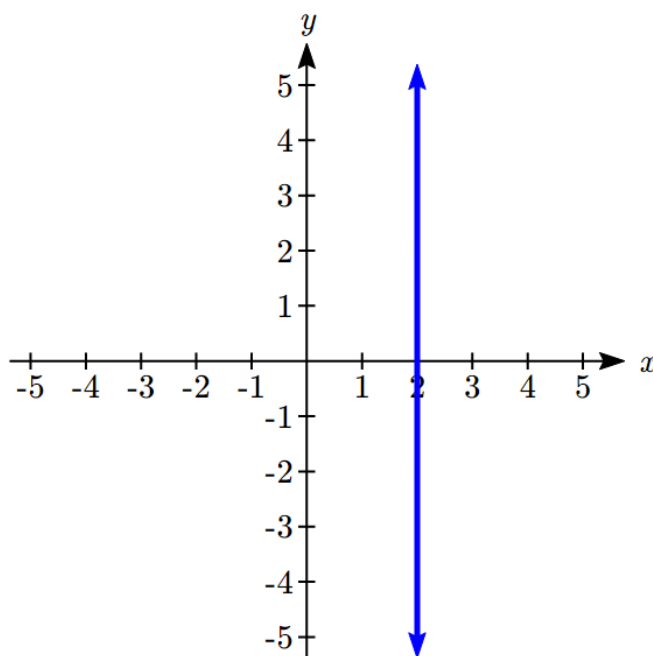
Notice a vertical line has a horizontal intercept, but no vertical intercept (unless it's the line $x = 0$).

Horizontal and Vertical Lines

Horizontal lines have equations of the form $f(x) = b$

Vertical lines have equations of the form $x = a$

Example 7.5E

Write an equation for the horizontal line graphed above.

This line would have equation $f(x) = 2$

Example 7.5F

Write an equation for the vertical line graphed above.

This line would have equation $x = 2$

Parallel and Perpendicular Lines

When two lines are graphed together, the lines will be **parallel** if they are increasing at the same rate – if the rates of change are the same. In this case, the graphs will never cross (unless they're the same line).

Parallel Lines

Two lines are **parallel** if the slopes are equal (or, if both lines are vertical). In other words, given two linear equations $f(x) = b + m_1 x$ and $g(x) = b + m_2 x$

The lines will be parallel if $m_1 = m_2$

Example 7.5G

Find a line parallel to $f(x) = 6 + 3x$ that passes through the point $(3, 0)$

We know the line we're looking for will have the same slope as the given line, $m = 3$. Using this and the given point, we can solve for the new line's vertical intercept:

$$g(x) = b + 3x$$

then at $(3, 0)$,

$$0 = b + 3(3)$$
$$b = -9$$

The line we're looking for is $g(x) = -9 + 3x$

If two lines are not parallel, one other interesting possibility is that the lines are perpendicular, which means the lines form a right angle (90 degree angle – a square corner) where they meet. In this case, the slopes when multiplied together will equal -1. Solving for one slope leads us to the definition:

Perpendicular Lines

Given two linear equations $f(x) = b + m_1 x$ and $g(x) = b + m_2 x$

The lines will be **perpendicular** if $m_1 m_2 = -1$, and so $m_2 = \frac{-1}{m_1}$

We often say the slope of a perpendicular line is the "negative reciprocal" of the other line's slope.

Example 7.5H

Find the slope of a line perpendicular to a line with:
a) a slope of 2.
b) a slope of -4.
c) a slope of $\frac{2}{3}$.

If the original line had slope 2, the perpendicular line's slope would be

$$m_2 = \frac{-1}{2}$$

If the original line had slope -4, the perpendicular line's slope would be

$$m_2 = \frac{-1}{-4} = \frac{1}{4}$$

If the original line had slope $\frac{2}{3}$, the perpendicular line's slope would be

$$m_2 = \frac{\frac{-1}{2}}{3} = \frac{-3}{2}$$

Example 7.5I

Find the equation of a line perpendicular to $f(x) = 6 + 3x$ and passing through the point $(3, 0)$

The original line has slope $m = 3$. The perpendicular line will have slope $m = \frac{-1}{3}$. Using this and the given point, we can find the equation for the line.

$$g(x) = b - \tfrac{1}{3}x$$

then at (3, 0),

$$0 = b - \tfrac{1}{3}(3)$$
$$b = 1$$

The line we're looking for is $g(x) = 1 - \tfrac{1}{3}x$

Give It Some Thought

2. Given the line $h(t) = -4 + 2t$, find an equation for the line passing through $(0, 0)$ that is:
a) parallel to $h(t)$.
b) perpendicular to $h(t)$.

Example 7.5J

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

From the two given points on the reference line, we can calculate the slope of that line:

$$m_1 = \frac{5-6}{4-(-2)} = \frac{-1}{6}$$

The perpendicular line will have slope

$$m_2 = \frac{-1}{\frac{-1}{6}} = 6$$

We can then solve for the vertical intercept that makes the line pass through the desired point:

$$g(x) = b + 6x$$

then at $(4, 5)$,

$$5 = b + 6(4)$$
$$b = -19$$

Giving the line $g(x) = -19 + 6x$

Intersections of Lines

The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

Example 7.5K

Find the intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$

Setting $h(t) = j(t)$,

$$3t - 4 = 5 - t$$
$$4t = 9$$
$$t = \tfrac{9}{4}$$

This tells us the lines intersect when the input is $\tfrac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input

$$j\left(\tfrac{9}{4}\right) = 5 - \tfrac{9}{4} = \tfrac{11}{4}$$

These lines intersect at the point $\left(\tfrac{9}{4}, \tfrac{11}{4}\right)$. Looking at the graph, this result seems reasonable.
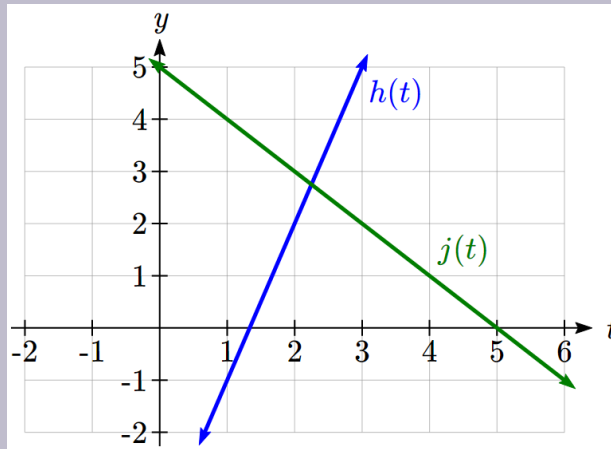
Two parallel lines can also intersect if they happen to be the same line. In that case, they intersect at every point on the lines.

Give It Some Thought

3. Look at the graph in Example 7.5K above and answer the following for the function $j(t)$:

a. Vertical intercept coordinates

b. Horizontal intercepts coordinates

c. Slope

d. Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)

e. Is $j(t)$ an Increasing or Decreasing function (or neither)

f. Write a transformation description from the identity toolkit function $f(x) = x$

Finding the intersection allows us to answer other questions as well, such as discovering when one function is larger than another.

Example 7.5L

Using the functions from the previous example, for what values of t is $h(t) > j(t)$

To answer this question, it is helpful first to know where the functions are equal, since that is the point where $h(t)$ could switch from being greater to smaller than $j(t)$ or vice-versa. From the previous example, we know the functions are equal at $t = \frac{9}{4}$.

By examining the graph, we can see that $h(t)$, the function with positive slope, is going to be larger than the other function to the right of the intersection. So $h(t) > j(t)$ when $t > \frac{9}{4}$

Give It Some Thought Answers

1. $(-3, 7)$ found by starting at the vertical intercept, going up 2 units and 3 in the negative horizontal direction. You could have also answered: $(-6, 9)$ or $(-9, 11)$ etc...

2. Parallel: $f(t) = 2t$

Perpendicular: $g(t) = -\frac{1}{2}t$

3. Given $j(t) = 5 - t$

a. (0,5)

b. (5,0)

c. Slope -1

d. Neither parallel nor perpendicular

e. Decreasing function

f. Given the identity function, perform a vertical flip (over the t axis) and shift up 5 units.

Section 7.5 Exercises

1. Sketch the graph of the following function.

   $f(x) = \frac{2}{3}x + 1$

2. Find the $x-$ and y-intercepts of the following lines and graph the lines:

   1. $\frac{x}{2} - \frac{y}{3} = 600$
   2. $0.09x - 0.06y = 54$

3. Find the slope of the following lines and graph the lines:

   1. $3x - 5y = 1.5$
   2. $y + 11 = 0$

4. Find the equations of the lines passing through the following points, identify – and -intercepts and slope for each, and sketch their graphs.

   1. $(-2, 0)$ and $(0, 3)$
   2. $\left(\frac{3}{4}, -3\right)$ and $\left(-5, \frac{1}{8}\right)$

5. Consider the points $(3, 2)$ and $(5, -2)$ and the linear function whose graph passes through them.

   1. Find the slope of the line.
   2. Find the slope-intercept form of the equation of the line.
   3. Write the equation of the line in the standard form.

6. Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope. Using the fact that the slope $-\frac{2}{3}$ could also be written as $\frac{2}{-3}$, find a point on the graph that has a negative x value.

7. Consider the graph of $j(t) = 5 - t$ and determine the following for the function :

   1. Vertical intercept coordinates
   2. Horizontal intercepts coordinates
   3. Slope
   4. Is $j(t)$ an increasing or decreasing function (or neither)?

7.6: Modeling with Linear Functions

When modeling scenarios with a linear function and solving problems involving quantities changing linearly, we typically follow the same problem solving strategies that we would use for any type of function:

Problem solving strategy

1. Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3. Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5. When needed, find a formula for the function.
6. Solve or evaluate using the formula you found for the desired quantities.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Example 7.6A

A company purchased $120,000 in new office equipment. Then expect the value to depreciate (decrease) by $16,000 per year. Find a linear model for the value, then find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

In the problem, there are two changing quantities: time and value. The remaining value of the equipment depends on how long the company has had it. We can define our variables, including units.

Output: V, value remaining, in dollars
Input: t, time, in years

Reading the problem, we identify two important values. The first, $120,000, is the initial value for V. The other value appears to be a rate of change – the units of dollars per year match the units of our output variable divided by our input variable. The value is depreciating, so you should recognize that the value remaining is decreasing each year and the slope is negative.

Using the intercept and slope provided in the problem, we can write the equation:

$$V(t) = 120,000 - 16,000t$$

To find the horizontal intercept, we set the output to zero, and solve for the input:

$$0 = 120,000 - 16,000t$$
$$t = \frac{120,000}{16,000} = 7.5$$

The horizontal intercept is 7.5 years. Since this represents the input value where the output will be zero, interpreting this, we could say: The equipment will have no remaining value after 7.5 years.

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than zero. This model is also not valid after the horizontal intercept.

The domain represents the set of input values and so the reasonable domain for this function is $0 \leq t \leq 7.5$. The range represents the set of output values and the value starts at $120,000 and ends with $0 after 7.5 years so the corresponding range is $0 \leq V(t) \leq 120,000$.

Most importantly remember that domain and range are tied together, and whatever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

Example 7.6B

Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of $20,

then 59 cents a mile. The second, Budget, charges an up-front fee of $16, then 63 cents a mile[7]. When will U-Haul be the better choice for Jamal?

The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input: m, miles driven

Outputs:
$Y(m)$: cost, in dollars, for renting from U-Haul
$B(m)$: cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: $0.59 a mile for U-Haul, and $0.63 a mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where the $Y(m)$ function is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$
$$B(m) = 16 + 0.63m$$

These graphs are sketched to the below, with $Y(m)$ drawn dashed.

---

7. Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/
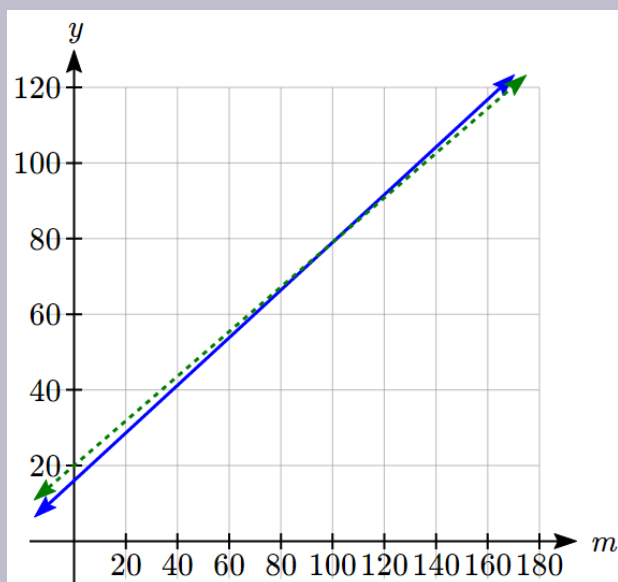
To find the intersection, we set the equations equal and solve:

$$Y(m) = B(m)$$
$$20 + 0.59m = 16 + 0.63m$$
$$4 = 0.04m$$
$$m = 100$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $Y(m)$ is growing at a slower rate, we can conclude that U-Haul will be the cheaper price when more than 100 miles are driven.

Example 7.6C

A company's revenue has been growing linearly. In 2012 their revenue was $1.45 million. By 2015 the revenue had grown to $1.81 million. If this trend continues,

a. Predict the revenue in 2020

b. When will revenue reach $3 million?

The two changing quantities are the revenue and time. While we could use the actual year value as

the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2012:

Input: t, years since 2012

Output: $R(t)$, the company's revenue, in millions of dollars

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2012 would correspond to $t = 0$, giving the point $(0, 1.45)$. Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2015 would correspond to $t = 3$, giving the point $(3, 1.81)$.

To predict the population in 2020 ($t = 8$), we would need an equation for the population. Likewise, to find when the revenue would reach $3 million, we would need to solve for the input that would provide an output of 3. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{1.81 - 1.45}{3 - 0} = \frac{0.36}{3} = 0.12 \text{ million dollars per year}$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$R(t) = 1.45 + 0.12t$$

To predict the revenue in 2020, we evaluate our function at $t = 8$

$$R(8) = 1.45 + 0.12(8) = 2.41$$

If the trend continues, our model predicts a revenue of $2.41 million in 2020.

To find when the population will reach $3 million, we can set $R(t) = 3$ and solve for t.

$$15000 = 6200 + 380t$$
$$8800 = 380t$$
$$t \approx 23.158$$
$$3 = 1.45 + 0.12t$$
$$1.55 = 0.12t$$
$$t \approx 12.917$$

Our model predicts the revenue will reach $3 million in a little under 13 years after 2012, or just before the year 2025.

In business, a very common application of functions is to model cost, revenue, and profit.

Cost, Revenue, Profit

When a company produces q items, the **total cost** is the cost of total cost of producing those items. The total cost includes both **fixed costs**, which are startup costs, like equipment and buildings, and **variable costs**, which are costs that depend on the number of items produced, like materials and labor.

In the most simple case, **Total Cost = (Fixed Costs) + (Variable Costs)** $\cdot\, q$

Revenue is the amount of money a company brings in from sales.
In the most simple case, **Revenue = (Revenue per item)** $\cdot\, q$

Profit is the amount of money brings in, after expenses.

Profit = Revenue − Costs

We often talk about the **break-even** point. This is the level of production where Revenue equals Cost, or equivalently where Profit is zero. This is typically the minimum level of sales necessary for the company to make a profit.

Example 7.6D

A tech startup is looking at developing and launching a new mobile app. Initial development of the app will cost $300,000, and they estimate marketing and support for each user will cost $0.50. While the app will be free, they estimate they will be able to bring in $2 per user on average from in-app purchases. How many users will the company need to break even?

We start by modeling the cost, revenue, and profit. Let q = number of users.

The fixed (initial) costs are $300,000, and the variable (per-item) costs as $0.50 per user. We can write the total cost equation:

$$TC(q) = 300,000 + 0.50q$$

The revenue will be $2 per user, so the revenue equation will be:

$$R(q) = 2q$$

We could find the break-even point by setting the total cost equal to the revenue, which is equivalent to finding the intersection of the lines.
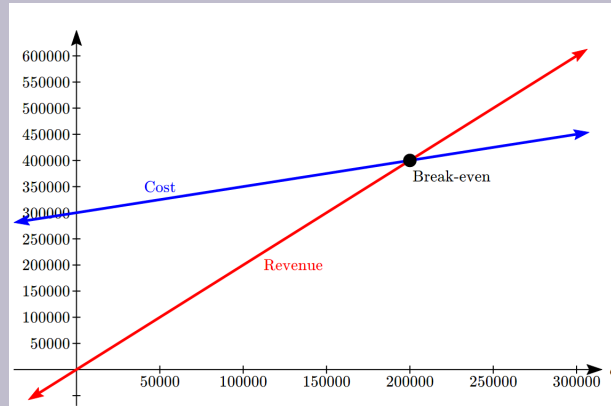
Alternatively, we could go ahead and find a profit equation first:

$$P(q) = R(q) - TC(q) = 2q - (300,000 + 0.50q) = 1.5q - 300,000$$

The break even point can be found by setting the profit equal to zero:

$$0 = 1.5q - 300,000$$
$$q = 200,000$$

The company will have to acquire 200,000 users to break even.

Give It Some Thought

1. A donut shop estimates their fixed daily expenses are $600. If each donut costs about $0.05 to make and sells for $0.60, how many donuts do they need to sell to break even?

In economics, there is a model for how prices are determined in a free market which states that **supply and demand** for a product is related to the price. The demand relationship shows the quantity of a certain product that consumers are willing to buy at a certain price. Typically the quantity demanded will decrease for an item if the price increases. The supply relationship shows the quantity of a product that suppliers are willing to produce at a certain sales price. Typically the supply demanded will increase if the price increases. Economic theory says that supply and demand will interact, and the intersection will be the equilibrium price, or market price, where the quantity supplied and demanded will be equal.

Supply and Demand

If p is the price of a product, then

Qd is the quantity demanded

Qs is the quantity supplied

The demand curve is a decreasing function, while the supply curve is an increasing function.

The intersection of the curves is the **equilibrium price and quantity**, also called the **market price and quantity**. This point is often notated as $p*, Q*$.

In later chapters you will explore supply and demand curves that are non-linear, but in this chapter we will focus on linear supply and demand functions.

In most economic books, you will see the supply and demand curve written with price as the input and quantity as the output, like $Q_d = 140 - 2p$. However, supply and demand graphs are drawn with price on the vertical axis and quantity on the horizontal. In an effort to avoid confusion, most of the time in this text we will instead write supply and demand curves with price as the output, to match its placement on the vertical axis.

Example 7.6E

At a price of $2.50 per gallon, there is a demand in a certain town for 42.5 thousand gallons of gas and a supply of 20 thousand gallons. At a price of $3.50, there is demand for 25.5 thousand gallons and a supply of 28 thousand gallons. Assuming supply and demand are linear, find the equilibrium price and quantity.

We start by finding a linear equation for both supply and demand. We will use price, p in dollars, as the output and quantity, q in thousands of gallons, as the input.

For supply, we have the points $(20, 2.50)$ and $(28, 3.50)$.

Finding the slope:

$$m = \frac{3.50 - 2.50}{28 - 20} = \frac{1}{8}$$

We know the equation will look like $p = \frac{1}{8}q + b$, so substituting in $(20, 2.50)$,

$$2.5 = \tfrac{1}{8}(20) + b$$
$$2.5 = 2.5 + b$$
$$b = 0$$

The supply equation is:

$$p = \tfrac{1}{8}q$$

For demand, we have the points $(42.5, 2.50)$ and $(25.5, 3.50)$. Using a similar approach, we can find the demand equation is:

$$p = -\tfrac{1}{17}q + 5$$

To find the equilibrium, we set the supply equal to the demand:

$$\tfrac{1}{8}q = -\tfrac{1}{17}q + 5$$

Multiplying through by 8(17) = 136 to clear the fractions

$$136\left(\tfrac{1}{8}q\right) = 136\left(-\tfrac{1}{17}q + 5\right)$$
$$17q = -8q + 680$$

Now we solve for q

$$25q = 680$$
$$q = 27.2$$

To find the equilibrium price, we can substitute that value back into either equation:

$$p = \tfrac{1}{8}(27.2) = 3.4$$

The equilibrium quantity will be 27.2 thousand gallons of gas at a price of $3.40.

418 | FUNCTIONS AND LINES
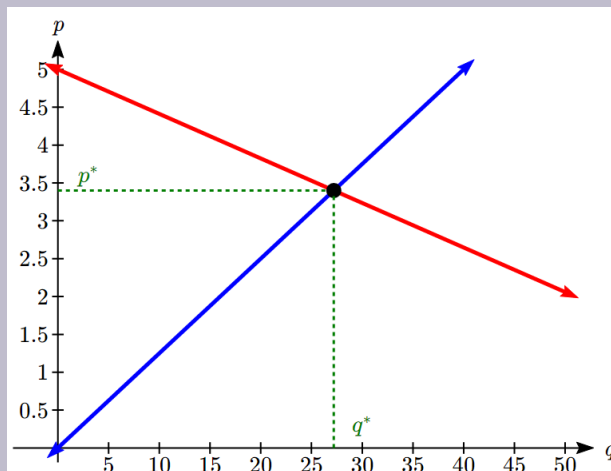
Give It Some Thought

2. A company estimates that at a price of $140 there will demand for 4000 items, and for each $5 increase in price the demand will drop by 200 items. The supply curve is $p = \frac{1}{20}q$. Find the equilibrium price and quantity.

Give It Some Thought Answers

1. Revenue: $R = 0.60q$

Break even when $R = C$, at a quantity of about 1091 donuts per day.

2. Demand: $p = -\frac{1}{40}q + 240$

Supply = Demand when $q = 3200, p = \$160$

Section 7.6 Exercises

1. A company purchased $120,000 in new office equipment. Then expect the value to depreciate (decrease) by $16,000 per year. Find a linear model for the value V of the equipment t years after purchase, then find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

2. At a price of $2.28 per bushel, the supply of barley is 7,500 million bushels and the demand is 7,900 million bushels. At a price of $2.37 per bushel, the supply is 7,900 million bushels and the demand is 7,800 bushels.

   a. Assuming that price and supply are linearly related, determine the price p in terms of supply x (the *price-supply equation*).

   b. Assuming that price and demand are linearly related, determine the price p in terms of demand x (the *price-demand equation*).

   c. Find the equilibrium point (price and the number of units for which supply and demand are

equal).

    d. Graph the price-supply equation, price-demand equation and the equilibrium point in the same coordinate system.

3. A plant can manufacture 50 tennis racquets per day for a total daily cost of $3,855 and 60 tennis racquets per day for a total daily cost of $4,245.

    a. Assuming that daily cost and production are linearly related, find the total daily cost C of producing x tennis racquets.

    b. Interpret the slope and the vertical intercept of this cost equation.

    c. Graph the total daily cost for $0 \le x \le 100$.

4. NewTech Wireless company offers a monthly calling plan where the total cost is linearly related to the number of minutes used. Given that the total monthly cost for 100 minutes used is $35.00 and that for 200 minutes the cost is $45.00:

    1. Express the cost C in terms of the number of minutes used t.

    2. What are the domain and the range of this function?

    3. What is the basic cost for the plan and what is the cost per minute?

    4. Sketch the graph of this function.

    5. What will be the cost if 400 minutes are used in a month?

    6. If the total cost for a month was $40.00, how many minutes were used?

5. A security company purchases a new security van for $53,000 and assumes that in 5 years it will have a trade-in value of $28,000.

    1. Find the linear model for the depreciated value V of the van after t years.

    2. What is the depreciated value of the van after 3 years?

    3. When will the depreciated value fall below $23,000?

6. The manufacturer of a new type of frying pan has calculated the monthly fixed costs to be $83,000 and variable costs of $7.35 for each frying pan produced. The pans are sold to a distributor for $20 per pan. The monthly manufacturing capacity is 20,000 units.

    1. Write the monthly cost function C in terms of number of units produced x. Determine its domain and range and graph it.

    2. Write the monthly revenue function R in terms of number of units produced x. Determine its domain and range and graph it.

    3. Write the monthly profit function P in terms of number of units produced x, assuming that all units produced are sold. Determine its domain and range and graph it.

    4. What will be their profit/loss if they are running at 25% capacity? At 75% capacity?

    5. How many units must they produce and sell in a month to break even? What percent is that of production capacity? What must be the sales to break even?

    6. How many units must they produce and sell in a month to make $100,000 in profit?

7. A manufacturing company, under contract to deliver a new line of beer bottles, estimates that it would

cost $41,000 to produce 80,000 bottles while it would cost $59,000 to produce 120,000 bottles. Based on the cost analysis of previous production runs, they determined that that a linear model would best represent the costs of production.

1. Assuming that the cost C and the number of bottles produced x are linearly related, determine the cost function $C(x)$.
2. What are the fixed costs in this model and what is the variable cost per bottle?

7.7: Fitting Linear Models to Data

In the real world, rarely do things follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behaviour of the data and allows us to use the linear model to make predictions about the data, inside and outside of the data range.

Example 7.7A

The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit[8]. Plot this data, and determine whether the data appears to be linearly related.

chirps	44	35	20.4	33	31	35	18.5	37	26
Temp	80.5	70.5	57	66	68	72	52	73.5	53

Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so.

---

8. Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010
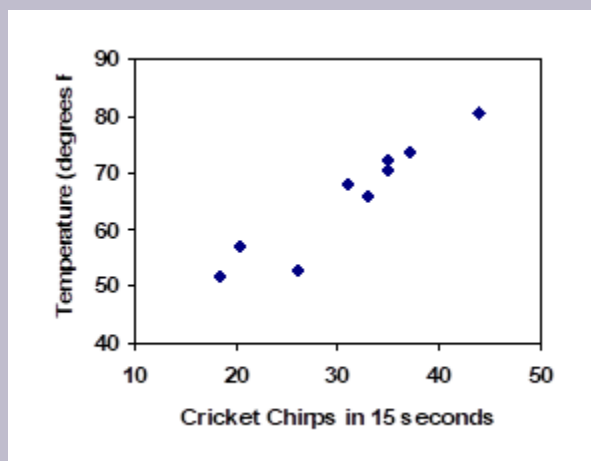
The simplest way to find an equation to approximate this data is to try to "eyeball" a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

Give It Some Thought

1. a. What descriptive variables would you choose to represent Temperature & Chirps?
b. Which variable is the independent variable and which is the dependent variable?
c. Based on this data and the graph, what is a reasonable domain & range?
d. Based on the data alone, is this function one-to-one, explain?

Example 7.7B

Using the table of values from the previous example, find a linear function that fits the data by "eyeballing" a line that seems to fit.

On a graph, we could try sketching in a line. Note the scale on the axes have been adjusted to start at zero to include the vertical axis and vertical intercept in the graph.
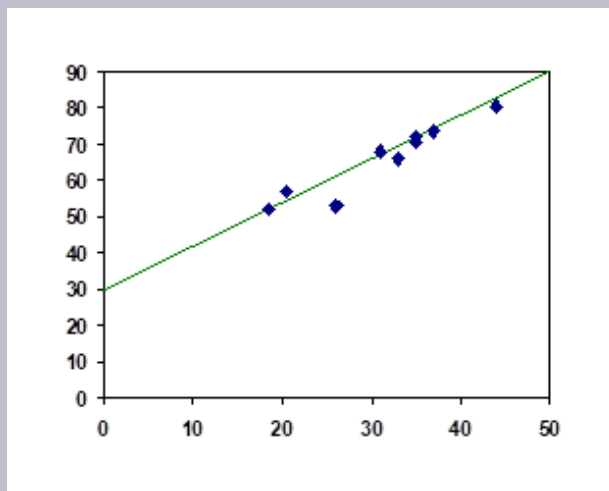
Using the starting and ending points of our "hand drawn" line, points $(0, 30)$ and $(50, 90)$, this graph has a slope of $m = \frac{60}{50} = 1.2$ and a vertical intercept at 30, giving an equation of

$$T(c) = 30 + 1.2c$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit.

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. There is a difference, though, between making predictions inside the domain and range of values we have data for, and outside that domain and range.

Interpolation and Extrapolation

Interpolation: When we predict a value inside the domain and range of the data

Extrapolation: When we predict a value outside the domain and range of the data

For the Temperature as a function of chirps in our hand drawn model above:

Interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44.

Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

Example 7.7C

a) Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

b) Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

With our cricket data, our number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$T(30) = 30 + 1.2(30) = 66 \text{ degrees.}$$

Based on the data we have, this value seems reasonable.

The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c$$
$$10 = 1.2c$$
$$c \approx 8.33$$

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

When our model no longer applies after some point, it is sometimes called **model breakdown**.

Give It Some Thought

2. What temperature would you predict if you counted 20 chirps in 15 seconds?

Fitting Lines with Technology

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values[9]. This technique is called **least-square regression**, and can be computed by many graphing calculators, spreadsheet software like Excel or Google Docs, statistical software, and many web-based calculators[10].

Example 7.7D

Find the least-squares regression line using the cricket chirp data from above.

---

9. Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

10. For example, http://www.shodor.org/unchem/math/lls/leastsq.html
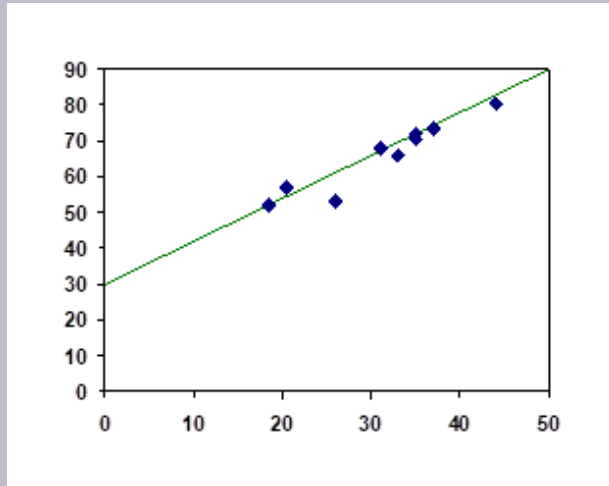
Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Notice that this line is quite similar to the equation we "eyeballed", but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$T(30) = 30.281 + 1.143(30) = 64.571 \approx 64.6 \text{ degrees.}$$

Most calculators and computer software will also provide you with the **correlation coefficient**, a measure of how closely the line fits the data.

Correlation Coefficient

The **correlation coefficient** is a value, r, between -1 and 1.
r> 0 suggests a positive (increasing) relationship
$r < 0$ suggests a negative (decreasing) relationship

The closer the value is to 0, the more scattered the data
The closer the value is to 1 or -1, the less scattered the data is

The correlation coefficient provides an easy way to get some idea of how close to a line the data falls.

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of r and the graph of the data, here are some large data sets with their correlation coefficients:

Examples of Correlation Coefficient Values
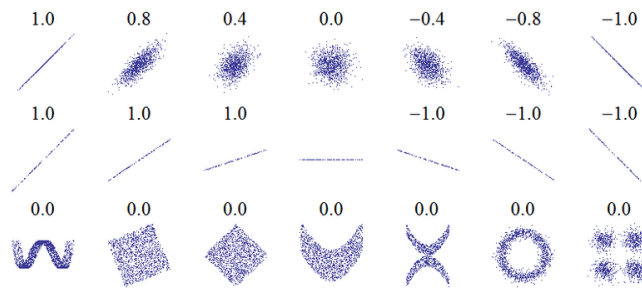
11

Example 7.7E

Calculate the correlation coefficient for our cricket data.

Because the data appears to follow a linear pattern, we can use technology to calculate $r = 0.9509$. Since this value is very close to 1, it suggests a strong increasing linear relationship.

Example 7.7F

Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004

11. http://en.wikipedia.org/wiki/File:Correlation_examples.png

is shown below[12]. Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136

To make things simpler, a new input variable is introduced, t, representing years since 1994.
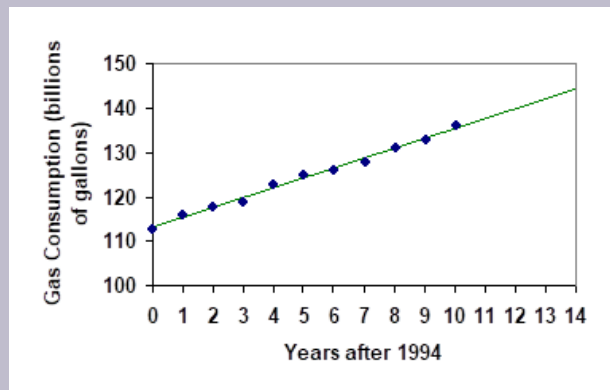
Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

The least-squares regression equation is:

$$C(t) = 113.318 + 2.209t.$$

Using this to predict consumption in 2008 $(t = 14)$,

$$C(14) = 113.318 + 2.209(14) = 144.244 \text{ billions of gallons}$$

The model predicts 144.244 billion gallons of gasoline will be consumed in 2008.

Give It Some Thought

---

12. http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html

3. Use the model created by technology in Example 7.7F to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Give It Some Thought Answers

1. a. T = Temperature, C = Chirps (answers may vary)

b. Independent (Chirps) , Dependent (Temperature)

c. Reasonable Domain $(18.5, 44)$, Reasonable Range $(52, 80.5)$ (answers may vary)

d. NO, it is not one-to-one, there are two different output values for 35 chirps.

2. 54 degrees Fahrenheit

3. 150.871 billion gallons, extrapolation

Section 7.7 Exercises

Insert review exercises here

Chapter 7 Summary

Key Takeaways

Section 7.1: Functions and Function Notation

- Definition of a function
- Input (independent variable)
- Output (dependent variable)
- Function notation
- Descriptive variables
- Functions in words, tables, graphs & formulas
- Vertical line test
- Evaluating a function at a specific input value
- Solving a function given a specific output value

Section 7.2: Domain and Range

- Definition of domain
- Definition of range
- Inequalities
- Interval notation
- Set builder notation
- Domain and Range from graphs
- Piecewise defined functions

Section 7.3: Rates of Change and Behaviour of Graphs

- Rate of Change
- Average Rate of Change
- Calculating Average Rate of Change using Function Notation
- Marginal cost/revenue/profit
- Increasing/Decreasing
- Local Maxima and Minima (Extrema)

Section 7.4: Linear Functions

- Definition of a linear function
- Structure of a linear function
- Increasing & Decreasing functions
- Finding the vertical intercept $(0, b)$
- Finding the slope/rate of change, m
- Interpreting linear functions

Section 7.5: Graphs of Linear Functions

- Methods for graphing linear functions
- Another name for slope = rise/run
- Horizontal intercepts $(a, 0)$
- Horizontal lines
- Vertical lines
- Parallel lines
- Perpendicular lines
- Intersecting lines

Section 7.6: Modeling with Linear Functions

- The problem solving process
  1. Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
  2. Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
  3. Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
  4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
  5. When needed, find a formula for the function.
  6. Solve or evaluate using the formula you found for the desired quantities.
  7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
  8. Clearly convey your result using appropriate units, and answer in full sentences when appropriate.
- Cost, revenue, profit equations
- Supply and demand
- Equilibrium price and quantity

Section 7.7: Fitting Linear Models to Data

- Fitting linear models to data by hand
- Fitting linear models to data using technology
- Interpolation
- Extrapolation
- Correlation coefficient

8.

SYSTEMS OF EQUATIONS AND LINEAR PROGRAMMING

Chapter 8 Topics

8.1 Systems of Equations

8.2 Inequalities in One Variable

8.3 Linear Inequalities

8.4 Graphical Solutions

8.1: Systems of Equations

Back when studying linear equations, we found the intersection of two lines. Doing so allowed us to solve interesting problems by finding a pair of values that satisfied two different equations. While we didn't call it this at the time, we were solving a system of equations. To start out, we'll review an example of the type of problem we've solved before.

Example 8.1 A

A small business produces soap and lotion gift baskets. Labor, utilities, and other fixed expenses cost $6,000 a month. Each basket costs $8 to produce, and sells for $20. How many baskets does the company need to sell each month to break even?

In business terms, "break even" means for revenue (money brought in) to equal costs. While this problem can be approached in several ways, we'll approach here by first creating two linear functions, one for the costs, and another for revenue.

Let's define n to be the number of gift baskets the company sells in a month. There are $6,000 of fixed costs each month, and costs increase by $8 for each basket, so we can write the linear function for costs, C, as:

$$C(n) = 6000 + 8n$$

Each sale brings in $20, so the revenue, R, after selling n baskets will be:

$$R(n) = 20n$$

To find the break-even point, we are looking for the number of baskets where the revenue will equal the costs. In other words, if we were to graph the two linear functions, we are looking for the point that lies on both lines; the solution is the point that satisfies both equations.
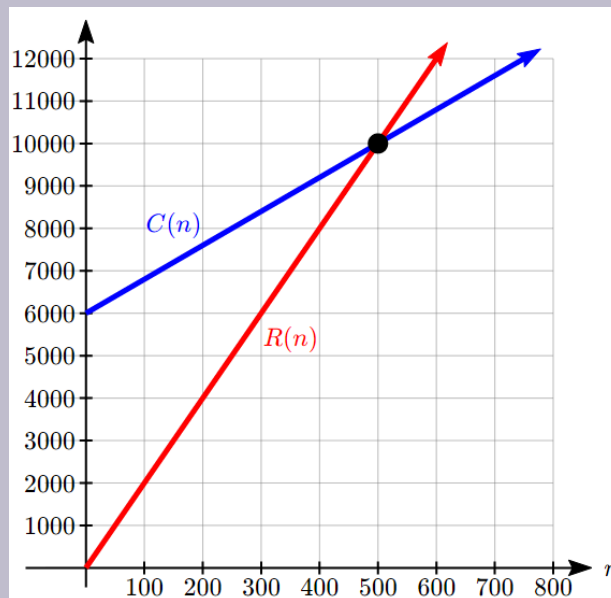
In this case we could probably solve the problem from the graph itself, but we can also solve it algebraically by setting the equations equal:

$$R(n) = C(n)$$
$$20n = 6000 + 8n$$

Subtract 8n from both sides:

$$12n = 6000$$

Divide:

$$n = \frac{6000}{12} = 500$$

Evaluate either function at this input:

$$R(500) = C(500) = 10,000$$

The break even point is at 500 baskets. The company must sell 500 baskets a month, at which point their revenue of $10,000 will cover their total costs of $10,000.

The example above illustrates one type of system of equations, one where both equations are given in functional form. When the equations are written this way, it is easy to solve the system using substitution, by setting the two outputs equal, and solving for the input. However, many system of equations problems aren't written this way.

Example 8.1 B

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. If the company wants to fully utilize all staffed hours, how many of each item should they produce?

Notice first that this problem has two variables, or two unknowns – the number of basic products to make, and the number of premium products to make. There are also two constraints – the hours

of assembly and the hours of painting available. This is going to give us two equations in two unknowns, what we call a 2 by 2 system of equations.

We'll start by defining our variables:
b: the number of basic products produced
p: the number of premium products produced

Now we can create our equations based on the constraints. Each basic product requires 20 minutes of assembly, so producing b items will require $20b$ minutes. Each premium product requires 30 minutes of assembly, so producing p items will require $30p$ minutes. Together we have 3,900 minutes available, giving us the equation:

$$20b + 30p = 3900$$

Using the same approach for painting gives the equation:

$$15b + 30p = 3300$$

Together, these form our system of equations. They are sometimes written as a pair with a curly bracket on the left to indicate that they should be considered as connected equations.

$$\begin{cases} 20b + 30p = 3900 \\ 15b + 30p = 3300 \end{cases}$$

As before, our goal is to find a pair of values, (b, p), that satisfies both equations. We'll return to this problem and solve it shortly.

While it may not be clear, the equation $20b + 30p = 3900$ we constructed above is a linear equation, like the linear equations from the first example, it's just written differently. We could, if desired, solve this equation for p to get it written in slope-intercept form:

$$30p = 3900 - 20b,$$

so

$$p = 130 - \tfrac{2}{3}b$$

We typically don't do this, since it often makes the system harder to solve then when using other techniques. To dive into this further, let's first clarify what it means to find a solution to a system of linear equations.

System of Linear Equations

A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously.

A solution to a system is a set of numerical values for each variable in the system that will satisfy all equations in the system at the same time.

Not every system will have exactly one solution, but we'll look more closely at that later. To check to see if an ordered pair is a solution to a system of equations, you would:

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 8.1 C

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{cases} x + 3y = 8 \\ 2x - 9 = y \end{cases}$$

Substitute the ordered pair $(5, 1)$ into both equations.

$$(5) + 3(1) = 8$$
$$8 = 8$$
$$\text{True}$$
$$2(5) - 9 = (1)$$
$$1 = 1$$
$$\text{True}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

There are three common methods for solving systems of linear equations with two variables. The first is solving by graphing. In the first example above we graphed both equations, and the solution to the system was the intersection of the lines.

Example 8.1 D

Solve the following system of equations by graphing.

$$2x + y = -8$$
$$x - y = -1$$

Solve the first equation for y.

$$2x + y = -8$$
$$y = -2x - 8$$

Solve the second equation for y.

$$x - y = -1$$
$$y = x + 1$$

Graph both equations on the same set of axes.
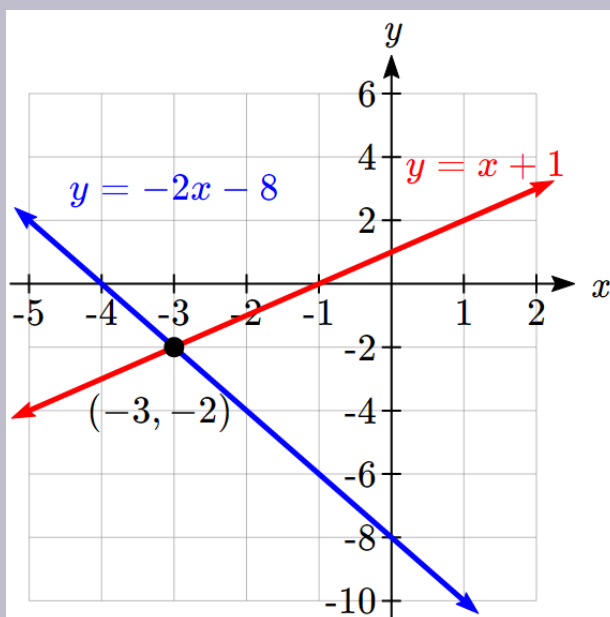
The lines appear to intersect at the point $(-3, -2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$2(-3) + (-2) = -8$$
$$-8 = -8$$
$$\text{True}$$
$$(-3) - (-2) = -1$$
$$-1 = -1$$
$$\text{True}$$

The solution to the system is the ordered pair (-3, -2).

Give It Some Thought

1. Solve the following system of equations by graphing.

$$2x - 5y = -25$$
$$-4x + 5y = 35$$

While this method can work well enough when the solution values are both integers, it is not very useful when the intersection is not at a clear point. Additionally, it requires solving both equations for y, which adds extra steps. Because of these limitations, solving by graphing is rarely used, but can be useful for checking whether your algebraic answers are reasonable.

Solving a System by Substitution

Another method for solving a system of equations is the substitution method, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable.

Solving a system using substitution

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining

variable.

3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.

4. Check the solution in both equations.

The problem we did in Example 8.1A was technically done by substitution, but it was made easier since both equations were already solved for one variable, y. An example of a more typical case is shown next.

Example 8.1 E

Solve the following system of equations by substitution.

$$-x + y = -5$$
$$2x - 5y = 1$$

First, we will solve the first equation for y.

$$-x + y = -5$$
$$y = x - 5$$

Now we can substitute the expression $x - 5$ for y in the second equation.

$$2x - 5y = 1$$
$$2x - 5(x - 5) = 1$$
$$2x - 5x + 25 = 1$$
$$-3x = -24$$
$$x = 8$$

Now, we substitute $x = 8$ into the first equation and solve for y.

$$-(8) + y = -5$$
$$y = 3$$

Our solution is $(8, 3)$.

We can check the solution by substituting $(8, 3)$ into both equations.

$$-x + y = -5$$
$$-(8) + (3) = -5$$
$$\text{True}$$
$$2x - 5y = 1$$
$$2(8) - 5(3) = 1$$
$$\text{True}$$

Give It Some Thought

2. Solve the following system of equations by substitution.
$$x = y + 3$$
$$4 = 3x - 2y$$

Substitution can always be used, but is an especially good choice when one of the variables in one of the equations has a coefficient of 1 or -1, making it easy to solve for that variable without introducing fractions. This is fairly common in many applications.

Example 8.1 F

Julia has just retired, and has $600,000 in her retirement account that she needs to reallocate to produce income. She is looking at two investments: a very safe guaranteed annuity that will provide 3% interest, and a somewhat riskier bond fund that averages 7% interest. She would like to invest as little as possible in the riskier bond fund, but needs to produce $40,000 a year in interest to live on. How much should she invest in each account?

Notice there are two unknowns in this problem: the amount she should invest in the annuity and the

amount she should invest in the bond fund. We can start by defining variables for the unknowns:

a: The amount (in dollars) she invests in the annuity

b: The amount (in dollars) she invests in the bond fund.

Our first equation comes from noting that together she is going to invest $600,000:

$$a + b = 600,000$$

Our second equation will come from the interest. She earns 3% on the annuity, so the interest earned in a year would be $0.03a$. Likewise, the interest earned on the bond fund in a year would be $0.07b$. Together, these need to total $40,000, giving the equation:

$$0.03a + 0.07b = 40,000$$

Together, these two equations form our system. The first equation is an ideal candidate for the first step of substitution – we can easily solve the equation for a or b:

$$a = 600,000 - b$$

Then we can substitute this expression for a in the second equation and solve.

$$0.03(600,000 - b) + 0.07b = 30,000$$
$$18,000 - 0.03b + 0.07b = 40,000$$
$$0.04b = 22,000$$
$$b = 550,000$$

Now substitute this back into the equation $a = 600,000 - b$ to find a

$$a = 600,000 - 550,000$$
$$a = 50,000$$

In order to reach her goal, Julia will have to invest $550,000 in the bond fund, and $50,000 in the annuity.

Solving a System by the Addition Method

A third method of solving systems of linear equations is the addition method, also called the elimination method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

Solving a System by the Addition Method

1. Write both equations with $x-$ and y-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Often using the addition method will require multiplying one or both equations by a constant so terms will eliminate. Using these approaches, we can revisit the equation from Example 8.1B.

Example 8.1 G

In Example 8.1B, we set up the system below. Solve it.

$$\begin{cases} 20b + 30p = 3900 \\ 15b + 30p = 3300 \end{cases}$$

Adding the equations would not eliminate a variable, but we notice that the coefficients on p are the same, so multiplying one of the equations by -1 will change the sign of the coefficients. Multiplying the second equation by -1 gives the system

$$20b + 30p = 3900$$
$$-15b - 30p = -3300$$

Adding these equations gives

$$5b = 600$$
$$b = 120$$

Substituting b = 30 into the first equation,

$$20(120) + 30p = 3900$$
$$2400 + 30p = 3900$$
$$30p = 1500$$
$$p = 50$$

The solution is $b = 120$, $p = 50$, meaning the company should produce 120 basic products and 50 premium products to full utilize staffed hours.

Checking our answer in the second equation:

$$15(120) + 30(50) = 3300$$
$$1800 + 1500 = 3300$$
$$3300 = 3300$$

Give It Some Thought

3. Solve the system of equations by addition.

$$2x - 7y = 2$$
$$3x + y = -20$$

Example 8.1 H

Solve the given system of equations in two variables by addition.

$$2x + 3y = -16$$
$$5x - 10y = 30$$

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$-5\left(2x + 3y\right) = -5\left(-16\right)$$
$$-10x - 15y = 80$$
$$2\left(5x - 10y\right) = 2\left(30\right)$$
$$10x - 20y = 60$$

Then, we add the two equations together.

$$\begin{aligned} -10x - 15y &= 80 \\ \underline{10x - 20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y = -4$ into the original first equation.

$$2x + 3\left(-4\right) = -16$$
$$2x - 12 = -16$$
$$2x = -4$$
$$x = -2$$

The solution is $\left(-2, -4\right)$. Check it in the other equation.

$$5x - 10y = \ 30$$
$$5\left(-2\right) - 10\left(-4\right) = 30$$
$$-10 + 40 = 30$$
$$30 = 30$$

Give It Some Thought

4. Solve the system of equations by addition.

$$2x + 3y = 8$$
$$3x + 5y = 10$$

Dependent and Inconsistent Systems

Up until now, we have only considered cases where there is exactly one solution to the system. We can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the examples we just explored. The two lines have different slopes and intersect at one point in the plane.

A consistent system is considered to be a **dependent system** if the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

Types of Linear Systems

- An **independent system** has exactly one solution pair (x,y). The point where the two lines intersect is the only solution.
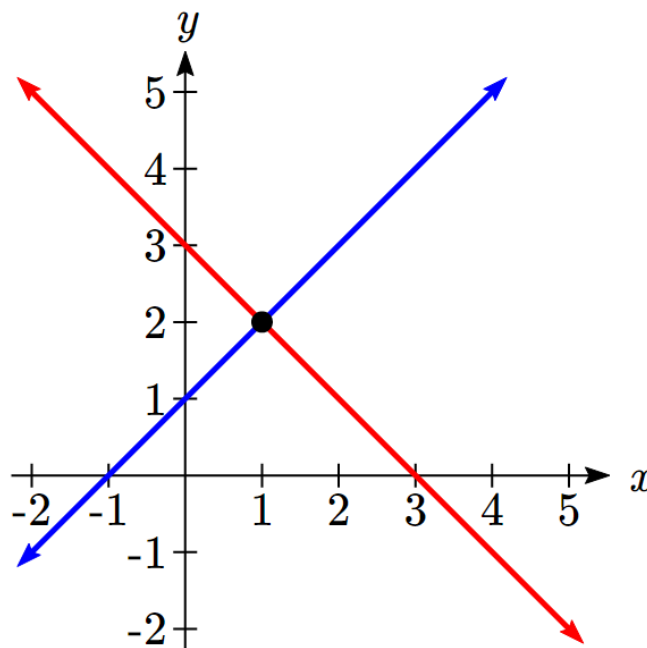
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
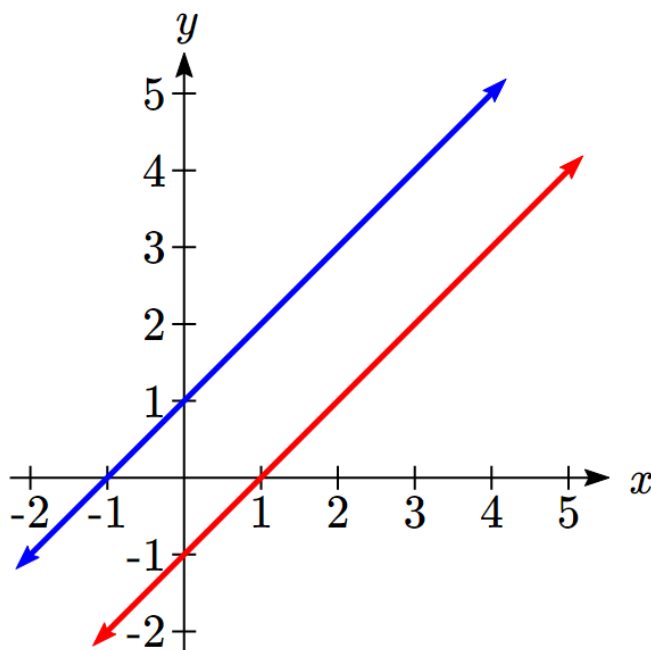
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.
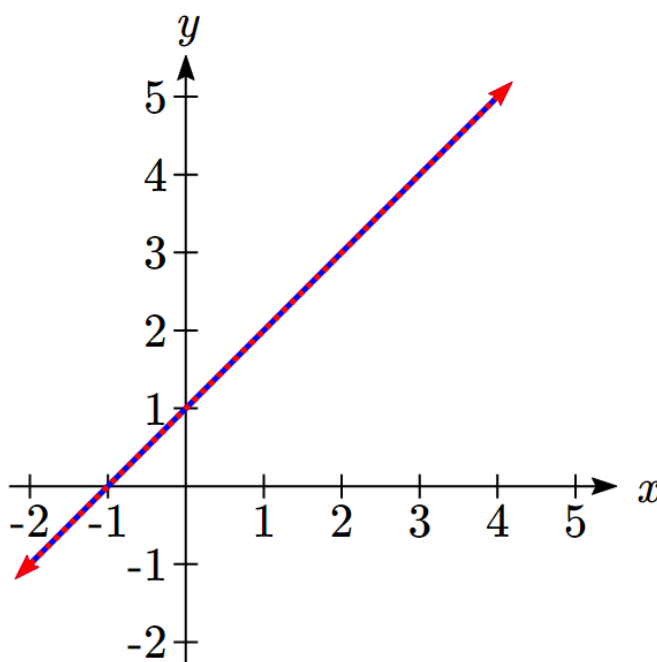
We can use substitution or addition to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Example 8.1 I

Solve the following system of equations.

$$x = 9 - 2y$$
$$x + 2y = 13$$

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$x + 2y = 13$$
$$(9 - 2y) + 2y = 13$$
$$9 + 0y = 13$$
$$9 = 13$$

Clearly, this statement is a contradiction (a false statement) because $9 \neq 13$. Therefore, the system has no solution, and the system is inconsistent.

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as

Example 8.1 J

Find a solution to the system of equations using the addition method.

$$x + 3y = 2$$
$$3x + 9y = 6$$

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$x + 3y = 2$$

Multiply both sides of the equation by -3

$$(-3)(x + 3y) = (-3)(2)$$
$$-3x - 9y = -6$$
$$-3x - 9y = -6$$
$$\underline{3x + 9y = 6}$$
$$0 = 0$$

We can see that there will be an infinite number of solutions that satisfy both equations. In some cases, realizing there are an infinite number of solutions is enough, and we can stop there. In other cases, we will want to describe the set of solutions.

One way is to simply say it's the set of points that satisfy $x + 3y = 2$, but often we would solve that equation for y and describe the solution as set of points $\left(x, -\frac{1}{3}x + \frac{2}{3}\right)$.

Give It Some Thought

5. Solve the systems:

a.

$$2y - 2x = 2$$
$$2y - 2x = 6$$

b.

$$y - 2x = 5$$
$$-3y + 6x = -15$$

Systems with 3 variables in 3 unknowns

In systems of two variables, a solution was an ordered pair (x, y) that satisfied both equations. The **solution set** to a three-by-three system is an ordered triple (x, y, z). Graphically, the ordered triple defines the point

that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Example 8.1K

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$x + y + z = 2$$
$$6x - 4y + 5z = 31$$
$$5x + 2y + 2z = 13$$

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$x + y + z = 2$ $(3) + (-2) + (1) = 2$ True	$6x - 4y + 5z = 31$ $6\,(3) - 4\,(-2) + 5\,(1) = 31$ $18 + 8 + 5 = 31$ True	$5x + 2y + 2z = 13$ $5\,(3) + 2\,(-2) + 2\,(1) = 13$ $15 - 4 + 2 = 13$ True

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

We can utilize the techniques we learned in the last section to solve 3-by-3 systems of equations by reducing the problem to one we already know how to solve.

Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 8.1 L

Find a solution to the following system. The equations are numbered so we can refer to them more easily.

$$x - 2y + 3z = 9 \quad (1)$$
$$-x + 3y - z = -6 \quad (2)$$
$$2x - 5y + 5z = 17 \quad (3)$$

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$x - 2y + 3z = 9 \quad (1)$$
$$\underline{-x + 3y - z = -6 \quad (2)}$$
$$y + 2z = 3 \quad (4)$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x:

$$\text{multiplied by} -2$$
$$-2x + 4y - 6z = -18 \quad (1)$$
$$\underline{2x - 5y + 5z = 17 \quad (3)}$$
$$-y - z = -1 \quad (5)$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for **z** by adding the two equations.

$$y + 2z = 3 \quad (4)$$
$$\underline{-y - z = -1 \quad (5)}$$
$$z = 2 \quad (6)$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$x - 2y + 3z = 9 \quad (1)$$
$$y + 2z = 3 \quad (4)$$
$$z = 2 \quad (6)$$

Next, we back-substitute $z = 2$ into equation (4) and solve for y.

$$y + 2\,(2) = 3$$
$$y + 4 = 3$$
$$y = -1$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

$$x - 2\,(-1) + 3\,(2) = 9$$
$$x + 2 + 6 = 9$$
$$x = 1$$

The solution is the ordered triple $(1, -1, 2)$.

Give It Some Thought

6. Solve the system of equations in three variables.

$$2x + y - 2z = -1$$
$$3x - 3y - z = 5$$
$$x - 2y + 3z = 6$$

Many problems in real life depend on more than two unknowns.

Example 8.1 M

Chad is trying to plan a meal to meet specific nutritional goals. He wants to construct a dish containing rice, tofu, and peanuts that will provide 30g of protein, 14g of fat, and 50g of carbohydrates. How much of each ingredient should he use?

First off, we're assuming any other ingredients used in the recipe aren't contributing significantly enough to the nutrition to be considered. To answer this question, we'll first need to know the nutritional content for the ingredients. Looking these up:

White rice: 1 cup provides: 0g fat, 44g carbohydrates, 4g protein

Tofu: 1 cup provides: 10g fat, 5g carbohydrates, 20g protein

Peanuts: 1 cup provides: 72g fat, 31g carbohydrates, 35g protein

Now we can define our variables. We are interested in the amount of each ingredient to use, so we'll define our variables as the quantity of each ingredient:

r: cups of rice, t: cups of tofu, p: cups of peanuts.

Now for each nutrient, we can create an equation. Since 1 cup of rice provides 44g carbohydrates, r cups will provide $44r$ grams of carbohydrates. Likewise t cups of tofu will provide $5t$ grams, and p cups of peanuts will provide $31p$ grams. Together we want our recipe to provide 50g of carbohydrates, giving the equation:

$$44r + 5t + 31p = 50$$

Doing the same for fat and protein gives the full system:

$$44r + 5t + 31p = 50$$
$$10t + 72p = 14$$
$$4r + 20t + 35p = 30$$

Now we can solve the system.

Step 1. Notice that the section equation already does not involve the variable r. To make things simpler, a first step might be to interchange the last two equations so the two equations with three variables will line up.

$$44r + 5t + 31p = 50 (1)$$
$$4r + 20t + 35p = 30 (2)$$
$$10t + 72p = 14 (3)$$

Step 2. Since r is already eliminated in the last equation, we'll eliminate r from the first two equations. Multiply equation (2) by -11.

$$44r + 5t + 31p = 50 (1)$$
$$-44r - 220t - 385p = -330 (2)$$
$$10t + 72p = 14 (3)$$

Step 3. Add equations (1) and (2), writing the result as row 2.

$$44r + 5t + 31p = 50(1)$$
$$-215t - 384p = -280(2)$$
$$10t + 72p = 14(3)$$

Step 4. Multiply equation (2) by 2 and equation (3) by 43.

$$44r + 5t + 31p = 50(1)$$
$$-430t - 708p = -560(2)$$
$$430t + 3096p = 602(3)$$

Step 5. Add equations (2) and (3), writing the result in row 3.

$$44r + 5t + 31p = 50(1)$$
$$-430t - 708p = -560(2)$$
$$2388p = 42(3)$$

Step 6. Solve for p in equation 3. For a real life problem like this, decimal approximations are probably fine.

$$2388p = 42$$
$$p = \frac{42}{2388} \approx 0.0176$$

Step 7. Back substitute the value for p into equation (2) to solve for t.

$$-430t - 708(0.0176) = -560$$
$$-430t - 12.4608 = -560$$
$$-430t = -547.5392$$
$$t = \frac{-547.5392}{-430} \approx 1.273$$

Step 8. Back substitute the values for p and t into equation (1) and solve for r.

$$44r + 5(1.273) + 31(0.0176) = 50$$
$$44r = 43.0894$$
$$r \approx 0.979$$

To meet his nutritional goals, Chad should use 0.979 cups of rice, 1.273 cups of tofu, and 0.0176 cups of peanuts.

This particular system was rather unpleasant to solve. In the rest of the chapter we will learn some other techniques for solving complex systems.

Give It Some Thought Answers

1. The solution to the system is the ordered pair $(-5, 3)$.
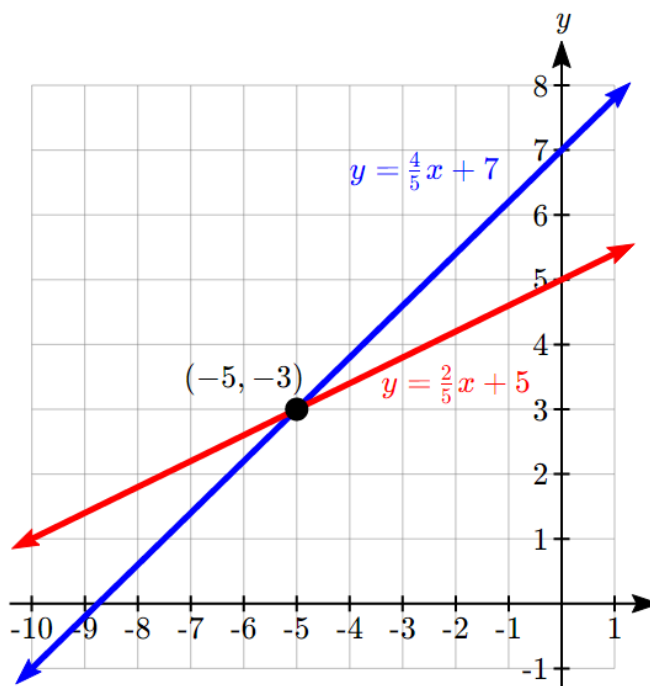
2. $(-2, -5)$

3. $(-6, -2)$

4. $(10, -4)$

5. a. No solution. The system is inconsistent.

b. The system is dependent so there are infinite solutions of the form $(x, 2x + 5)$.

6. $(1, -1, 1)$

Section 8.1 Exercises

1. Solve each of the following systems of equations.

   a. $3x + 2y = -1$
   $5x + 3y = -2$

   b. $4x - 5y = 25$
   $3x + 2y = 13$

   c. $y = -10x$
      $3y = 29 - x$

   d. $2y = 3x + 17$
      $3x = 11 - 5y$

   e. $48a - 32b = 128$
      $16a + 48b = 32$

   f. $0.5m + 0.3n = 54$
      $0.3m + 0.7n = 74$

   g. $\dfrac{3}{4}m + \dfrac{5}{8}n = \dfrac{3}{4}$
      $\dfrac{5}{6}n + \dfrac{2}{3}m = \dfrac{7}{9}$

2. Kayla and Amina agree to form a partnership which requires Amina to invest $6,465 less than three-quarters of what Kayla is to invest. If the total partnership value is $61,243, how much did each partner invest?

3. A college theatre collected $1,300 from the sale of 450 tickets. If the tickets were sold for $2.50 and $3.50, how many tickets were sold at each price?

4. An employee gets paid $16 per hour. Last week, the employee's weekly total of regular hours and overtime hours at time-and-a-half were 44 and the total compensation was $800. How many regular and overtime hours did the employee work?

5. A furniture company has been producing 93 bedside tables day working two shifts. The second shift has produced 33 bedside tables fewer than five-halves of the number of bedside tables produced by the first shift. Determine the number of bedside tables each shift has produced.

6. Eyad diversified his investment so that part of his investment earns interest at rate of 4% per annum, and part earns 6% per annum. If the total annual interest on the investment is $560, and he invests twice as much at 4% as he invests at 6%, how much did Eyad invest at each rate?

7. The Kirkland Company markets two products. Each unit of Product A requires five units of labour, while each unit of Product B requires two units of labour. Two hundred units of labour are available per time period. How many of each product can they market?

8. Nancy's sales last week were $140 less than three times Andrea's sales. Together they sold $940. How much did each sell last week?

9. A political candidate has budgeted $10,000 to spend on radio and television advertising. A radio ad costs $200 per 30-second spot, and a television ad costs $800 per 30-second spot. How many radio and television spots can the candidate purchase for $10,000 if they want to run as many radio spots as television spots?

10. A restaurant offered two dinner specials. The difference between seven times the orders for the first dinner special and four times the orders for the second special is 12. The sum of the three-fourths of the orders for the first special and two-thirds of the orders for the second special is 21. Find the number of

orders for each special.

11. A manufacturing plant makes two types of inflatable boats – a two-person boat and a four-person boat. Each two-person boat requires 0.9 labour-hours in the cutting department and 0.8 labour hours in the assembly department. Each four-person boat requires 1.8 labour-hours in the cutting department and 1.2 labour hours in the assembly department. The monthly total number of labour-hours in the cutting and the assembly departments are 864 and 672 respectively. How many of each type of boat can the plant manufacture?

12. A human resource manager is trying to estimate the cost of a workforce accident. These costs usually consist of direct costs (such as medical bills, equipment damage, and legal expenses) and indirect costs (such as decreased output, production delays, and fines). From past experience, she knows that indirect costs average six times as much as direct costs. If she estimates the cost of an accident to be $21,000, what is the direct cost of the accident?

13. St. Boniface Hospital raises funds for research through its Mega Lottery program. In this program, 16,000 tickets are available for purchase at a price of one for $100 or three for $250. This year, the lottery sold out with sales of $1,506,050. To better plan next year's lottery, the marketing manager wants to know how many tickets were purchased under each option this year.

14. An accountant is trying to allocate production costs from two different products to their appropriate ledgers. Unfortunately, the production log sheet for last week has gone missing. However, from other documents he was able to figure out that 1,250 units in total were produced last week. The production machinery was run for 2,562.5 minutes, and he knows that Product A takes 1.5 minutes to manufacture while Product B takes 2.75 minutes to manufacture. How many units of each product were produced last week?

15. A 240 mL bottle of an orange drink claims that it is made with real fruit juice. Upon examination of the ingredient list, only 15% of the contents is actually fruit juice. How many milliliters of "other ingredients" are in the bottle?

16. Goodyear Tires just completed a "Buy Three Get One Free" promotion on its ultra-grip SUV tires, regularly priced at $249.99 per tire. Over the course of the promotion, 1,405 tires were sold resulting in sales of $276,238.95. How many tires were sold at the regular price and how many tires were sold at the special promotional price?

17. A local baseball team sells tickets with two price zones. Seats behind the plate from first base to third base are priced at $20 per ticket. All other seats down the base lines and in the outfield are priced at $10 per ticket. At last night's game, 5,332 fans were in attendance and total ticket revenue was $71,750. How many tickets in each zone were sold?

18. Marianne, William, Hendrick, and Charlotte have all decided to go into business together. They need $175,000 in initial capital funding. William was able to contribute 20% less than Marianne, Hendrick contributed 62.5% more than William, and Charlotte contributed $5,000 less than half as much as Marianne. How much did each partner contribute to the initial funds?

19. A mall is being constructed and needs to meet the legal requirements for parking availability. Parking laws require one parking stall for every 100 square feet of retail space. The mall is designed to have 1,200,000 square feet of retail space. Of the total parking stalls available, 2% need to be handicap accessible, there need to be three times as many small car spaces as handicap spaces, RV spaces need to be one-quarter of the number of small car spaces, and the rest of the spaces are for regular stalls. How many of each type of parking space does the mall require?

8.2: Inequalities in One Variable

When learning about domain and range, you learned about inequalities and using set-builder and interval notation to represent them. In this section, we will explore how to solve linear and absolute value inequalities in one variable. The process is very similar to solve equations, but instead of the solution being a single value, the solution will be an inequality.

Notice that if an inequality is true, like 2 < 5, then these operations result in a true statement as well, just like with equations:

Adding a number to both sides:

$$2 + 4 < 5 + 4$$
$$6 < 9$$
$$\text{True}$$

Subtracting a number on both sides:

$$2 - 3 < 5 - 3$$
$$-1 < 2$$
$$\text{True}$$

Multiplying a positive number on both sides:

$$2\,(3) < 5\,(3)$$
$$6 < 15$$
$$\text{True}$$

Dividing by a positive number on both sides:

$$\frac{2}{2} < \frac{5}{2}$$
$$1 < 2.5$$
$$\text{True}$$

We can use these operations just like when solving equations.

Example 8.2A

Solve $3x + 7 \geq 1$

$$3x + 7 \geq 1$$
$$\text{Subtract } 7 \text{ from both sides}$$
$$3x \geq -6$$
$$\text{Divide both sides by } 3$$
$$x \geq -2$$

This inequality represents the solution set. It tells us that all numbers greater than or equal to -2 will satisfy the original inequality. We could also write this solution in interval notation, as $[-2, \infty)$

To understand what is happening, we could also consider the problem graphically. If we were to graph the equation $y = 3x + 7$, then solving $3x + 7 \geq 1$ would correspond with asking "for what values of x is $y \geq 1$".
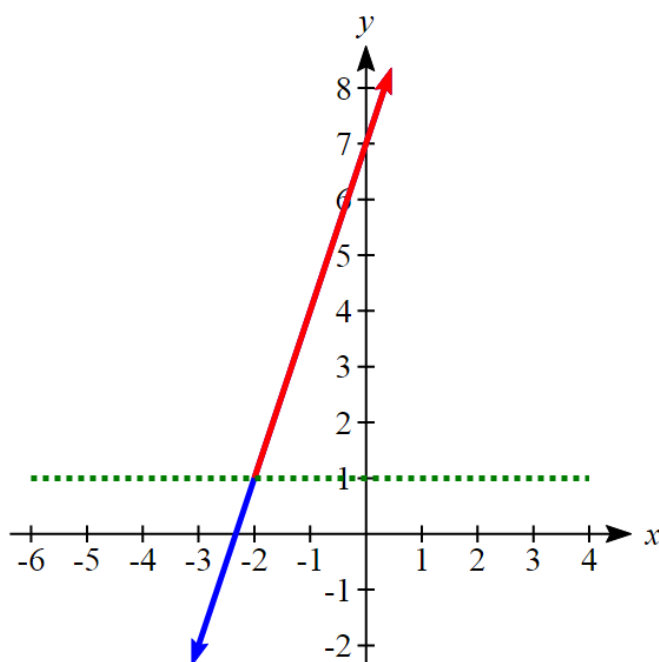
Notice that the part of the graph where this is true corresponds to where $x \geq -2$.

While most operations in solving inequalities are the same as in solving equations, we run into a problem when multiplying or dividing both sides by a negative number. Notice, for example:

$$2\left(-3\right) < 5\left(-3\right)$$
$$-6 < -15$$
$$\text{Not True}$$

To account for this, when multiplying or dividing by a negative number, we must reverse the sign of the inequality.

Rules for Solving Linear Inequalities

1. You may add or subtract a positive or negative number to both sides of the inequality.
2. You can multiply or divide both sides of the inequality by a positive number.
3. You can multiply or divide both sides of the inequality by a negative number, but you must reverse the direction of the inequality.

Example 8.2B

Solve $12 - 4x < 6$

$$12 - 4x < 6$$

Subtract 12 from both sides

$$-4x < -6$$

Divide both sides by -4, and reverse the inequality direction

$$x > \frac{-6}{-4}$$

Simplify

$$x = \frac{3}{2}$$

Give It Some Thought

1. Solve: $6 + 2x \leq 18 + 5x$

Example 8.2C

A company spends $1200 per day on overhead and labor, and each item they produce costs $5 for

materials. If they sell the items for $15 each, how many items will they need to sell each day for their profits to be positive?

While we could solve this problem using equations, it also lends itself to inequalities, since we want the profit to be positive: $P > 0$.

Costs: $C(q) = 1200 + 5q$

Revenue: $R(q) = 10q$

Profit: $P(q) = 10q - (1200 + 5q) = 5q - 1200$

Solving $P(q) > 0$:

$$5q - 1200 > 0$$
$$5q > 1200$$
$$q > 240$$

The company will need to less at least 240 items a day to make a profit.

Compound Inequalities

Compound inequalities are inequalities that consist of more than one part. The most common type is called a tripartite inequality. The basic version looks like this: $-1 < 3x + 5 < 14$.

When we write these it is important that both inequalities point in the same direction and that the "outside" inequality is also true – in this case $-1 < 14$ is true, so this is valid. Expressions like $10 < x < 2$ and $1 < x < 5$ are not valid notation.

The most universal way to solve a tripartite inequality is to:

1. Break it into two separate inequalities
2. Solve each inequality separately
3. Combine the solutions if possible.

Example 8.2D

Solve $-1 < -3x + 5 < 14$

First we separate this into two inequalities:

$$-1 < -3x + 5$$

$$\text{and}$$

$$-3x + 5 < 14$$

Now we solve each:

$$-6 < -3x$$
$$2 > x$$
$$\text{and}$$
$$-3x < 9$$
$$x > -3$$

Now we can combine these solution sets. The numbers where both $2 > x$ and $x > -3$ are true is the set:

$$2 > x > -3$$

While this solution is valid and correct, it is more common to write the solution to tripartite inequalities with the smaller number on the left. We could rewrite the solution as:

$$-3 < x < 2$$

This also has the advantage of corresponding better with the answer in interval notation: $(-3, 2)$.

With this particular inequality, it would also be possible to skip the step of breaking it apart, and instead just subtract 5 from all three "parts" of the inequality. This works for simple problems like this, but may fail if the inequality has variables in more than one "part" of the inequality.

Give It Some Thought

2. Solve: $4 \leq 2x + 6 < 16$

Absolute Value

So far in this section we have been looking at inequalities that are linear. We will now turn to absolute value inequalities. The absolute value function is a piecewise-defined function made up of two linear functions.

Absolute Value Function

The absolute value function can be defined as

$$f(x) = |x| = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

The graph of the absolute value looks like a V:
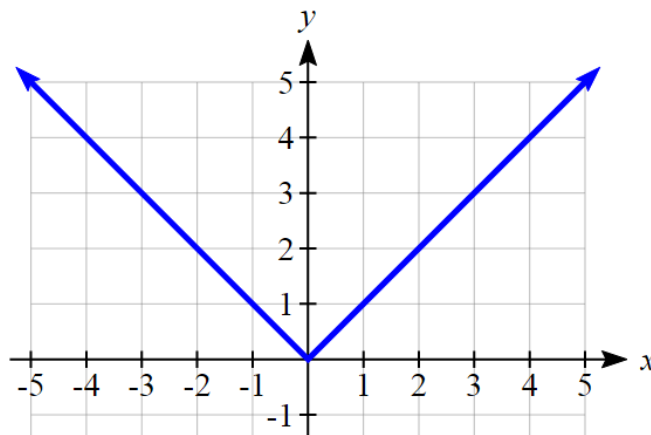
The absolute value function is commonly used to determine the distance between two numbers on the number line. Given two values a and b, then $|a - b|$ will give the distance, a positive quantity, between these values, regardless of which value is larger.

Example 8.2E

Describe all values, x, within a distance of 4 from the number 5.

We want the distance between x and 5 to be less than or equal to 4. The distance can be represented using the absolute value, giving the expression

$$|x - 5| \leq 4$$

Example 8.2 F

A 2010 poll reported 78% of Americans believe that people who are gay should be able to serve in the US military, with a reported margin of error of 3%[1]. The margin of error tells us how far off the actual value could be from the survey value[2]. Express the set of possible values using absolute values.

Since we want the size of the difference between the actual percentage, p, and the reported percentage to be less than 3%,

$$|p - 78| \leq 3$$

Give It Some Thought

---

1. http://www.pollingreport.com/civil.htm, retrieved August 4, 2010
2. Technically, margin of error usually means that the surveyors are 95% confident that actual value falls within this range.

3. Students who score within 20 points of 80 will pass the test. Write this as a distance from 80 using the absolute value notation.

Solving Absolute Value Equations

To solve an equation like $8 = |2x - 6|$, we can notice that the absolute value will be equal to 8 if the quantity inside the absolute value were 8 or -8. This leads to two different equations we can solve independently:

$$2x - 6 = 8$$
$$2x = 14$$
$$x = 7$$

or

$$2x - 6 = -8$$
$$2x = -2$$
$$x = -1$$

Solutions to Absolute Value Equations

An equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$.

Example 8.2G

Solve: $0 = |4x + 1| - 7$

$$0 = |4x + 1| - 7$$

Isolate the absolute value on one side of the equation

$$7 = |4x + 1|$$

Now we can break this into two separate equations:

$$7 = 4x + 1$$
$$6 = 4x$$
$$x = \frac{6}{4} = \frac{3}{2}$$

or

$$-7 = 4x + 1$$
$$-8 = 4x$$
$$x = \frac{-8}{4} = -2$$

There are two solutions: $x = \frac{3}{2}$ and $x = -2$

Example 8.2H

Solve $1 = 4|x - 2| + 2$

Isolating the absolute value on one side the equation,

$$1 = 4|x - 2| + 2$$
$$-1 = 4|x - 2|$$
$$-\frac{1}{4} = |x - 2|$$

At this point, we notice that this equation has no solutions – the absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value.

Give It Some Thought

4. Find the horizontal & vertical intercepts for the function

$$f(x) = -|x + 2| + 3$$

Solving Absolute Value Inequalities

When absolute value inequalities are written to describe a set of values, like the inequality $|x - 5| \leq 4$ we wrote earlier, it is sometimes desirable to express this set of values without the absolute value, either using inequalities, or using interval notation.

We will explore two approaches to solving absolute value inequalities:

1. Using a graph
2. Using test values

Example 8.21

Solve $|x - 5| \leq 4$

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x - 5| = 4$. We do this because the absolute value is a nice friendly function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x - 5| = 4$,

$$x - 5 = 4$$
$$x = 9$$

or

$$x - 5 = -4$$
$$x = 1$$

To use a graph, we can sketch the function $f(x) = |x - 5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched.
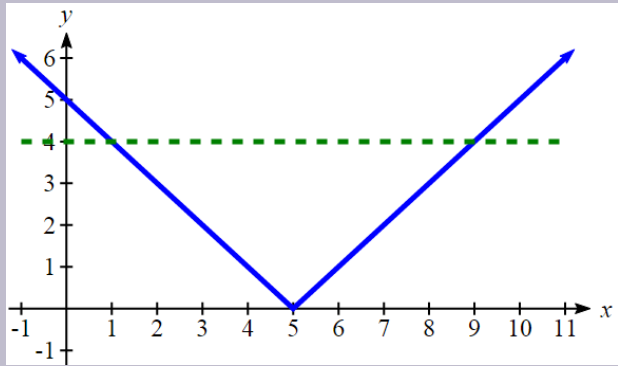
On the graph, we can see that indeed the output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$. Based on the shape of the graph, we can determine the absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval $[1, 9]$.

As an alternative to graphing, after determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can only change from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals: $x < 1$, $1 < x < 9$, and $x > 9$. To determine when the function is less than 4, we could pick a value in each interval and see if the output is less than or greater than 4.

Interval	Test x	$f(x)$	<4 or >4?
$x < 1$	0	$|0 - 5| = 5$	greater
$1 < x < 9$	6	$|6 - 5| = 1$	less
$x > 9$	11	$|11 - 5| = 6$	greater

Since $1 < x < 9$ is the only interval in which the output at the test value is less than 4, we can conclude the solution to $|x - 5| \leq 4$ is $1 \leq x \leq 9$.

Example 8.2J

Given the function $f(x) = -\frac{1}{2}|4x - 5| + 3$, determine for what x values the function values are negative.

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x - 5| + 3 < 0$. We begin by isolating the absolute value:

$$-\tfrac{1}{2}|4x - 5| < -3$$

when we multiply both sides by -2, it reverses the inequality

$$|4x - 5| > 6$$

Next we solve for the equality $|4x - 5| = 6$

$$4x - 5 = 6$$
$$4x = 11$$
$$x = \tfrac{11}{4}$$

or

$$4x - 5 = -6$$
$$4x = -1$$
$$x = \tfrac{-1}{4}$$

We can now either pick test values or sketch a graph of the function to determine on which intervals the original function value are negative. Notice that it is not even really important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x = \frac{-1}{4}$ and $x = \frac{11}{4}$, and that the graph has been flipped.
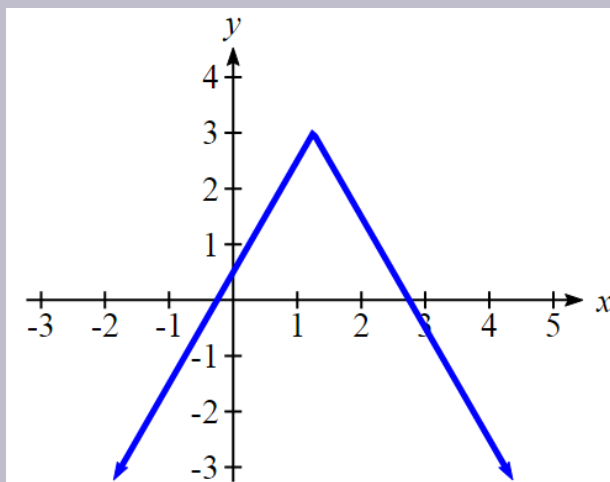
From the graph of the function, we can see the function values are negative to the left of the first horizontal intercept at $x = \frac{-1}{4}$, and negative to the right of the second intercept at $x = \frac{11}{4}$. This gives us the solution to the inequality:

$$x < \frac{-1}{4} \text{ or } x > \frac{11}{4}$$

In interval notation, this would be $\left(-\infty, \frac{-1}{4}\right) \cup \left(\frac{11}{4}, \infty\right)$

Give It Some Thought

6. Solve $-2|k - 4| \leq -6$

There is a third approach to solving absolute value inequalities that is formulaic. While it works, and you are welcome to use it, it is much more likely that you will remember the other approaches.

Solutions to Absolute Value Inequalities

To solve $|A| < B$, solve: $-B < A < B$

To solve $|A| > B$, solve: $A > B$ or $A < -B$

Example 8.2K

Solve $3|x + 4| - 2 \geq 7$

We need to start by isolating the absolute value:

$$3|x + 4| - 2 \geq 7$$
$$\text{Add } 2 \text{ to both sides}$$
$$3|x + 4| \geq 9$$
$$\text{Divide both sides by } 3$$
$$|x + 4| \geq 3$$

Now we can break this apart and solve each piece separately:

$$x + 4 \geq 3$$
$$x \geq -1$$

or

$$x + 4 \leq -3$$
$$x \leq -7$$

In interval notation, this would be $(-\infty, -7] \cup [-1, \infty)$

Give It Some Thought Answers

1. $x \geq -4$

2. $4 \leq wx + 6 < 16$

3. Using the variable p, for passing, $|p - 80| \leq 20$

4. $f(x) = -|x + 2| + 3$

5. $f(0) = 1$, so the vertical intercept is at $(0, 1)$.
$f(x) = 0$ when $x = -5$ and $x = 1$ so the horizontal intercepts are at $(-5, 0)$ and $(1, 0)$.

6. $k < 1$ or $k > 7$; in interval notation this would be $(-\infty, 1) \cup (7, \infty)$

Section 8.2 Exercises

1. Solve each inequality and write the solution set using interval form and graph it.
   a. $2x + 1 < 5$
   b. $-2 - 4x \geq 6$
   c. $\dfrac{1}{2} - x < \dfrac{x}{5} + \dfrac{1}{4}$
   d. $\dfrac{5 + x}{3} \geq -1$
   e. $\dfrac{3 - 5x}{3} \leq 0$

2. Use graph solution notation to write the following using single interval, if possible.
   a. $(-\infty, 0)(-\infty, 5)$
   b. $(4, 7)(2, \infty)$
   c. $(-3, \infty)(2, \infty)$
   d. $(-\infty, 4)[4, 5]$
   e. $[3, 6](5, 7]$
   f. $[-5, \infty)(9, \infty)$

3. Solve the following inequalities and describe the solutions using set, interval and graph notation:
   1. $2(t - 5) > 15 - (4 - 2t)$
   2. $\dfrac{4}{3}(3x - 2) + \dfrac{3}{5}(4x - 3) \geq \dfrac{11}{60} + 3x$

4. Solve each compound inequality. Write the solution set using interval notation and graph it.
   a. $5 > 7 - x$ and $2 + 1.5x < 4$
   b. $5 - x < 4$ and $0.2x - 3 < 1$
   c. $\dfrac{2x - 5}{-2} < 2$ and $\dfrac{2x + 1}{3} > 0$
   d. $1 - x < 7 + x$ or $4x + 3 > x$
   e. $5 + x > 3 - x$ or $2x - 3 > x$
   f. $\dfrac{1}{2}(x + 1) > 3$ or $0 < 7 - x$
   g. $1 - \dfrac{3}{2}x < 4$ and $\dfrac{1}{4}x - 2 \leq -3$
   h. $\dfrac{3}{5}x - 1 > 2$ and $5 - \dfrac{2}{5}x \geq 3$

    i. $1 \leq 3x - 5 < 8$

    j. $-2 < 4x + 9 \leq 16$

    k. $-2 \leq 3 - 6x < 22$

    l. $-13 < 5 - 9x < 41$

5. Write an inequality to describe the following situation: Kate will buy the jacket provided the cost C is at most $117.

6. A company spends $1200 per day on overhead and labor, and each item they produce costs $5 for materials. If they sell the items for $15 each, how many items will they need to sell each day for their profits to be positive?

7. Consider the following rectangle. For what values of (measured in cm) will the rectangle have the perimeter of at least 242 cm?

$6x + 9$

$3x + 4$
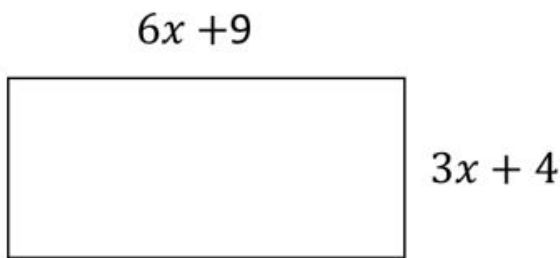
8. For what values of x is each of the following expressions a real number?

    a. $\sqrt{6x - 9}$

    b. $\dfrac{5}{x + 1}$

    c. $\dfrac{6}{\sqrt{18 - 3x}}$

9. A student scored 67 points on his Psychology 101 midterm. If the average of his midterm and final must be between 79 and 90 inclusive for a B, then for what range of scores on the final exam would a student get a B? Both tests have a maximum of 100 points. Write your answer in interval

10. To receive a B in a math course, an average between 80 and 89, inclusive, is required. Leah has scores of 76, 82 and 90 on her first three math tests. After the fourth test, she states that she now has a B average. In what range was her score on the fourth test for this statement to be true? Write your answer in interval notation.

11. To mail an envelope, Canada Post charges 37 cents for the first gram, and 25 cents for each additional gram. What is the weight of the envelope that can be mailed for $8.35? Write your answer in interval notation.

12. A car rents for $45 per day plus 21 ¢ per kilometer. You are at a daily budget of $66. How many kilometers can you cover to stay within your budget?

13. Anna is considering writing and publishing her own book. She estimates her revenue to be $R = 6.41x$ and her cost to be $C = 10039 + 1.09x$ where x is the number of books sold. Find the minimum

number of books she must sell to make a profit.

14. Jan scored 65 points on the calculus midterm. If the final exam counts twice as much as the midterm exam, then what range of the final exam scores would result in the final average between 79 and 90? Both tests have a maximum of 100 points. Write your answer in interval form.

15. Canada post defines girth of a box as the sum of the twice the width and twice the width. The maximum length, height and width allowed are 2 m and length plus girth must be at most 3 m. If the box is 1 m long and 75 cm wide, in what range can the height be?

8.3: Linear Inequalities

In the systems of equations section, we looked at a company that produces a basic and premium version of its product, and we determined how many of each item they should produce to fully utilize all staffed hours. In some cases, though, it might not make the most sense for the company to utilize all the staffed hours; if the premium product has high demand and a high price, it might make more sense for the company to make as many of those, even if some staff hours go unused. In this case, the company might be most interested in what combinations of basic and premium product production are possible. For that, we need linear inequalities.

A linear **equation** in two variables is an equation like $f(x) = 2x + 1$, which is sometimes written without function notation as $y = 2x + 1$. Recall that the graph of this equation is a line, formed by all the points (x, y) that satisfy the equation. A linear **inequality** in two variables is similar, but involves an inequality. Some examples:

$$y < 2x + 1$$

$$y > 4x - 1$$

$$y \leq \tfrac{2}{3}x + 4$$

$$y \geq 5 - 2x$$

Linear inequalities can also be written with both variables on the same side of the equation, like $2x - 3y < 4$.

The solution set to a linear inequality will be all the points (x, y) that satisfy the inequality. Notice that the line $y = 2x + 1$ divides the coordinate plane into two halves; on one half $y < 2x + 1$, and on the other $y > 2x + 1$. The solution set to a linear inequality will be a half plane, and to show the solution set we shade the part of the coordinate plane where the points lie in the solution set.

To graph the solution to a linear inequality

1. If necessary, rewrite the linear inequality into a form convenient for graphing, like slope-intercept form.
2. Graph the corresponding linear equation.

   a. For a strict inequality (< or >), draw a dashed line to show that the points on the line are not part of the solution

   b. For an inequality that includes the equal sign (≤ or ≥), draw a solid line to show that the points on the line are part of the solution.

3. Choose a test point, not on the line.

   a. Substitute the test point into the inequality

   b. If the inequality is true at the test point, shade the half plane on the side including the test point

   c. If the inequality is not true at the test point, shade the half plane on the side that doesn't include the test point.

Example 8.3A

Graph the solution to $y < 2x + 1$

Since this is a strict inequality, we will graph $y = 2x + 1$ as a dashed line.
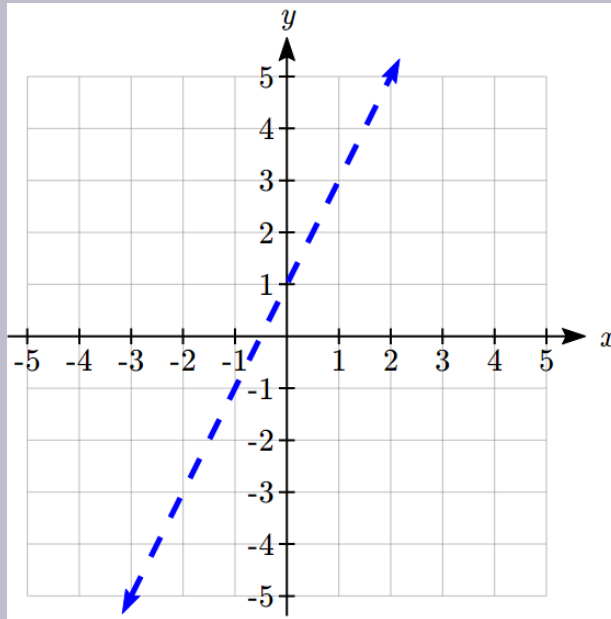
Now we choose a test point not on the line. Usually picking a point where one coordinate is zero will make things easier. Let's use $(3, 0)$.

Substituting $(3, 0)$ into the inequality, we get

$$0 < 2(3) + 1$$
$$0 < 7$$

This is a true statement, so we will shade the side of plane that includes $(3, 0)$
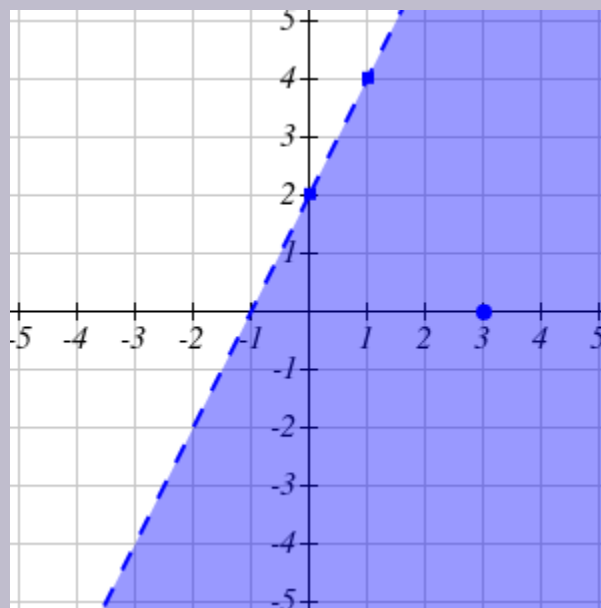
Example 8.3B

Graph the solution to $x \geq 2y + 4$

This is not written in a form we are used to use for graphing, so we might solve it for y first.

$$x \geq 2y + 4$$

Subtract 2y from both sides

$$x - 2y \geq 4$$

Subtract x from both sides

$$-2y \geq 4 - x$$

Divide by -2, reversing the order of the inequality

$$y \leq -2 + \tfrac{1}{2}x$$

Since this inequality includes the equals sign, we will graph $y = -2 + \frac{1}{2}x$ as a solid line.
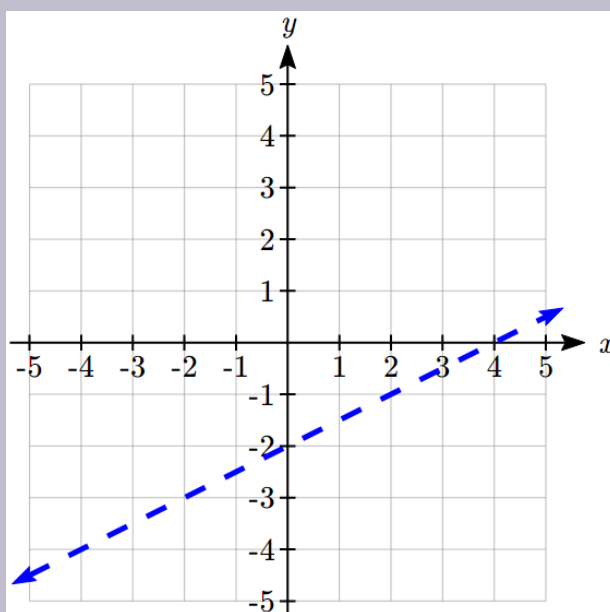
Now we choose a test point not on the line. $(0, 0)$ is a convenient choice. Substituting $(0, 0)$ into the inequality, we get

$$0 \leq -2 + \tfrac{1}{2}(0)$$
$$0 \leq -2$$

This is a false statement, so we shade the half of the plane that does not include $(0, 0)$.
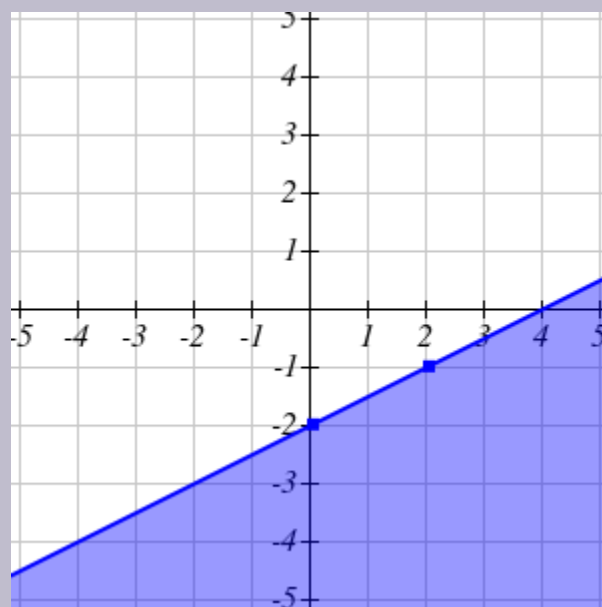
Give It Some Thought

1. Graph the solution to $y \geq -\frac{1}{2}x + 1$

Example 8.3C

A store sells peanuts for $4/pound and cashews for $8/pound, and plans to sell a new mixed-nut blend in a jar. What combinations of peanuts and cashews are possible, if they want the mix to cost $6 or less?

We start by defining our variables:

p: The number of pounds of peanuts in 1 pound of mix

c: The number of pounds of cashews in 1 pound of mix

The cost of one pound of mix will be $4p + 8c$, so all the mixes costing $6 or less will satisfy the inequality $4p + 8c \leq 6$.

We can graph the equation $4p + 8c = 6$ fairly easily by finding the intercepts:

When p = 0,

$$4\left(0\right) + 8c = 6$$
$$8c = 6$$
$$c = \tfrac{6}{8} = \tfrac{3}{4}$$

So the point $\left(0, \tfrac{3}{4}\right)$ is on the line.

When $c = 0$, $4p = 6$, so $p = \tfrac{6}{4} = \tfrac{3}{2}$, giving the point $\left(\tfrac{3}{2}, 0\right)$.

Notice the test point $\left(0, 0\right)$ will satisfy the inequality, so we will shade the side of the line including the origin.
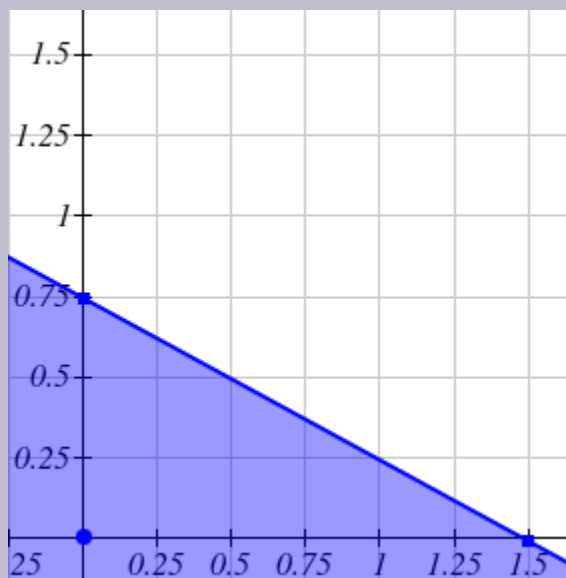
Because of the context, only values in the first quadrant are reasonable. The graph shows all the possible combinations the store could use, including 1 pound of peanuts with 1/4 pound of cashews, or 1/2 a pound of each.

Systems of Linear Inequalities

In the systems of equations chapter, we looked for solutions to a system of linear equations – a point that would satisfy all the equations in the system. Likewise, we can consider a system of linear inequalities. The solution to a system of linear inequalities is the set of points that satisfy all the inequalities in the system.

With a single linear inequality, we can show the solution set graphically. Likewise, with a system of linear inequalities we show the solution set graphically. We find it by looking for where the regions indicated by the individual linear inequalities overlap.

Example 8.3D

Graph the solution to the system of linear inequalities

$$y \leq x + 2$$
$$y \geq 1 - x$$

If we graph the solution set to each inequality individually, we get the two solutions sets shown here.
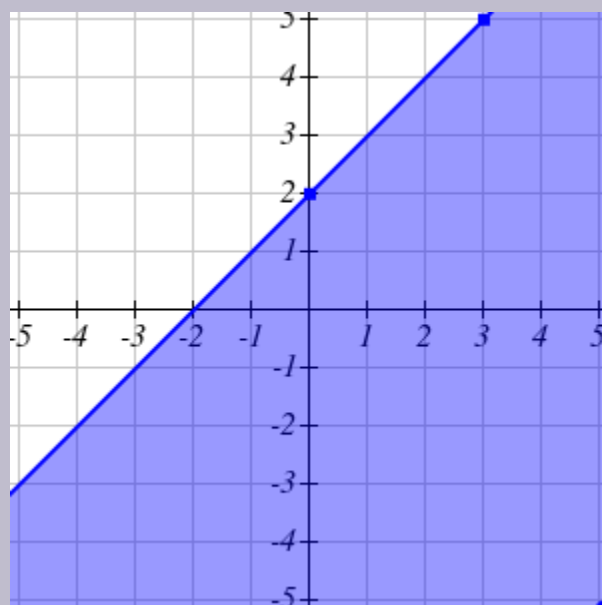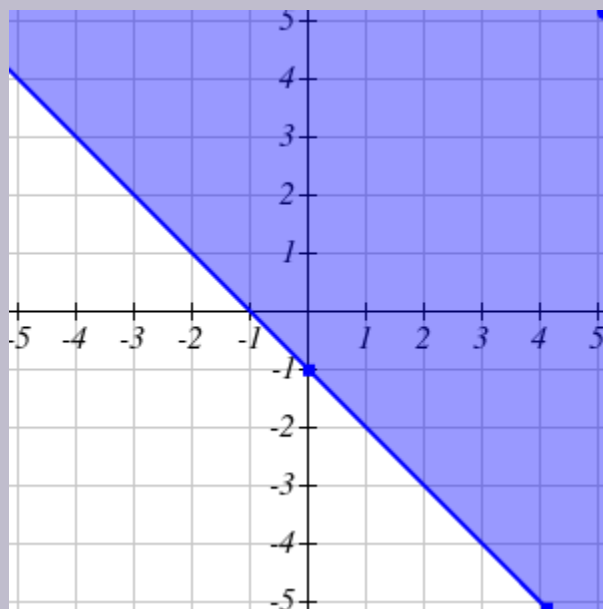
Graphing these solution sets on the same axes reveals the solution to the system of inequalities as the region where the two overlap.

The solution set, where the regions overlap:
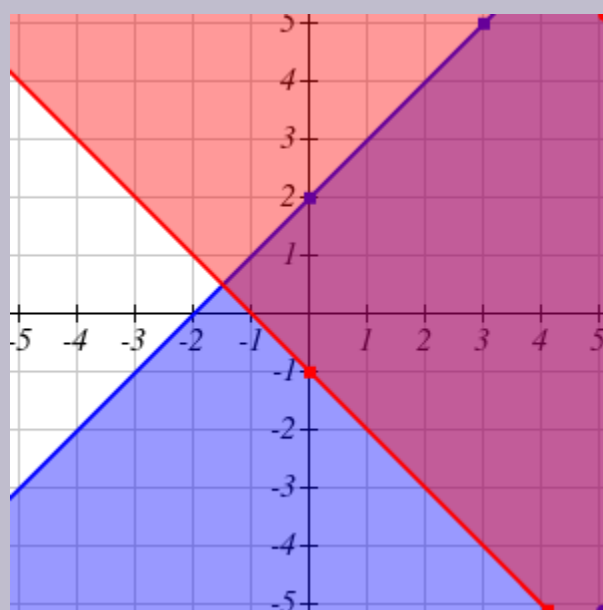
The solution set, drawn alone:
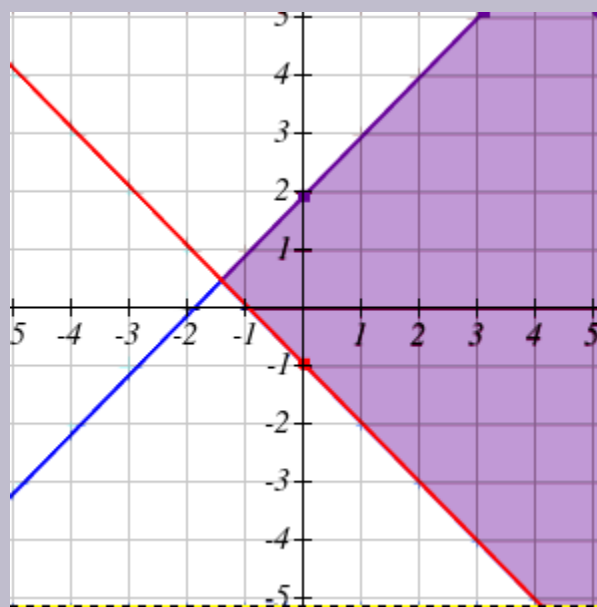
Give It Some Thought

2. Graph the solution to the system of linear inequalities

$$y \leq 3 - 2x$$
$$y \geq \tfrac{1}{2}x + 1$$

The following question is similar to a problem we solved using systems in the systems chapter.

Example 8.3E

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. How many items can then produce within the limits of their staffing?

Notice this problem is different than the question we asked in the first section, since we are no longer concerned about fully utilizing staffing, we are only interested in what is possible. As before, we'll define

b: The number of basic products made

p: The number of premium products made.

Just as we created equations in the first section, we can now create inequalities, since we know the hours used in production needs to be less than or equal to the hours available. This leads to two inequalities:

$$20b + 30p \leq 3900$$
$$15b + 30p \leq 3300$$

Graphing these inequalities gives us the solution set.
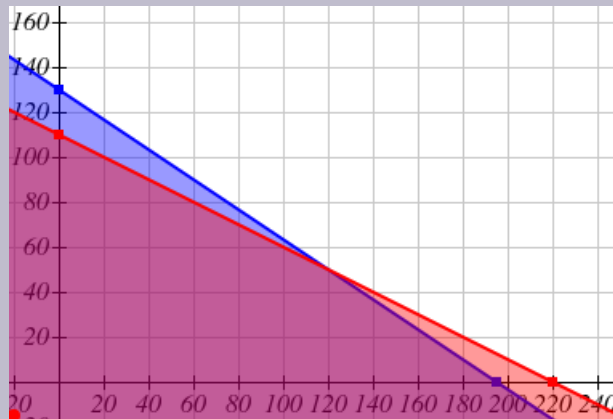
Since it isn't reasonable to consider negative numbers of items, we can further restrict the solutions to the first quadrant. This could also be represented by adding the inequalities $b \geq 0$ and $p \geq 0$.

In the solution set we can see the solution to the system of equations we solved in the previous

section: 120 basic products and 50 premium products. The solution set shows that if the company is willing to not fully utilize the staffing, there are many other possible combinations of products they could produce.

The techniques we used above are key to a branch of mathematics called linear programming, which is used extensively in business. We will explore linear programming further in the next section.

Give It Some Thought Answers

1.
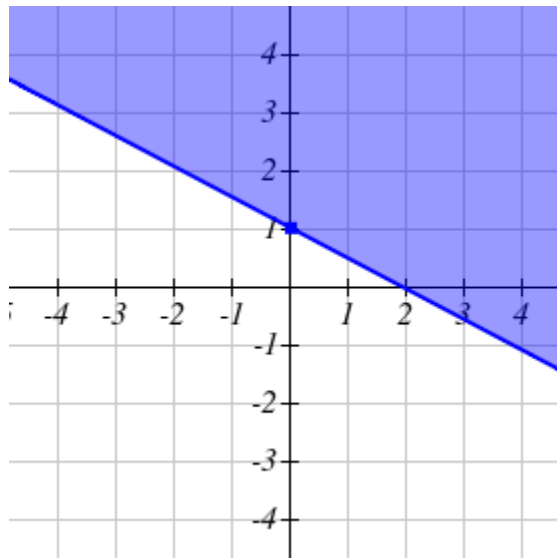
2.
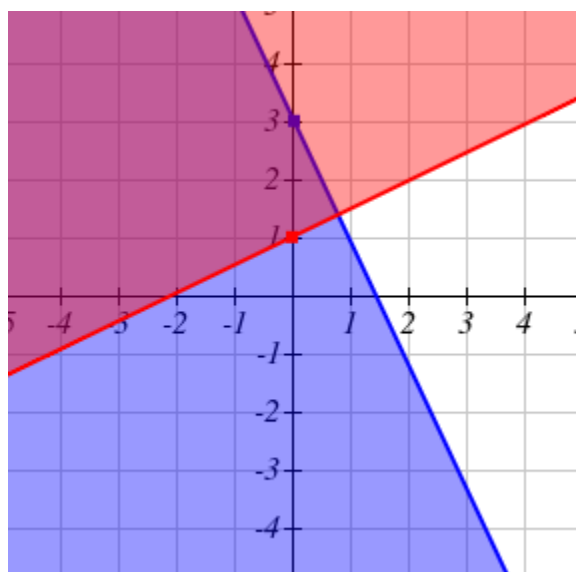
Section 8.3 Exercises

1. Graph by hand the solution region of each of the following systems of inequalities, identify its corner points and state whether the region is bounded or unbounded.

   a. $x + y \leq 17$
      $4x + y \geq 5$

   Answer

   b. $x + y < -2$
      $4x + 2y \geq 5$

   Answer

   c. $2x - 3y > 10$
      $4x + 3y > 5$

   Answer

8.4: Graphical Solutions

Wouldn't it be nice if we could simply produce and sell infinitely many units of a product and thus make a never-ending amount of money? In business (and in day-to-day living) we know that some things are just unreasonable or impossible. Instead, our hope is to maximize or minimize some quantity, given a set of constraints.

In order to have a linear programming problem, we must have:

- Constraints, represented as inequalities
- An **objective function**, that is, a function whose value we either want to be as large as possible (want to maximize it) or as small as possible (want to minimize it).

Consider this extension of the example from the end of the last section.

Example 8.4A

A company produces a basic and premium version of its product. The basic version requires 20 minutes of assembly and 15 minutes of painting. The premium version requires 30 minutes of assembly and 30 minutes of painting. If the company has staffing for 3,900 minutes of assembly and 3,300 minutes of painting each week. They sell the basic products for a profit of $30 and the premium products for a profit of $40. How many of each version should be produced to maximize profit?

Let b = the number of basic products made, and p = the number of premium products made. Our objective function is what we're trying to maximize or minimize. In this case, we're trying to maximize profit. The total profit, P, is

$$P = 30b + 40p$$

In the last section, the example developed our constraints. Together, these define our linear programming problem:

Objective function: $P = 30b + 40p$

Constraints:

$$20b + 30p \leq 3900$$
$$15b + 30p \leq 3300$$
$$b \geq 0, p \geq 0$$

In this section, we will approach this type of problem graphically. We start by graphing the constraints to determine the **feasible region** – the set of possible solutions. Just showing the solution set where the four inequalities overlap, we see a clear region.
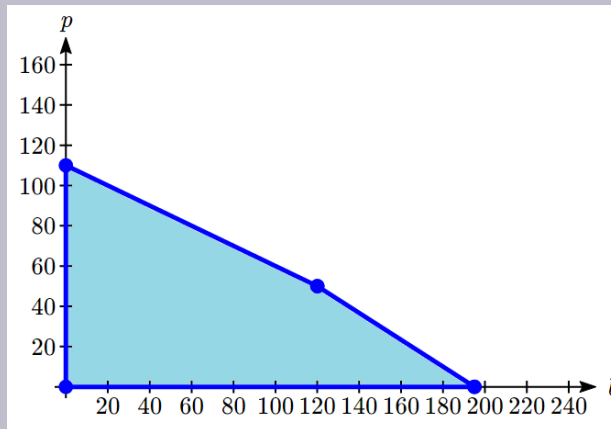
To consider how the objective function connects, suppose we considered all the possible production combinations that gave a profit of P = $3000, so that $3000 = 30b + 40p$. That set of combinations would form a line in the graph. Doing the same for a profit of $5000 and $6500 would give additional lines. Graphing those on top of our feasible region, we see a pattern:
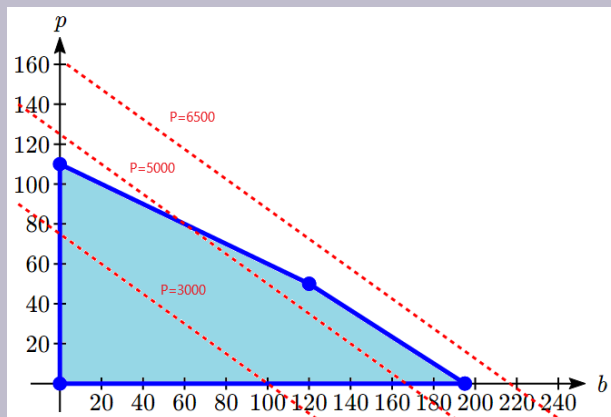
Notice that all the constant-profit lines are parallel, and that in general the profit increases as we move up to upper right. Notice also that for a profit of $5000 there are some production levels inside the feasible region for that profit level, but some are outside. That means we could feasibly make $5000 profit by producing, for example, 167 basic items and no premium items, but we can't make $5000 by producing 125 premium items and no basic items because that falls outside our constraints.

The solution to our linear programming problem will be the largest possible profit that is still feasible. Graphically, that means the line furthest to the upper-right that still touches the feasible region on at least point. That solution is the one below:
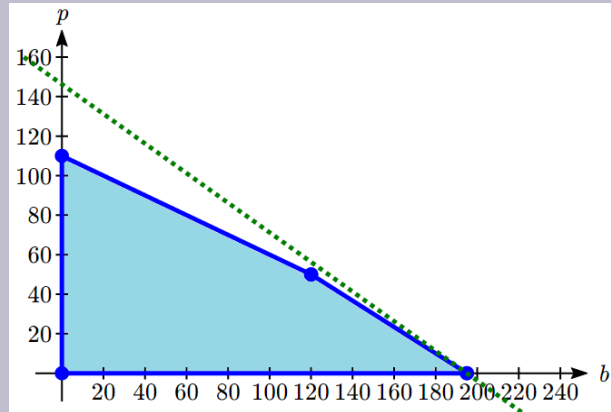
This profit line touches the feasible region where $b = 195$ and $p = 0$, giving a profit of P = 30(195) + 40(0) = $5850.

Notice that this is slightly larger than the profit that would be made by completely utilizing all staffing at $b = 120, p = 50$, where the profit would be $5600.

The objective function along with the four corner points above forms a **bounded** linear programming problem. That is, imagine you are looking at three fence posts connected by fencing (black point and lines, respectively). If you were to put your dog in the middle, you could be sure it would not escape (assuming the fence is tall enough). If this is the case, then you have a bounded linear programming problem. If the dog could walk infinitely in any one direction, then the problem is unbounded.

In the past example, you can see that the line of maximum profit will always touch the boundary of the feasible region. That observation inspires the fundamental theorem of linear programming.

Fundamental Theorem of Linear Programming

- If a solution exists to a bounded linear programming problem, then it occurs at one of the corner points.
- If a feasible region is unbounded, then a maximum value for the objective function does not exist.
- If a feasible region is unbounded, and the objective function has only positive coefficients, then a minimum value exists.

In the last example we solve the problem somewhat intuitively by "sliding" the profit line up. Typically we use a more procedural approach.

Solving a Linear Programming Problem Graphically

1. Define the variables to be optimized. The question asked is a good indicator as to what these will be.
2. Write the objective function, first in words, then convert to a mathematical equation
3. Write the constraints, first in words, then convert to mathematical inequalities
4. Graph the constraints inequalities, and shade the feasible region
5. Identify the corner points by solving systems of linear equations whose intersection represents a corner point.
6. Test all corner points in the objective function. The "winning" point is the point that optimizes the objective function (biggest if maximizing, smallest if minimizing)

Give It Some Thought

1. Maximize $P = 14x + 9y$ subject to the constraints:

$$x + y \le 9$$
$$3x + y \le 15$$
$$x \ge 0, y \ge 0$$

Example 8.4B

A health-food business would like to create a high-potassium blend of dried fruit in the form of a box of 10 fruit bars. It decides to use dried apricots, which have 407 mg of potassium per serving, and dried dates, which have 271 mg of potassium per serving. The company can purchase its fruit in bulk for a reasonable price. Dried apricots cost $9.99/lb. (about 3 servings) and dried dates cost $7.99/lb.

(about 4 servings). The company would like the box of bars to have at least the recommended daily potassium intake of about 4700 mg, and contain at least 1 serving of each fruit. In order to minimize cost, how many servings of each dried fruit should go into the box of bars?

We begin by defining the variables. Let
x = number of servings of dried apricots
y = number of servings of dried dates

We next work on the objective function.

For apricots, there are 3 servings in one pound. This means that the cost per serving is $9.99/3 = $3.33. The cost for x servings would thus be $3.33x$.

For dates, there are 4 servings per pound. This means that the cost per serving is $7.99/4 = $2.00. The cost for y servings would thus be $2.00y$.

The total cost, C, for apricots and dates would be $C = 3.33x + 2.00y$

Normally we would have constraints $x \geq 0$ and $y \geq 0$ since negative servings can't be used. But in this case, we're further restricted. In words:

- There must be at least 1 serving of each fruit
- The product must contain at least 4700 mg of potassium

Mathematically,

- Since there must be at least 1 serving of each fruit, $x \geq 1$ and $y \geq 1$
- There are $407x$ mg of potassium in x servings of apricots and $271y$ mg of potassium in y servings of dates. The sum should be greater than or equal to 4700 mg of potassium, or $407x + 271y \geq 4700$

Thus we have,
Objective function:

$$C = 3.33x + 2.00y$$

Constraints:

$$407x + 271y \geq 4700$$
$$x \geq 1, y \geq 1$$

We graph the constraints and shade the feasible region:
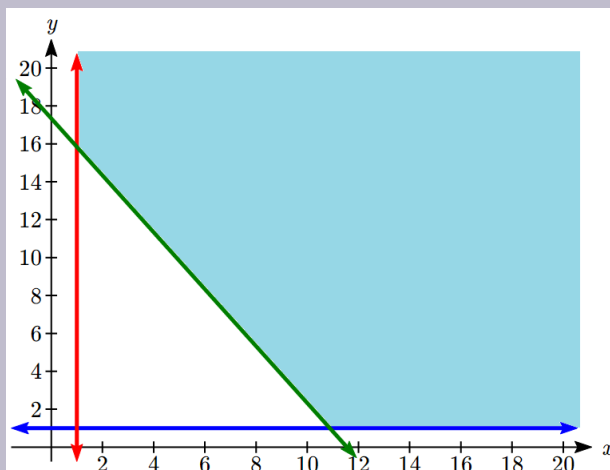
The region is unbounded, but we will be able to find a minimum still. We can see there are two corner points.

The one in the upper left is the intersection of the lines $407x + 271y = 4700$ and $x = 1$. Solving for the intersection using substitution:

$$407(1) + 271y = 4700$$
$$y \approx 15.8$$

$$\text{Point: } (1, 15.8)$$

The one in the lower right is the intersection of the lines $407x + 271y = 4700$ and $y = 1$.

$$407x + 271(1) = 4700$$
$$x \approx 10.9$$

$$\text{Point: } (10.9, 1)$$

Testing the objective function at each of these corner points:

Point	Cost, $C = 3.33x + 2.00y$
$(10.9, 1)$	$C = 3.33(10.9) + 2.00(1)$ = $38.30
$(1, 15.8)$	$C = 3.33(1) + 2.00(15.8)$ = $34.96

The company can minimize cost by using 1 serving of apricots and 15.8 servings of dates.

Give It Some Thought

2. A company makes two products. Product A requires 3 hours of manufacturing and 1 hour of assembly. Product B requires 4 hours of manufacturing and 2 hours of assembly. There are a total of 84 hours of manufacturing and 32 hours of assembly available. Determine the production to maximize profit if the profit on product A is $50 and the profit on product B is $60.

Example 8.4C

A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 40 hours; the second at most 42 hours; and the third at most 25 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is $20 per unit for a chair and $30 for a table, how many units of each should be manufactured to maximize profit?

We begin by defining the variables. Let
c = number of chairs made
t = number of tables made

The profit, P, will be $P = 20c + 30t$.

For cutting, c chairs will require $1c$ hours and t tables will require $2t$ hours. We can use at most 40 hours, so $c + 2t \leq 40$.

For assembly, c chairs will require $2c$ hours and t tables will require $1t$ hours. We can use at most 42 hours, so $2c + t \leq 42$.

For finishing, c chairs will require $1c$ hours and t tables will require $1t$ hours. We can use at most 25 hours, so $c + t \leq 25$.

Since we can't produce negative items, $c \geq 0, t \geq 0$.

Graphing the constraints, we can see the feasible region.
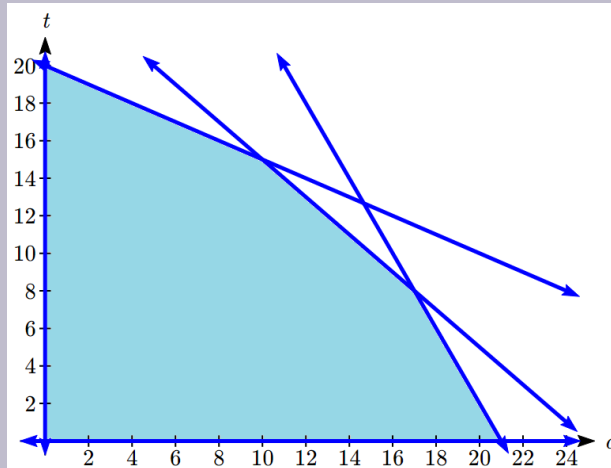
There are five corner points for this region.

Point 1: In the lower left, where $t = 0$ crosses $c = 0$.

$$\text{Point: } (0, 0)$$

Point 2: In the upper left, where $c = 0$ crosses $c + 2t = 40$.
Using substitution, $0 + 2t = 40$, so $t = 20$.

$$\text{Point: } (0, 20)$$

Point 3: In the lower right, where $t = 0$ crosses $2c + t = 42$.
Using substitution, $2c + 0 = 42$, so $c = 21$.

$$\text{Point: } (21, 0)$$

Point 4: Where $c + 2t = 40$ crosses $2c + t = 42$.

We can solve this as a system using any techniques we know. We could solve the second equation for c, giving $c = 25 - t$, then substitute into the first equation:

$$(25 - t) + 2t = 40$$
$$25 + t = 40$$
$$t = 15$$

Then $c = 25 - 15 = 10$.

$$\text{Point: } (10, 15)$$

Point 5: Where $2c + t = 42$ crosses $c + t = 25$.

We can solve this as a system using any techniques we know. Using a different technique this time, we could multiply the bottom equation by -1 then add it to the first:

$$2c + t = 42$$
$$\underline{-c - t = -25}$$
$$c = 17$$

Then using $c + t = 25$, we have $17 + t = 25$, so $t = 8$.

$$\text{Point: } (17, 8)$$

Testing the objective function at each of these corner points:

Point	Profit, $P = 20c + 30t$
$(0, 0)$	$P = 20(0) + 30(0)$ = $0
$(0, 20)$	$P = 20(0) + 30(20)$ = $600
$(21, 0)$	$P = 20(21) + 30(0)$ = $420
$(10, 15)$	$P = 20(10) + 30(15)$ = $650
$(17, 8)$	$P = 20(17) + 30(8)$ = $580

The profit will be maximized by producing 10 chairs and 15 tables.

Give It Some Thought Answers

1. Maximize $P = 14x + 9y$ subject to:

$$x + y \leq 9$$
$$3x + y \leq 15$$
$$x \geq 0, y \geq 0$$

Graphing the feasible region, we see there are three corner points of potential interest: $(0, 9)$, $(5, 0)$, and the intersection of the two lines at $(3, 6)$. We then evaluate the objective function at each corner point.
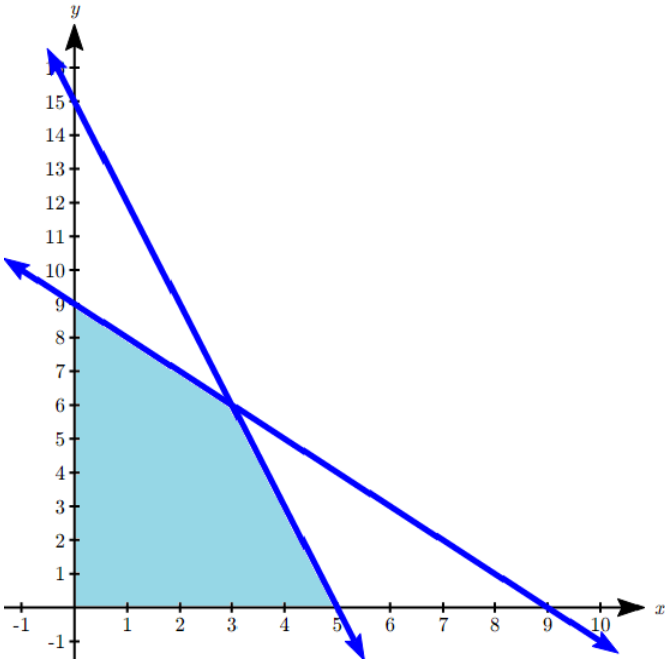
Point	$P = 14x + 9y$
$(0, 9)$	81
$5, 0)$	70

P is maximized when $x = 3$, $y = 6$.

2. Let a be the number of product A produced, and b be the number of product B.

Our goal is to maximize profit: $P = 50a + 60b$.

From manufacturing we get the constraint:

$$3a + 4b \leq 84$$

From assembly we get the constraint:

$$1a + 2b \leq 32$$

We have corner points at $(0, 16)$, $(28, 0)$. The third is at the intersection of the two lines. To find that we could multiply the second equation by -2 and add:

$$3a + 4b = 84$$
$$-2a - 4b = -64$$
$$\overline{\, a = 20}$$

The third corner point is at $(20, 6)$.
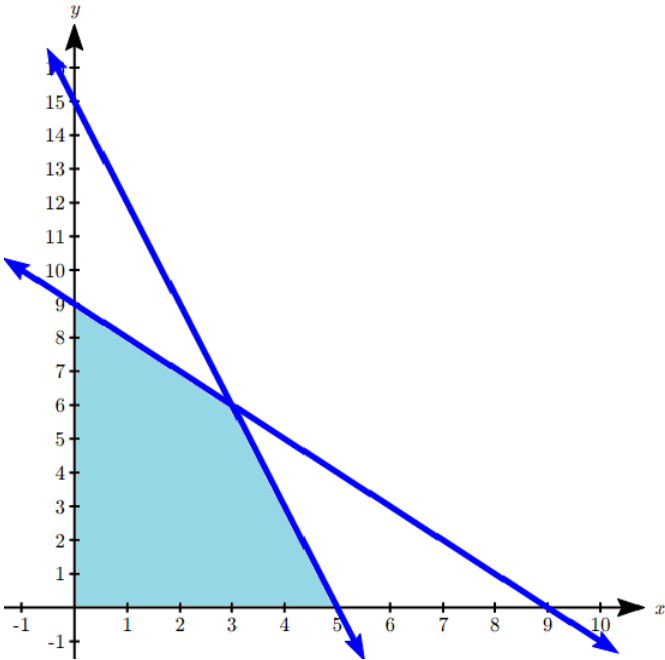
Evaluating the objective function at each of these points:

Point	P
$(0, 16$	$P = 50\,(0) + 60\,(16) = 960$

Profit will be maximized by producing 28 units of product A and 0 units of product B.

Section 8.4 Exercises

Insert review exercises here

Chapter 8 Summary

Key Takeaways

Section 8.1: Systems of Equations

- Solving systems
  Graphing
  Elimination
  Addition
- Setting up systems of equations
- Inconsistent and dependent systems
- Solving 3 by 3 systems

Section 8.2: Inequalities in One Variable

- The properties of the absolute value function
- Solving absolute value equations
- Finding intercepts
- Solving absolute value inequalities

Section 8.3: Linear Inequalities

- Graphing linear inequalities
  Dashed line for strict inequalities, solid for ≤ or ≥
- Graphing systems of linear inequalities

Section 8.4: Graphical Solutions

- Objective function
- Constraint equations
- Feasible region
- Corner points
- Fundamental Theorem of Linear Programming
- Solving a linear programming problem using a graph

9.

POLYNOMIAL AND RATIONAL FUNCTIONS

9.1 Quadratic Functions

In this section, we will explore the quadratic functions, a type of polynomial function. Quadratics commonly arise from problems involving areas, as well as revenue and profit, providing some interesting applications.

Example 9.1A: Understanding Quadratic Functions

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
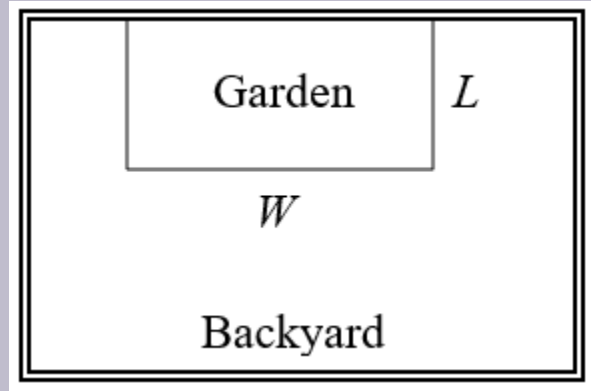
In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, *W*, to represent the side of fencing parallel to the 4th side or backyard fence.

Since we know we only have 80 feet of fence available, we know that , or more simply, . This allows us to represent the width, *W*, in terms of *L*:

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

This formula represents the area of the fence in terms of the variable length *L*.

Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**.

The **standard form** for a quadratic is $f(x) = ax^2 + bx + c$, but you will often see them written in the form $f(x) = a(x - h)^2 + k$. To see why, consider this example.

Example 9.1B: Vertex of the Quadratic

Sketch a graph of $g(x) = \frac{1}{2}(x+2)^2 - 3$

We can create a table of values, which we can use to plot several points and connect them with a smooth curve.

x	g(x)
-5	1.5
-4	-1
-3	-2.5
-2	-3
-1	-2.5
0	-1
1	1.5

Notice that the turning point of the graph, where it changes from decreasing to increasing, is at the point (-2, -3). We call this point the **vertex** of the quadratic. Notice that $g(x) = \frac{1}{2}(x+2)^2 - 3$ can also be written as $g(x) = \frac{1}{2}(x-(-2))^2 - 3$. Comparing that to the form $f(x) = a(x-h)^2 + k$, you can see that the vertex of the graph, (-2, -3), corresponds with the point *(h, k)*.

Forms of Quadratic Functions

The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$

The **vertex form** of a quadratic function is $f(x) = a(x-h)^2 + k$

The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the vertex form of the function.

When $a > 0$, the graph of the quadratic will open upwards.

When $a < 0$, the graph of the quadratic will open downwards.

$a > 0$ $a < 0$
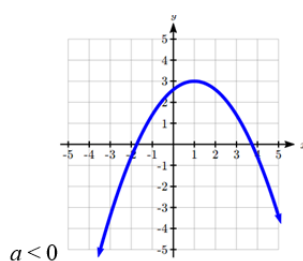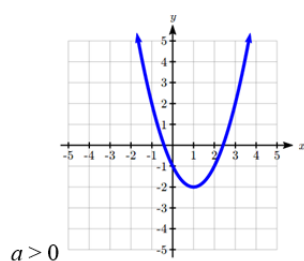
Give It Some Thought

1. A coordinate grid has been superimposed over the quadratic path of a basketball[1]. Find an equation for the path of the ball. Does he make the basket?
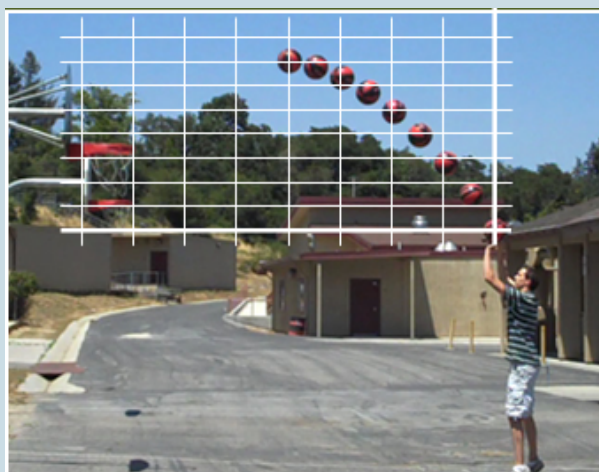
[1] From http://blog.mrmeyer.com/?p=4778, © Dan Meyer, CC-BY

Example 9.1C: Standard Polynomial Function

Write $g(x) = \frac{1}{2}(x+2)^2 - 3$ in standard form.

To write this in standard polynomial form, we could expand the formula and simplify terms:

$g(x) = \frac{1}{2}(x+2)^2 - 3$
$g(x) = \frac{1}{2}(x+2)(x+2) - 3$
$g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3$
$g(x) = \frac{1}{2}x^2 + 2x + 2 - 3$
$g(x) = \frac{1}{2}x^2 + 2x - 1 </div>$

In the previous example, we saw that it is possible to rewrite a quadratic function given in vertex form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process, since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$f(x) = a(x-h)^2 + k = a(x-h)(x-h) + k$
$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$

This should be equal to the standard form of the quadratic:

$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$-2ah = b$, so $h = -\frac{b}{2a}$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$ah^2 + k = c$, so $k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $k = f(h)$.

Finding the Vertex of a Quadratic

For a quadratic given in standard form, the vertex (h, k) is located at:

$h = -\frac{b}{2a}, k = f(h) = f\left(\frac{-b}{2a}\right)$

Example 4

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic into vertex form.

The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$

The vertical coordinate of the vertex will be at $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}$

Rewriting into vertex form, the value of *a* will remain the same as in the original quadratic.

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Give It Some Thought

2. Given the equation $g(x) = 13 + x^2 = 6x$ write the equation in standard form and then in vertex form.

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 9.1D: Finding the Vertex

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L) = 80L - 2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that *a* is the coefficient on the squared term, so *a* = -2, *b* = 80, and *c* = 0.

Finding the vertex:

$$h = -\frac{80}{2(-2)} = 20, k = A(20) = 80(20) - 2(20)^2 = 800$$

The maximum value of the function is an area of 800 square feet, which occurs when *L* = 20 feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 9.1E: Maximum Revenue

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, *C* for charge per subscription and *S* for the number subscribers, giving us the equation:

Revenue = *CS*

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently *S* = 84,000 and *C* = 30, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, *C* = 32 and *S* = 79,000. From this we can find a linear equation relating the two quantities. Treating *C* as the input and

S as the output, the equation will have form $S = mC + b$. The slope will be $m = \frac{79,000-84,000}{32-30} = \frac{-5,000}{2} = -2,500$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$$S = -2500C + b$$

Plug in the point *S* = 84,000 and *C* = 30

$$84,000 = -2500(30) + b$$

Solve for *b*:

$$b = 159,000$$

This gives us the linear equation $S = -2,500C + 159,000$ relating cost and subscribers. We now return to our revenue equation.

Substituting the equation for *S* from above:

Revenue = *CS*

Revenue = *C*(-2,500C+159,000)

Expanding:

Revenue = -2,500C^2+159,000*C*

We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

$$h = -\frac{159,000}{2(-2,500)} = 31.8$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:

Maximum Revenue = -2,500(31.8)^2+159,000(31.8)=2,528,100

Notice that the equation *S*= -2,500C+159,000 we found in the last example is essentially a demand function – a relationship between the price (*C*) and the demand (*S*). This is a common type of economic application.

Maximizing Revenue or Profit

To solve a problem involving maximizing revenue or profit given data on demand at different price levels,

1. Use the data provided to create a demand equation relating price and quantity demanded. This will usually be a linear function.

2. Create a Revenue equation. Start with Revenue = price times quantity, then substitute the demand equation to create a revenue equation just in terms of quantity.

3. If the problem is to maximize profit, create a Profit equation. Profit = revenue minus cost.

4. Find the vertex of the Revenue or Profit function to find the quantity that maximizes revenue or profit.

5. If needed, use that quantity with the demand function to find the price that maximizes revenue or profit.

Example 9.1F: Maximum and Total Profit

A company is planning to sell a new smart fitness device. Developing the product will cost $700,000, and each product will cost $30 to manufacture. Market research suggests that if they sell the device for $100, they will be able to sell 30,000 items. For each $10 they lower the price, they estimate they will sell 5,000 more items. Assuming quantity demanded is linearly related to price, determine the price that will maximize profit

Let p be the price per item, and q be the quantity the company can sell. We start by creating a linear demand function, of the form $p = mq + b$.

If they sell the device for $100, they will be able to sell 30,000 items, giving the point (30000, 100).

We are told that for each $10 they lower the price, they estimate they will sell 5,000 more items. We can either directly interpret this as a slope, or use it to create a second point. Lowering the price to $90 would raise the quantity sold to 35,000, giving a second point (35000, 90).

Finding the slope:

$$m = \frac{90-100}{35000-30000} = \frac{-10}{5000} = -\frac{1}{500} = -0.002$$

The demand equation will look like $p = -0.002q + b$. Substituting in (30000, 10) to solve for b:

100 = – 0.002 (30000) + b

100 = – 60 + b

b = 160

We now have our demand function: $p = -0.002q + 160$.

Now we construct our revenue function, using our demand curve

$R = pq$

Substituting in the demand function for p

$R = (-0.002q + 160)q = -0.002q^2 + 160q)$

Now looking at costs, we know the fixed costs are $700,000, and the per-item costs are $30, leading to the cost equation

$C = 700,00 + 30q$

Finally we can construct the profit equation:

Profit = Revenue – Cost

$$P = (-0.002q^2 + 160q) - (700,000 + 30q)$$
$$P = -0.002q^2 + 160q - 700,000 - 30q$$
$$P = -0.002q^2 + 130q - 700,000$$

Now to find the maximum we find the vertex of the quadratic:

$$h = -\frac{130}{2(-0.002)} = 32,500$$

To find the price that will produce a demand of 32,500 items, we use the demand function:

$P = -0.002(32500) + 160 = 95$

To maximize profit, the company should price the device at $95. At that price they should expect to sell 35,000 items, with a total profit of:

$P = -0.002(32500)^2 + 130(32500) - 70,000 = \$1,412,500$

Short run Behavior: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.
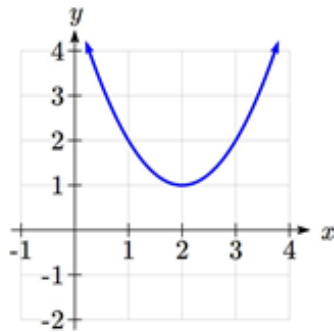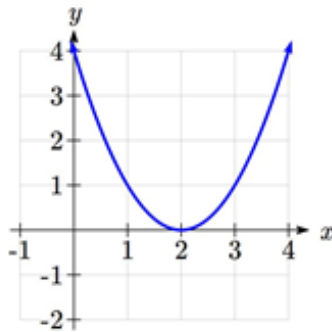
zero horizontal intercepts   one horizontal intercept   two horizontal intercepts
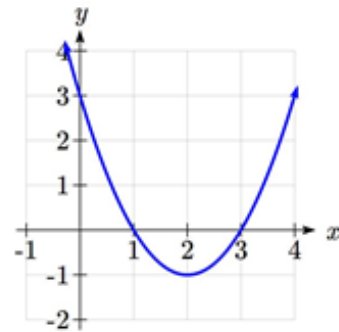
Example 9.1G: Vertical and Horizontal Intercepts

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

We can find the vertical intercept by evaluating the function at an input of zero:

$f(0) = 3(0)^2 + 5(0) - 2 = -2$ Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero $0 = 3x^2 + 5x - 2$

In this case, the quadratic can be factored easily, providing the simplest method for solution

0= (3x-1)(x+2)

$0 = 3x - 1$

$x = 1 \frac{}{3}$ or

$= x + 2$

x = - 2

Horizontal intercepts at and (-2,0)

Notice that in the standard form of a quadratic, the constant term *c* reveals the vertical intercept of the graph.

For quadratics that can't be factored, we need another tecnique. Based on our previous work we showed that any quadratic in standard form can be written into vertex form as:

$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

Solving for the horizontal intercepts using this general equation gives:

$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ start to solve for x by moving the constants to the other side divide both sides by a

$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$ find a common denominator to combine fractions

$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ combine the fractions on the left side of the equation

$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ take the square root of both sides

$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$ subtract $b/2a$ from both sides

$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$ combining the fractions

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Notice that this can yield two different answers for x

Quadratic Formula

For a quadratic function given in standard form , the quadratic formula gives the horizontal intercepts of the graph of this function.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 1.9H: Equilibrim

The supply for a certain product can be modeled by p=3q2 and the demand can be modeled by p=1620=2q2, where p is the price in dollars, and q is the quantity in thousands of items. Find the equilibrium price and quantity.

Recall that the equilibrium price and quantity is found by finding where the supply and demand curve intersect. We can find that by setting the equations equal:

$$3q^2 = 1620 - 2q^2$$

Add $2q^2$ to both sides

$5q^2 = 1620$ Divide by 5 on both sides

$q^2 = 324$ Take the square root of both sides

$$q = \pm\sqrt{324} = \pm 18$$

Since it doesn't make sense to talk about negative quantities, the equilibrium quantity is q = 18. To find the equilibrium price, we evaluate either function at the equilibrium quantity.

p= $3(18)^2$= 972

The equilibrium is 18 thousand items, at a price of $972.

Give It Some Thought

3. For these two equations determine if the vertex will be a maximum value or a minimum value.

a. $g(x) = -8x + x^2 + 7$

b. $g(x) = -3(3 - x)^2 + 2$

Give It Some Thought Answers

1. The path passes through the origin with vertex at (-4, 7). $h(x) = -\dfrac{7}{16}(x+4)^2 + 7$. To make the shot, h(-7.5) would need to be about 4. h(-7.5) = 1.64; he doesn't make it.

2. g(x) = x2 + 6x + 13 in Standard form; in Transformation form g(x) = (x − 3)2 + 4

3. a. Vertex is a minimum value

b. Vertex is a maximum value

9.2 Polynomial Functions

In the previous section we explored the short run behavior of quadratics, a special case of polynomials. In this section we will explore the behavior of polynomials in general. The basic building blocks of polynomials are power functions.

Power Functions

A **power function** is a function that can be represented in the form $f(x) = x^p$%

Where the base is a variable and the exponent, p, is a number.
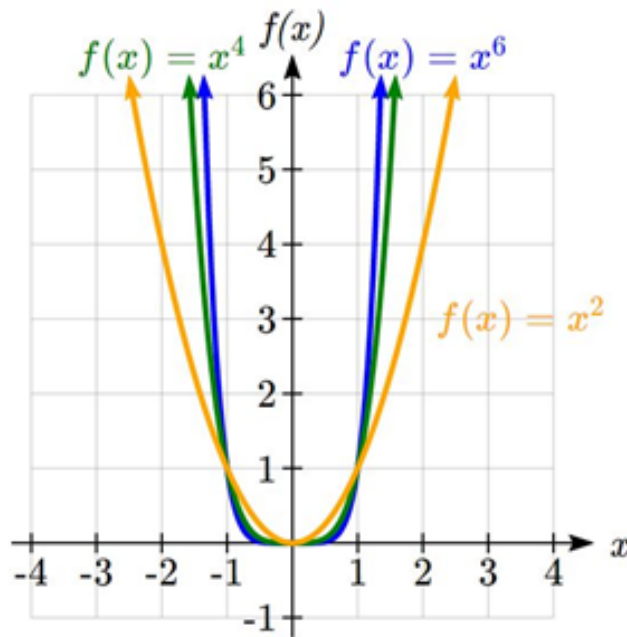
Characteristics of Power Functions

Shown to the right are the graphs of $f(x) = x^2$, $f(X) = x^4$, and $f(X) = x^6$, all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to a quadratic, but as the power increases the graphs flatten somewhat near the origin, and become steeper away from the origin.

To describe the behavior as numbers become larger and larger, we use the idea of infinity. The symbol for positive infinity is ∞, and $-\infty$ for negative infinity. When we say that "x approaches infinity", which can be symbolically written as $x \to \infty$, we are describing a behavior – we are saying that x is getting large in the positive direction.

With the even power function, as the input becomes large in either the positive or negative direction, the output values become very large positive numbers. Equivalently, we could describe this by saying that as x approaches positive or negative infinity, the *f(x)* values approach positive infinity. In symbolic form, we could write: as $x \to \pm\infty, f(x) \to \infty$.

Shown here are the graphs of f(x) = x^3, f(X) =x^5, and f(X) = x^7, all odd whole number powers. Notice all these graphs look similar, but again as the power increases the graphs flatten near the origin and become steeper away from the origin.

For these odd power functions, as x approaches negative infinity, *f(x)* approaches negative infinity. As x approaches positive infinity, *f(x)* approaches positive infinity. In symbolic form we write: as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$.

Long Run Behavior

The behavior of the graph of a function as the input takes on large negative values ($x \to -\infty$) and large positive values ($x \to \infty$) as is referred to as the **long run behavior** of the function.

Terminology of Polynomial Functions

A **polynomial** is function that can be written as $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Each of the *ai* constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any . Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending.

$$f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$$

Example 9.2A: Identifying Degrees, Leading Terms and Leading Coefficient

Identify the degree, leading term, and leading coefficient of these polynomials:

$$f(x) = 3 + 2x^2 - 4x^3$$
$$g(t) = 5t^5 - 2t^3 + 7t$$
$$h(p) = 6p - p^3 - 2$$

For the function *f(x)*, the degree is 3, the highest power on *x*. The leading term is the term containing that power, -4x^3. The leading coefficient is the coefficient of that term, -4.

For *g(t)*, the degree is 5, the leading term is 5t^5, and the leading coefficient is 5.

For *h(p)*, the degree is 3, the leading term is , so the leading coefficient is -1.

Long Run Behavior of Polynomials

For any polynomial, the **long run behavior** of the polynomial will match the long run behavior of the leading term.

Example 9.2B: Long Run Behavior and Degree

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?

Since the output grows large and positive as the inputs grow large and positive, we describe

the long run behavior symbolically by writing: as $x \to \infty$, $f(x) \to \infty$. Similarly, as $x \to -\infty$, $f(x) \to -\infty$.

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd, and the leading coefficient would be positive.

Short Run Behavior: Intercepts

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at the point $(0, a_0)$. The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Horizontal intercepts are also called **zeros**, or **roots** of the function.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4th degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

- 1) The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
- 2) The polynomial is given in factored form.
- 3) Technology is used to determine the intercepts.

Example 9.2C: Finding Horizontal Intercepts

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$x^6 - 3x^4 + 2x^2 = 0$ Factoring out the greatest common factor

$x^2 \left(x^4 - 3x^2 + 2\right) = 0$ Factoring the inside as a quadratic in $x2$

$x^2 \left(x^2 - 1\right) \left(x^2 - 2\right) = 0$ Then break apart to find solutions

$x^2 = 0$
$x = 0 < /div >$ or

$\left(x^2 - 1\right) = 0$
$x^2 = 1$ or
$x = \pm 1 < /div >$

$\left(x^2 - 2\right) = 0$
$x^2 = 2$
$x = \pm\sqrt{2}$

This gives us 5 horizontal intercepts.

Example 9.2D: Finding Vertical and Horizontal Intercepts

Find the vertical and horizontal intercepts of $g(t) = (t - 2)^2(2t + 3)$

The vertical intercept can be found by evaluating $g(0)$.

$g(0) = (0 - 2)^2(2(0) + 3) = 12$

The horizontal intercepts can be found by solving $g(t) = 0$

$(t - 2)^2(2t + 3) = 0$ Since this is already factored, we can break it apart:
$(t - 2)^2 = 0$
$t - 2 = 0$ or
$t = 2$
$(2t + 3) = 0$
$t = \frac{-3}{2}$

We can always check our answers are reasonable by graphing the polynomial.

Example 9.2E: Finding Horizontal Intercepts of Non-factorial Polynomial

Find the horizontal intercepts of $h(t) = t^3 + 4t^2 + t - 6$
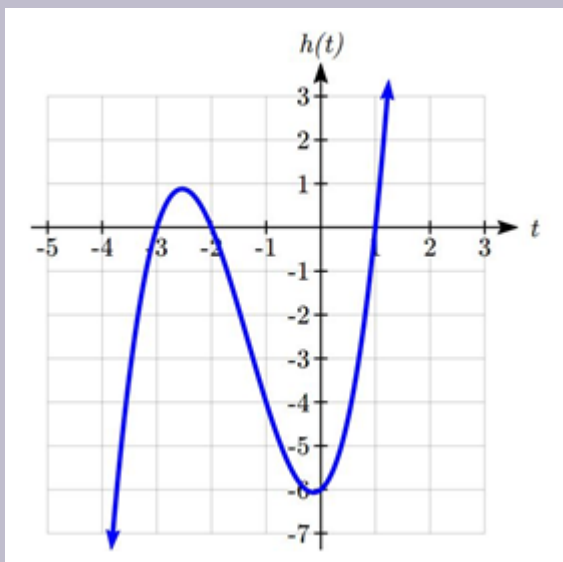
Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

Graphing this function, it appears there are horizontal intercepts at t = -3, -2, and 1.

We could check these are correct by plugging in these values for t and verifying that $h(-3) = h(-2) = h(1) = 0$.

Notice that the polynomial in the previous example was degree three, and had three horizontal intercepts and two turning points – places where the graph changes direction. We will now make a general statement without justifying it.

Intercepts and Turning Points of Polynomials

A polynomial of degree *n* will have:

At most *n* horizontal intercepts. An odd degree polynomial will always have at least one.

At most *n*-1 turning points

Give It Some Thought

1. Find the vertical and horizontal intercepts of the function $f(t) = t^4 - 4t^2$.

Graphical Behavior at Intercepts

If we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$, notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept *x* = -3, coming from the x+3 factor of the polynomial, the graph passes directly through the horizontal intercept. The factor is linear (has a power of 1), so the behavior near the intercept is like that of a line – it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.

At the horizontal intercept *x* = 2, coming from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off of the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated twice, so we call this a double zero. We could also say the zero has **multiplicity** 2.

At the horizontal intercept *x* = -1, coming from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the toolkit has. We call this a triple zero. We could also say the zero has multiplicity 3.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

Graphical Behavior of Polynomials at Horizontal Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the horizontal intercept h is determined by the power on the factor.

$p = 1$

Single zero
Multiplicity 1

$p = 2$

Double zero
Multiplicity 2

$p = 3$

Triple zero
Multiplicity 3
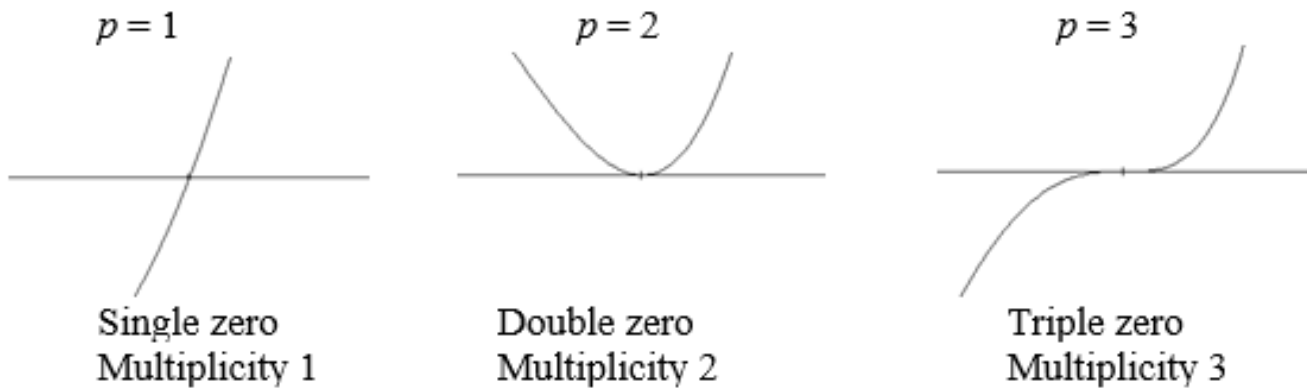
For higher even powers 4,6,8 etc.... the graph will still bounce off of the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc... the graph will still pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

Example 9.2F: Sketching Graph of Polynomials at Horizontal Intercepts

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

This graph has two horizontal intercepts. At x = -3, the factor is squared, indicating the graph will bounce at this horizontal intercept. At x = 5, the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:

As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

At (-3, 0) the graph bounces off of the horizontal axis and so the function must start increasing.

At (0, 90) the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at (5,0).

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4th quadrant.

Using technology we can verify that the resulting graph will look like:
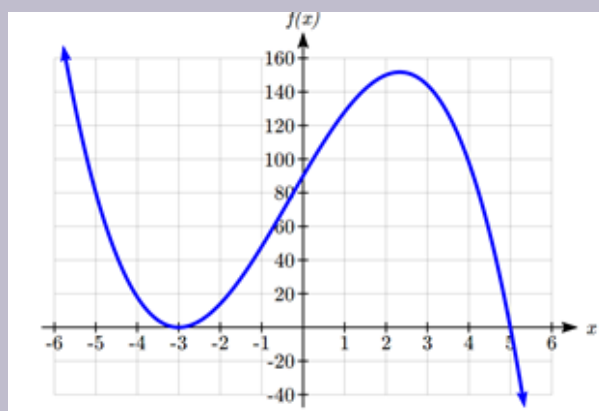
Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

Example 9.2G: Determining Positive Functions

Solve $(x + 3)(x + 1)^2(x - 4) > 0$

As with all inequalities, we start by solving the equality $(x + 3)(x + 1)^2(x - 4) = 0$, which has

solutions at x = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x) = (x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	$f($ test value$)$	>0 or <0?
$x < -3$	-4	72	> 0
$-3 < x < -1$	-2	-6	< 0
$-1 < x < 4$	0	-12	< 0
$x > 4$	5	288	> 0

On a number line this would look like:
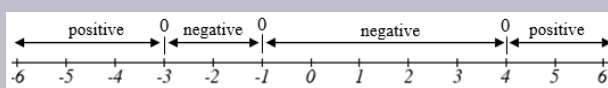
From our test values, we can determine this function is positive when $x < -3$ or $x > 4$, or in interval notation, $(-\infty, -3) \cup (4, \infty)$

We could have also determined on which intervals the function was positive by sketching a graph of the function. We illustrate that technique in the next example.

Example 9.2H: Finding Domain of a Function

Find the domain of the function $v(t) = \sqrt{6 - 5t - t^2}$.

A square root is only defined when the quantity we are taking the square root of, the quantity inside the square root, is zero or greater. Thus, the domain of this function will be when $6 - 5t - t^2 \geq 0$
.
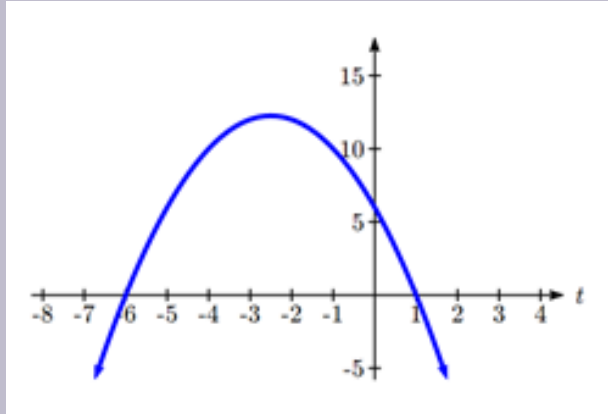
Again we start by solving the equality $6 - 5t - t^2 = 0$. While we could use the quadratic formula, this equation factors nicely to $(6 + t)(1 - t) = 0$, giving horizontal intercepts t = 1 and t = -6. Sketching a graph of this quadratic will allow us to determine when it is positive.

From the graph we can see this function is positive for inputs between the intercepts. So $6 - 5t - t^2 \geq 0$ for $-6 \leq t \leq 1$, and this will be the domain of the *v(t)* function.

Give it Some Thought

2. Given the function $g(x) = x^3 - x^2 - 6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Example 9.2I: Estimating Extrema

An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, *w*.
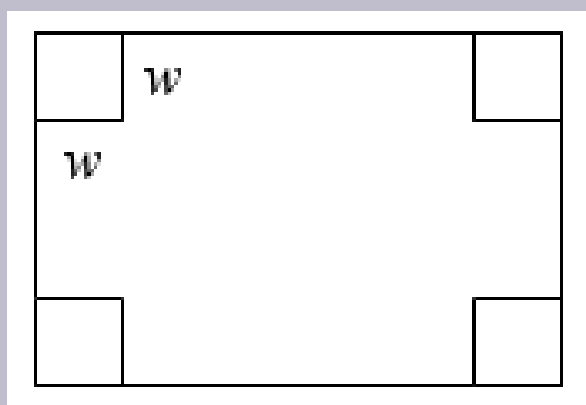
Notice that after a square is cut out from each end, it leaves a (14-2*w*) cm by (20-2*w*) cm rectangle for the base of the box, and the box will be *w* cm tall. This gives the volume:

$$V(w) = (14 - 2w)(20 - 2w)w = 280w - 68w^2 + 4w^3$$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for *w*: values from 0 to 7.
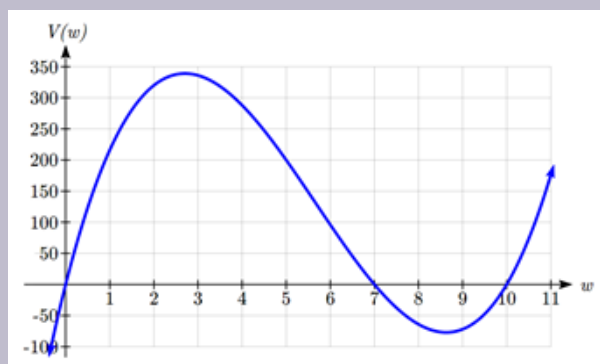
From this graph, we can estimate the maximum value is around 340, and occurs when the squares

are about 2.75cm square. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph.
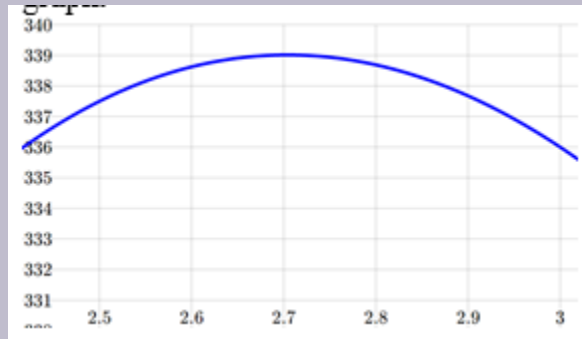
From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

Give It Some Thought

3. Use technology to find the maximum and minimum values on the interval [-1, 4] of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Give It Some Thought Answers

1. Vertical intercept (0, 0), Horizontal intercepts (0, 0), (-2, 0), (2, 0)

2. Vertical intercept (0, 0), Horizontal intercepts (-2, 0), (0, 0), (3, 0)

The function is negative on $(-x^p\%, -2)$ and (0, 3)

The function is positive on (-2, 0) and $(3, x^p\%)$

The leading term is so as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$
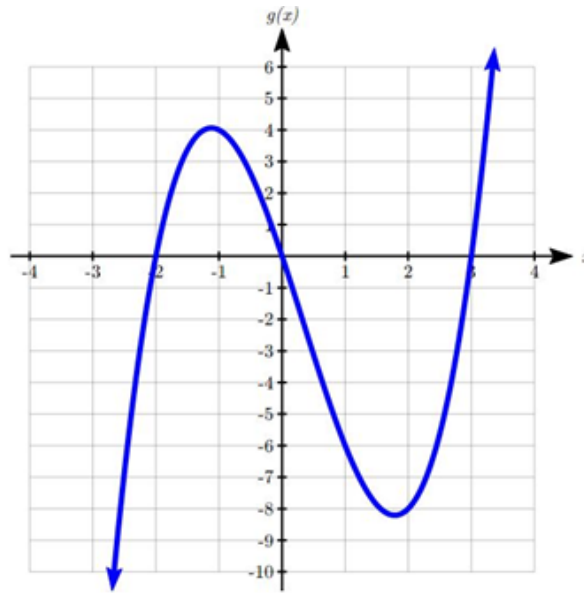
3. The minimum occurs at approximately the point (0, -6.5), and the maximum occurs at approximately the point (3.5, 7).

Section 7.6 Exercises

Insert review exercises here

9.3 Rational Functions

We will begin this section by looking at another economic concept. In addition to total cost and marginal cost, another metric of costs is **average cost**.

Average Cost

The **average cost** of producing q items with a total cost of C(q) is $AC(q) = \frac{C(q)}{q}$

Example 9.3A: Finding Average Cost

Suppose that manufacturing an item has fixed costs of $4000 and per-item (marginal) costs of $20 per item. Find the average cost per item when 250 items are produced.

The total cost function is $C(q) = 4000 + 20q$.

Then average cost is $AC(q) = \frac{4000 + 20q}{q}$.

When 250 items are produced, the average cost per item will be

$$AC(250) = \frac{4000 + 20(250)}{250} = 36$$

The average cost per item is $36.

Notice that the average cost per item distributes the fixed costs across all the items. If we were to graph the average cost, notice that average costs decrease as production increases, eventually levelling off towards the per-item cost of $20.
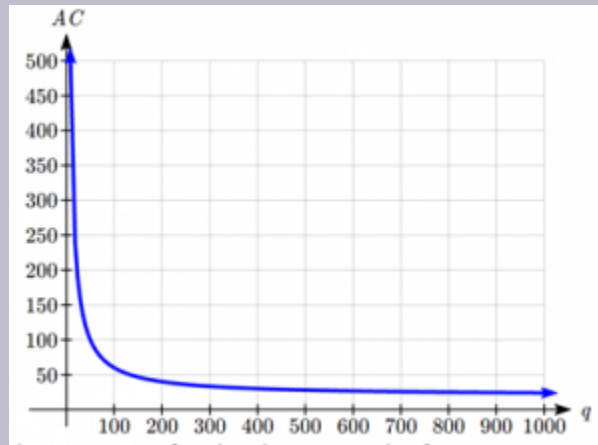
The average cost functions is one example of a **rational function**.

Rational Functions

A **rational function** is a function that can be written as the ratio of two polynomials, *P(x)* and *Q(x)*.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p}{b_0 + b_1 x + b_2 x^2 + \cdots + b_q x^q}$$

To get a feel for these functions, we will look at a graph.

Example 9.3B: Graphing Rational Function

Graph $f(x) = \frac{2x+4}{x-3}$

Notice that the graph is undefined at $x = 3$, since that would cause us to divide by 0. We can create a table of values to get a sense for the function.

x	$f(x)$
-5	.75
-3	.333
-2	0
0	-1.333
2	-8
2.9	-98
2.99	-998
3	undefined
3.1	102
4	12
5	7

Notice that as x gets close to 3 from the left, the function values are getting larger and larger in the negative direction. When x is close to 3 from the right, the function values are getting large and positive. We call $x = 3$ a **vertical asymptote** of the function.

When $x = -2$, the function value is 0, giving a **horizontal intercept**. Notice that when $x = -2$ the numerator of the rational function is zero.

To get a better sense of the long run behavior, we can evaluate the function at additional, larger inputs:

x	f(x)
-1000	1.99
-100	1.90
-10	1.23
10	3.43
100	2.10
1000	2.01

Notice that as *x* gets large on both ends, the function approaches 2 but never actually reaches 2, it just "levels off" as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line f(X)=2 as the input becomes very large in the negative and positive directions.

Sketching a graph of this function:
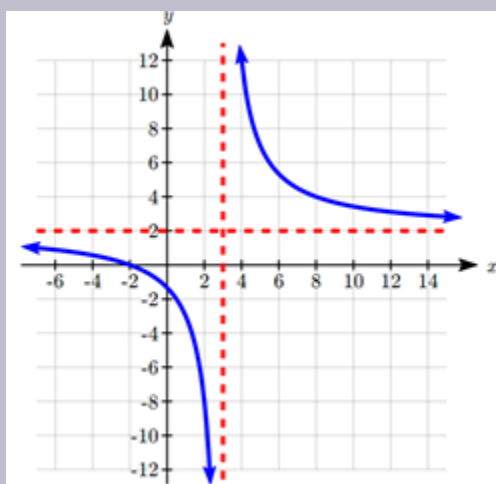
The graphs of rational functions are sometimes sketched with the horizontal and vertical asymptotes drawn as dashed lines. It's important to realize these are *not* part of the graph of the function itself.

Vertical and Horizontal Asymptotes

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach a.

A **horizontal asymptote** of a graph is a horizontal line where the graph approaches the line as the inputs get large.

In the first example looking at average cost, the function had a vertical asymptote at $q=0$, and a horizontal asymptote at $AC=20$.

Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator of the function is undefined. With one exception, a vertical asymptote will occur whenever the denominator is undefined.

Example 9.3C: Finding Verical Asymptotes of a Function

Find the vertical asymptotes of the function $k(x) = \frac{5+2x^2}{2-x-x^2}$
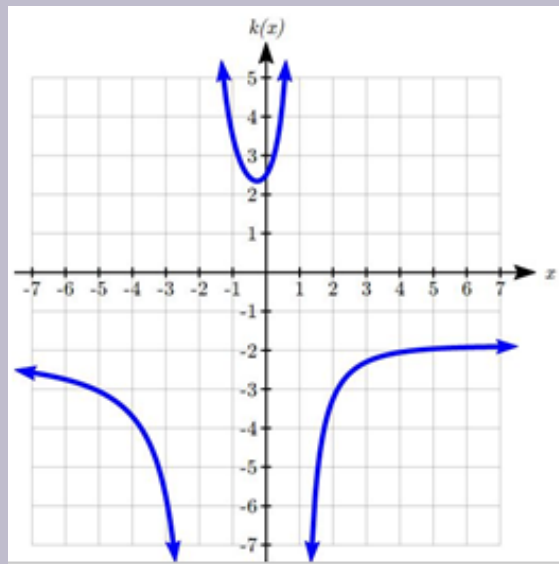
To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$2 - x - x^2 = 0$$
$$(2 + x)(1 - x) = 0$$
$$x = -2,\ 1 </div>$$

This indicates two vertical asymptotes, which a look at a graph confirms.

The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input.

Example 9.3D: Finding Vertical Asymptotes of a Complex Function

Find the vertical asymptotes of the function $k(x) = \frac{x-2}{x^2-4}$.

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = -2,\ 2 </div>$$

However, the numerator of this function is also equal to zero when $x = 2$. Because of this, the function will still be undefined at 2, since $\frac{0}{0}$ is undefined, but the graph will not have a vertical asymptote at $x = 2$.
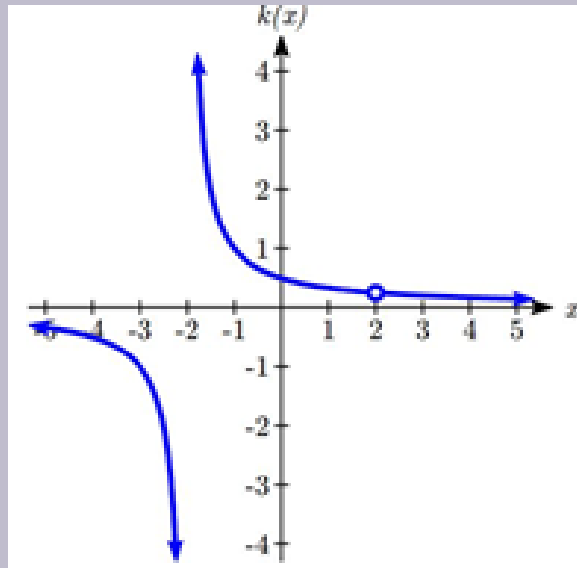
The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole: a single point where the graph is not defined, indicated by an open circle.

Vertical Asymptotes and Holes of Rational Functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. In this case, factor the numerator and denominator and simplify; if the simplified expression still has a zero in the denominator at the original input the original function has a vertical asymptote at the input, otherwise it has a hole.

To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

Case 1: The degree of the denominator > degree of the numerator

Example: $f(x) = \frac{3x+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x}{x^2} = \frac{3}{x}$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x) = \frac{3}{x}$. As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at y=0.

Case 2: The degree of the denominator < degree of the numerator

Example: $f(x) = \frac{3x^2+2}{x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x^2}{x} = 3x$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.

Ultimately, if the numerator is larger than the denominator, the long run behavior of the graph will mimic the behavior of the reduced long run behavior fraction. As another example if we had the function $f(x) = \frac{3x^5-x^2}{x+3}$ with long run behavior $f(x) \approx \frac{3x^5}{x} = 3x^4$, the long run behavior of the graph would look similar to that of an even polynomial.

Case 3: The degree of the denominator = degree of the numerator

Example: $f(x) = \frac{3x^2+2}{x^2+4x-5}$

In this case, the long run behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. This function has a horizontal asymptote at y=3.

Horizontal Asymptote of Rational Functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator > degree of numerator: Horizontal asymptote at y=0

Degree of denominator < degree of numerator: No horizontal asymptote

Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 9.3E: Finding Horizontal and Vertical Asymptotes of the function

Find the horizontal and vertical asymptotes of the function.

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at x = 1, -2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at y=0.

Give It Some Thought

1. Find the vertical and horizontal asymptotes of the function.

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Example 9.3F: Determining Market Share

Nadi Products, an Ebay retailer that resells items purchased from Fiji, has started selling a new-to-market product. Nadi is currently selling 2,000 units a month, which has been growing by 100 units a month. The total market for the product is currently 10,000 units, but is growing by 1000 units a month.

a) Determine Nadi's current market share.

b) Find the horizontal asymptote and interpret it in context of the scenario.

Notice that the sales per month are increasing linearly. Let t be the number of months, then Nadi's sales per month are $N = 2000 + 100t$, and the total market is $M = 10000 + 1000t$.

Nadi's market share, *MS*, the fraction of the total market that Nadi holds:

$$MS(t) = \frac{2000+100t}{10000+1000t}$$

a) Nadi's current market share is:

$$MS(0) = \frac{2000+100(0)}{10000+1000(0)} = \frac{2000}{10000} = 0.20 = 20\%$$

b) Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is $100t$, with coefficient 100. In the denominator, the leading term is $1000t$, with coefficient 1000. The horizontal asymptote will be at the ratio of these values, at $y = \frac{100}{1000} = 0.10$.

This tells us that as the input gets large, the output values will approach 0.10. In context, this means that as more time goes by, Nadi's market share will approach 10%.
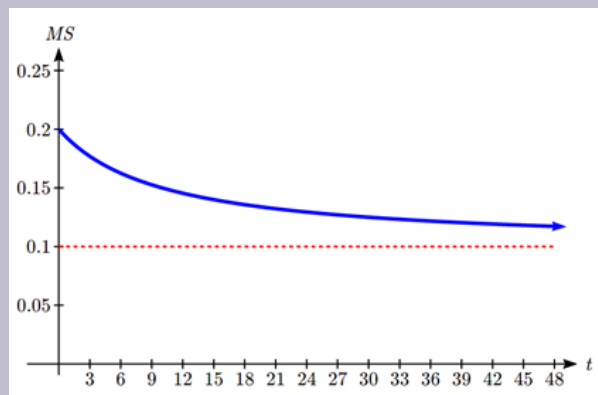
Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is

defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

Example 9.3G: Finding Inercepts by Evaluating the Function at Zero

Find the intercepts of $f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

We can find the vertical intercept by evaluating the function at zero

$f(0) = \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} = \frac{-6}{10} = -\frac{3}{5}$

The horizontal intercepts will occur when the function is equal to zero:

$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$ This is zero when the numerator is zero

$0 = (x-2)(x+3)$
$x = 2, -3 < /div >$ %

Example 9.3H: Sketching Graph by Evaluating the Function at Zero

Sketch a graph of $f(x) = \frac{6(x-1)}{(x+2)(x-3)}$.

We can start our sketch by finding intercepts and asymptotes. Evaluating the function at zero gives the vertical intercept:

$f(0) = \frac{6(0-1)}{(0+2)(0-3)} = \frac{-6}{-6} = 1$

Looking at when the numerator of the function is zero, we can determine the graph will have a horizontal intercept at $x = 1$.

Looking at when the denominator of the function is zero, we can determine the graph will have vertical asymptotes at $x = -2$ and $x = 3$.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.

To sketch the graph, we might start by plotting the intercepts and drawing the asymptotes as dashed lines.

We can now try to sketch a smooth curve passing through both intercepts. As the graph approaches $x=3$, the graph must approach negative infinity. As we sketch from $x=0$ towards $x=-2$, we'd expect the graph to approach positive infinity.
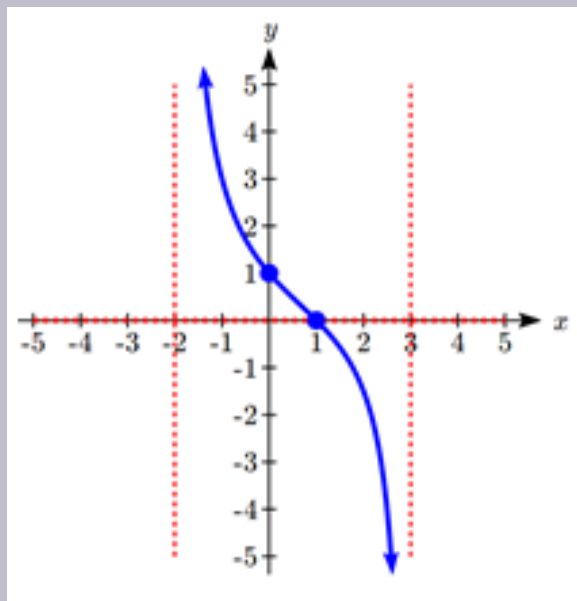
To finish the curve, we need to fill in the portions to the right of $x=3$ and to the left of $x=-2$. If we evaluate the function to the right of $x=3$ we'd see the output is positive, indicating the graph will decrease from positive infinity then level off towards the horizontal asymptote of $y=0$. To the left of $x=-2$ the graph will also level off at the horizontal asymptote, and will approach negative infinity to the left of the asymptote.
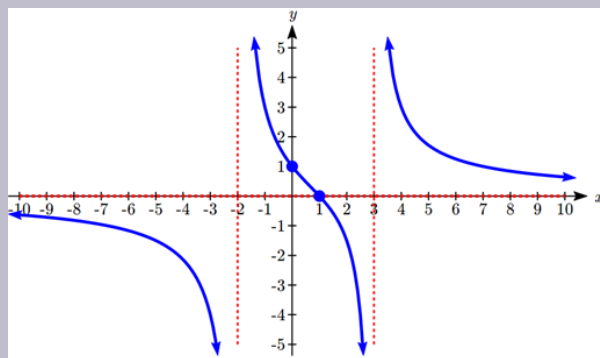
Give It Some Thought

2. Given the function $f(x) = \frac{2(x+2)}{x-1}$, find the intercepts and asymptotes and sketch the function.

Give It Some Thought Answers

1. Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$

2. Horizontal asymptote at $y = 2$.

Vertical asymptote is at $x = 1$.

Vertical intercept at $(0, 4)$.

Horizontal intercept $(2, 0)$.
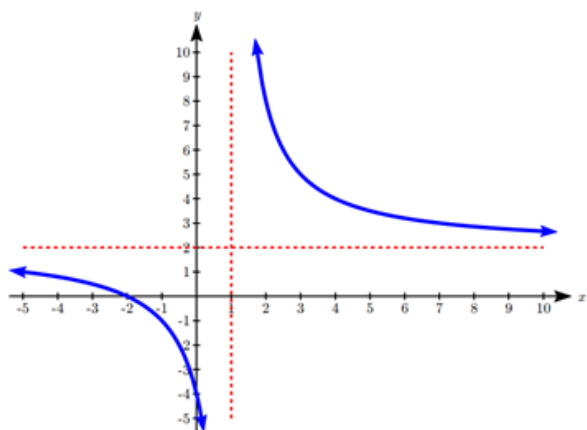
Key Takeaways

Section 9.1: Quadratic Functions

- Forms f Quadratic Functions
- Finding the Vertex of a Quadratic
- Maximizing Revenue or Profit
- Quadratic Formula

Section 9.2: Domain and Range

- Characteristics of Power Functions
- Polynomials
- Graphical Behavior at Intercepts
- Solving Polynomials Inequalities
- Estimating Extrema

Section 9.3: Rates of Change and Behaviour of Graphs

- Rational Functions
- Vertical and Horizontal Asymptotes
- Vertical Asymptotes and Holes of Rational Functions
- Horizontal Asymptote of Rational Functions
- Intercepts

10.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

10.1 Exponential Functions

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year[1]. We might ask if we can find a formula to model the population, P, as a function of time, t, in years after 2008, if the population continues to grow at this rate.

In linear growth, we had a constant rate of change – a constant *number* that the output increased for each increase in input. For example, in the equation $f(x) = 3x + 4\%$, the slope tells us the output increases by three each time the input increases by one. This population scenario is different – we have a *percent* rate of change rather than a constant number of people as our rate of change. To see the significance of this difference consider these two companies:

Company *A* has 100 stores, and expands by opening 50 new stores a year

Company *B* has 100 stores, and expands by increasing the number of stores by 50% of their total each year.

Looking at a few years of growth for these companies:

Year	Stores, company A		Stores, company B
0	100	Starting with 100 each	100
1	100 + 50 = 150	They both grow by 50 stores in the first year.	100 + 50% of 100 100 + 0.50(100) = 150
2	150 + 50 = 200	Store A grows by 50, Store B grows by 75	150 + 50% of 150 150 + 0.50(150) = 225
3	200 + 50 = 250	Store A grows by 50, Store B grows by 112.5	225 + 50% of 225 225 + 0.50(225) = 337.5

Notice that with the percent growth, each year the company is grows by 50% of the current year's total, so as the company grows larger, the number of stores added in a year grows as well.

To try to simplify the calculations, notice that after 1 year the number of stores for company B was:

$$100 + 0.50(100)$$

or equivalently by factoring

$$100(1 + 0.50) = 150$$

We can think of this as "the new number of stores is the original 100% plus another 50%".

After 2 years, the number of stores was:

$$150 + 0.50(150)$$

or equivalently by factoring

$$150(1 + 0.50)$$

now recall the 150 came from 100(1+0.50). Substituting that,

$$100(1 + 0.50)(1 + 0.50) = 100(1 + 0.50)^2 = 225$$

After 3 years, the number of stores was:

$$225 + 0.50(225)$$

or equivalently by factoring

$$225(1 + 0.50)$$

now recall the 225 came from

$$100(1 + 0.50)^2$$

. Substituting that,

$$100(1 + 0.50)^2(1 + 0.50) = 100(1 + 0.50)^3 = 337.5$$

From this, we can generalize, noticing that to show a 50% increase, each year we multiply by a factor of (1+0.50), so after n years, our equation would be

$$B(n) = 100(1 + 0.50)^n$$

In this equation, the 100 represented the initial quantity, and the 0.50 was the percent growth rate. Generalizing further, we arrive at the general form of exponential functions.

Exponential Function

An **exponential growth or decay function** is a function that grows or shrinks at a constant percent growth rate. The equation can be written in the form

$$f(x) = a(1 + r)^x \text{ or } f(x) = ab^x \text{ where } b = 1+r$$

Where

a is the initial or starting value of the function

r is the percent growth or decay rate, written as a decimal

b is the growth factor or growth multiplier. Since powers of negative numbers behave strangely, we limit b to positive values.

To see more clearly the difference between exponential and linear growth, compare the two tables and graphs below, which illustrate the growth of company A and B described above over a longer time frame if the growth patterns were to continue.

years	Company A	Company B
2	200	225
4	300	506
6	400	1139
8	500	2563
10	600	5767
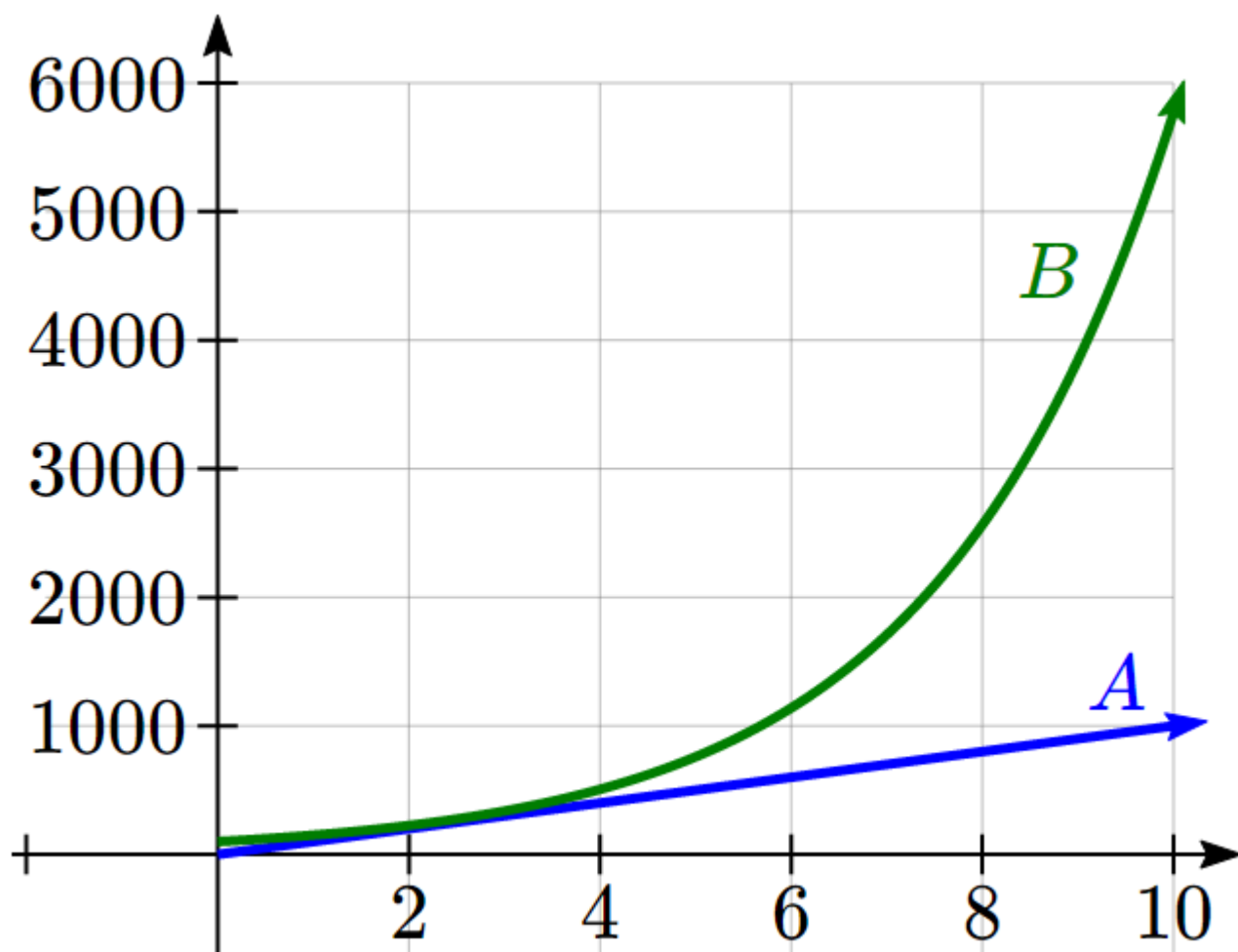
Example 1

Write an exponential function for India's population, and use it to predict the population in 2020.

At the beginning of the chapter we were given India's population of 1.14 billion in the year 2008 and a percent growth rate of 1.34%. Using 2008 as our starting time ($t = 0$), our initial population will be 1.14 billion. Since the percent growth rate was 1.34%, our value for r is 0.0134.

Using the basic formula for exponential growth

$f(x) = a(1 + r)^x$ we can write the formula,

$$f(t) = 1.14(1 + 0.0134)^t$$

To estimate the population in 2020, we evaluate the function at $t = 12$, since 2020 is 12 years after 2008.

$$f(12) = 1.14(1 + 0.0134)^{12} \approx 1.337 \text{ billion people in 2020}$$

Try it Now

1. Given the three statements below, identify which represents exponential functions.

A. The cost of living allowance for state employees increases salaries by 3.1% each year.

B. State employees can expect a $300 raise each year they work for the state.

C. Tuition costs have increased by 2.8% each year for the last 3 years.

Example 2

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. If a bank offers a 24 month CD with an annual interest rate of 1.2% compounded monthly, how much will a $1000 investment grow to over those 24 months?

First, we must notice that the interest rate is an annual rate, but is compounded monthly, meaning interest is calculated and added to the account monthly.

To find the monthly interest rate, we divide the annual rate of 1.2% by 12 since there are 12 months in a year: 1.2%/12 = 0.1%. Each month we will earn 0.1% interest. From this, we can set up an exponential function, with our initial amount of $1000 and a growth rate of $r = 0.001$, and our input m measured in months.

$$f(m) = 1000\left(1 + \tfrac{.012}{12}\right)^m$$

$$f(m) = 1000(1 + 0.001)^m$$

After 24 months, the account will have grown to

$$f(24) = 1000(1 + 0.001)^{24} = \$1024.28$$

$$B(t) = 900(1.075)^t$$

Try it Now

2. Looking at these two equations that represent the balance in two different savings accounts, which account is growing faster, and which account will have a higher balance after 3 years?

$$A(t) = 1000(1.05)^t$$

In all the preceding examples, we saw exponential growth. Exponential functions can also be used to model

quantities that are decreasing at a constant percent rate. An example of this is radioactive decay, a process in which radioactive isotopes of certain atoms transform to an atom of a different type, causing a percentage decrease of the original material over time.

Example 3

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is a = 100 mg, and our growth rate will be negative 13%, since we are decreasing: r = -0.13. This gives the equation:

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be

$$Q(7) = 100(0.87)^7 = 37.73$$

mg of Bismuth-210 remains.

Try it Now

3. A population of 1000 is decreasing 3% each year. Find the population in 30 years.

Example 4

T(q) represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2010,

Interpret all of the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$.

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2010 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. *T*(2) = 231.3056 means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,305 Android smart phone contracts.

Finding Equations of Exponential Functions

In the previous examples, we were able to write equations for exponential functions since we knew the initial quantity and the growth rate. If we do not know the growth rate, but instead know only some input and output pairs of values, we can still construct an exponential function.

Example 5

In 2002, a company had 80 retail stores. By 2008, the company had grown to 180 retail stores. If the company is growing exponentially, find a formula for the function.

By defining our input variable to be *t*, years after 2002, the information listed can be written as two input-output pairs: (0,80) and (6,180). Notice that by choosing our input variable to be measured as years after the first year value provided, we have effectively "given" ourselves the initial value for the function: *a* = 80. This gives us an equation of the form

$$f(t) = 80b^t.$$

Substituting in our second input-output pair allows us to solve for b:

$180 = 80b^6$

Divide by 80

$b^6 = \frac{180}{80} = \frac{9}{4}$

Take the 6th root of both sides.

$b = \sqrt[6]{\frac{9}{4}} = 1.1447$

This gives us our equation for the population:

$f(t) = 80(1.1447)^t$

Recall that since $b = 1+r$, we can interpret this to mean that the population growth rate is $r = 0.1447$, and so the population is growing by about 14.47% each year.

In this example, you could also have used (9/4)^(1/6) to evaluate the 6th root if your calculator doesn't have an nth root button. We chose to use the $f(x) = ab^x$ form of the exponential function rather than the $f(x) = a(1 + r)^x$ form, but this choice was entirely arbitrary – either form would be fine to use.

When finding equations, the value for b or r will usually have to be rounded to be written easily. To preserve accuracy, it is important to not over-round these values. Typically, you want to be sure to preserve at least 3 significant digits in the growth rate. For example, if your value for b was 1.00317643, you would want to round this no further than to 1.00318.

Try it now

4. Find the formula for an exponential passing through (0, 10) and (3, 7).

Graphs of Exponentials

To get a sense for the behavior of exponentials, let us begin by looking more closely at the function $f(x) = 2^x$
.

Listing a table of values for this function:

x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Notice that:

1. This function is positive for all values of x.
2. As x increases, the function grows faster and faster (the rate of change increases)
3. As x decreases, the function values grow smaller, approaching zero.

This is an example of exponential growth.

Looking at the function

$$g(x) = \left(\tfrac{1}{2}\right)^x$$

x	-3	-2	-1	0	1	2	3
$g(x)$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Note this function is also positive for all values of x, but in this case grows as x decreases, and decreases towards zero as x increases. This is an example of exponential decay. You may notice from the table that this function appears to be the horizontal reflection of the $f(x) = 2^x$ table. This is in fact the case:

$$f(-x) = 2^{-x} = (2^{-1})^x = \left(\tfrac{1}{2}\right)^x = g(x)$$

Looking at the graphs also confirms this relationship:
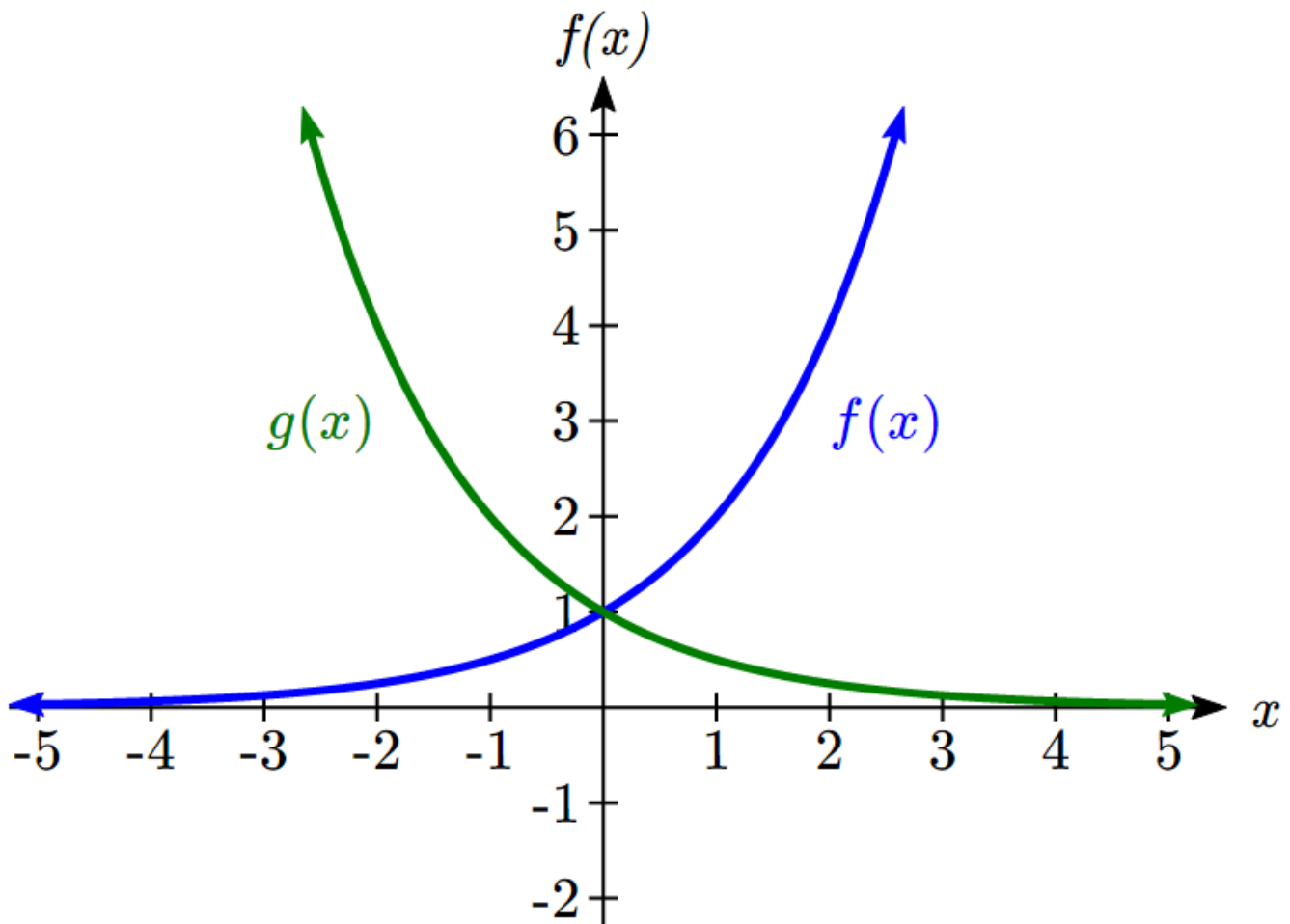
Consider a function for the form $f(x) = ab^x$. Since a, which we called the initial value in the last section, is the function value at an input of zero, a will give us the vertical intercept of the graph. From the graphs above, we can see that an exponential graph will have a horizontal asymptote on one side of the graph, and can either increase or decrease, depending upon the growth factor. This horizontal asymptote will also help us determine the long run behavior and is easy to determine from the graph.

The graph will grow when the growth rate is positive, which will make the growth factor b larger than one. When it's negative, the growth factor will be less than one.

Graphical Features of Exponential Functions

Graphically, in the function $f(x) = ab^x$

a is the vertical intercept of the graph

b determines the rate at which the graph grows. When a is positive,

the function will increase if $b > 1$

the function will decrease if $0 < b < 1$

The graph will have a horizontal asymptote at $y = 0$

The graph will be concave up if $a > 0$; concave down if $a < 0$.

The domain of the function is all real numbers

The range of the function is

$(0, \infty)$

When sketching the graph of an exponential function, it can be helpful to remember that the graph will pass through the points $(0, a)$ and $(1, ab)$.

The value b will determine the function's long run behavior:

If $b > 1$, as , and as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to 0$.

If $0 < b < 1$, as $x \to \infty$, $f(x) \to 0$ and as $x \to -\infty$, $f(x) \to \infty$.

Example 6
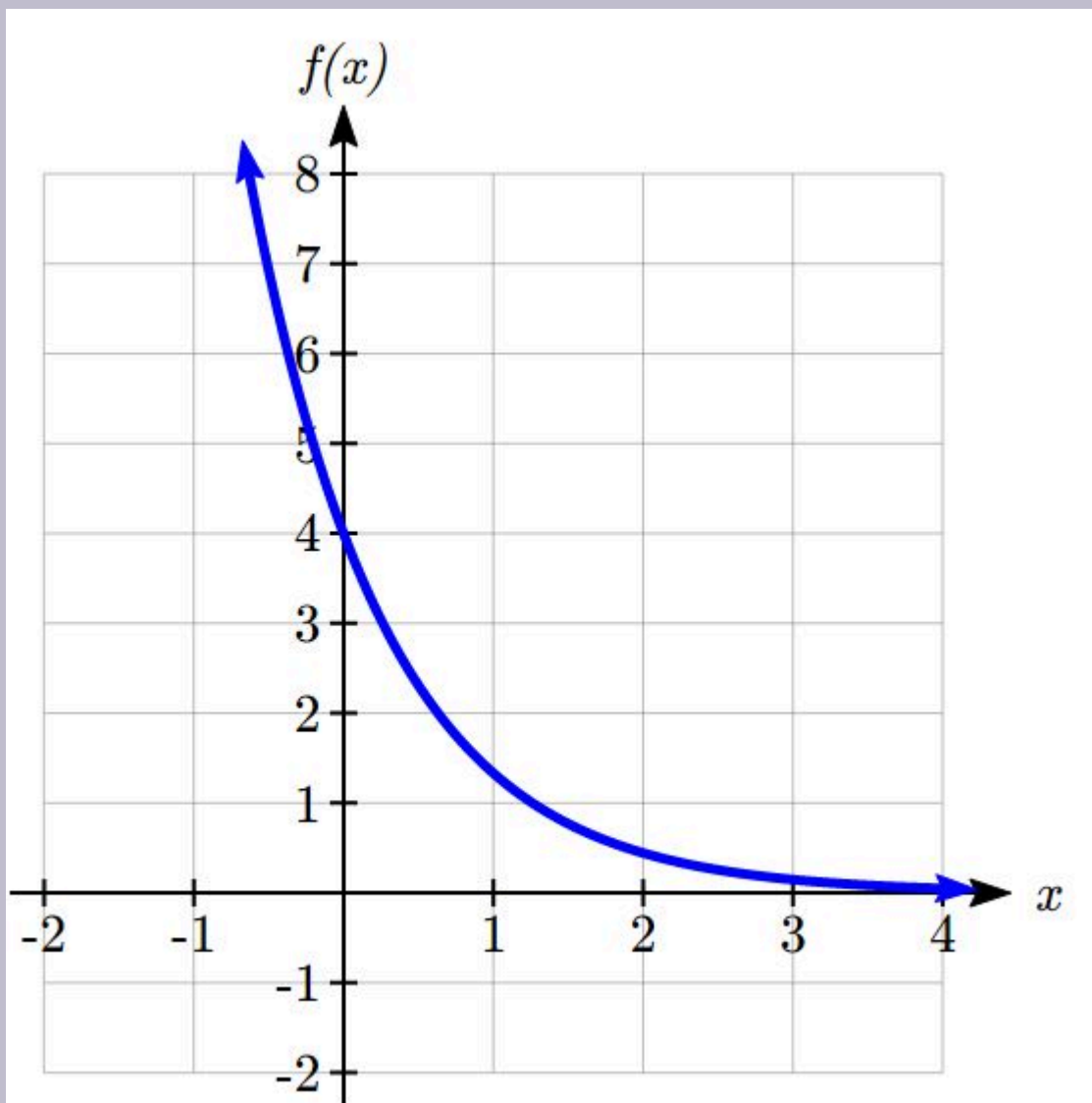
Sketch a graph of $f(x) = 4\left(\frac{1}{3}\right)^x$

This graph will have a vertical intercept at (0,4), and pass through the point $\left(1, \frac{4}{3}\right)$. Since $b < 1$, the graph will be decreasing towards zero. Since $a > 0$, the graph will be concave up.

We can also see from the graph the long run behavior: as , and as , .

To get a better feeling for the effect of a and b on the graph, examine the sets of graphs below. The first set shows various graphs, where a remains the same and we only change the value for b.
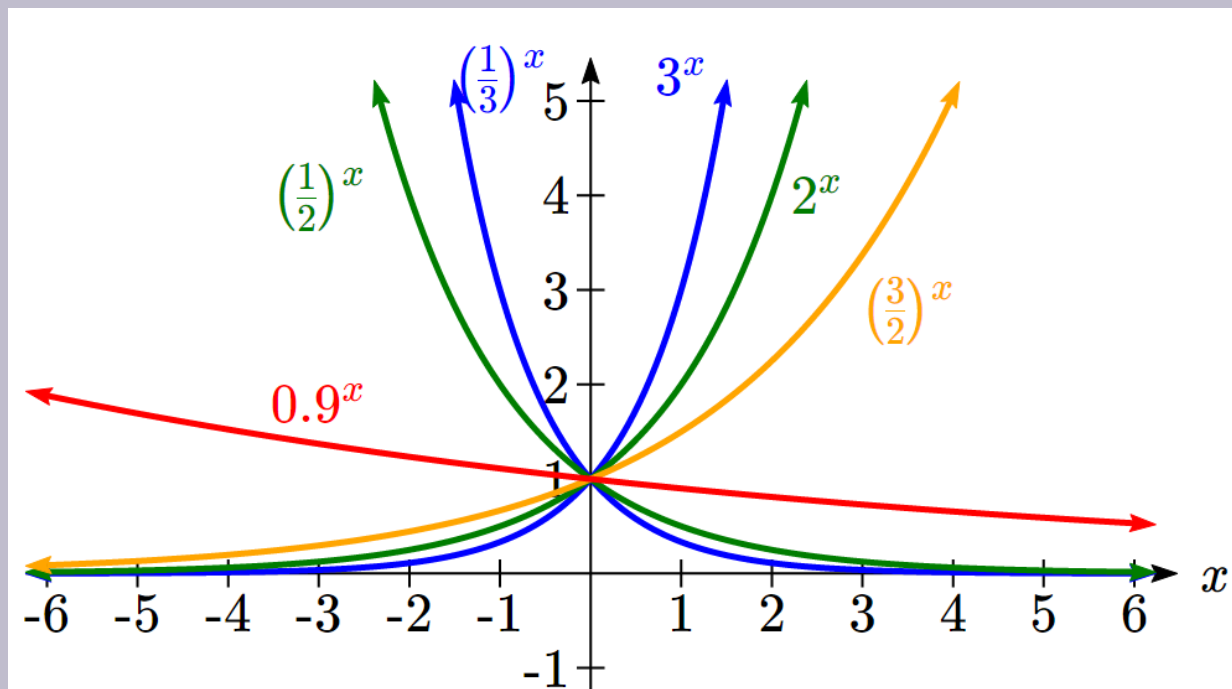
Notice that the closer the value of *b* is to 1, the less steep the graph will be.

In the next set of graphs, *a* is altered and our value for *b* remains the same.
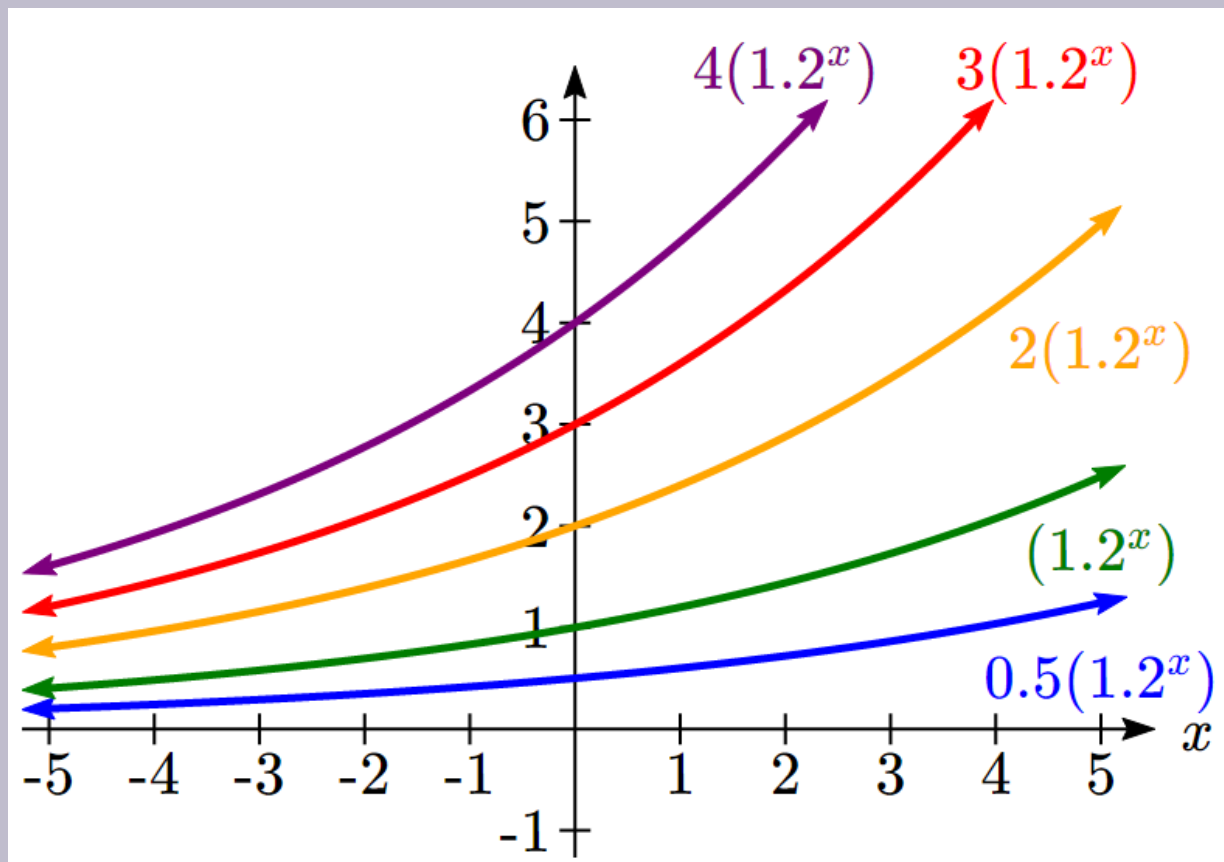
Notice that changing the value for *a* changes the vertical intercept. Since *a* is multiplying the b^x term, *a* acts as a vertical stretch factor, not as a shift. Notice also that the long run behavior for all of these functions is the same because the growth factor did not change and none of these *a* values introduced a vertical flip.

Example 7
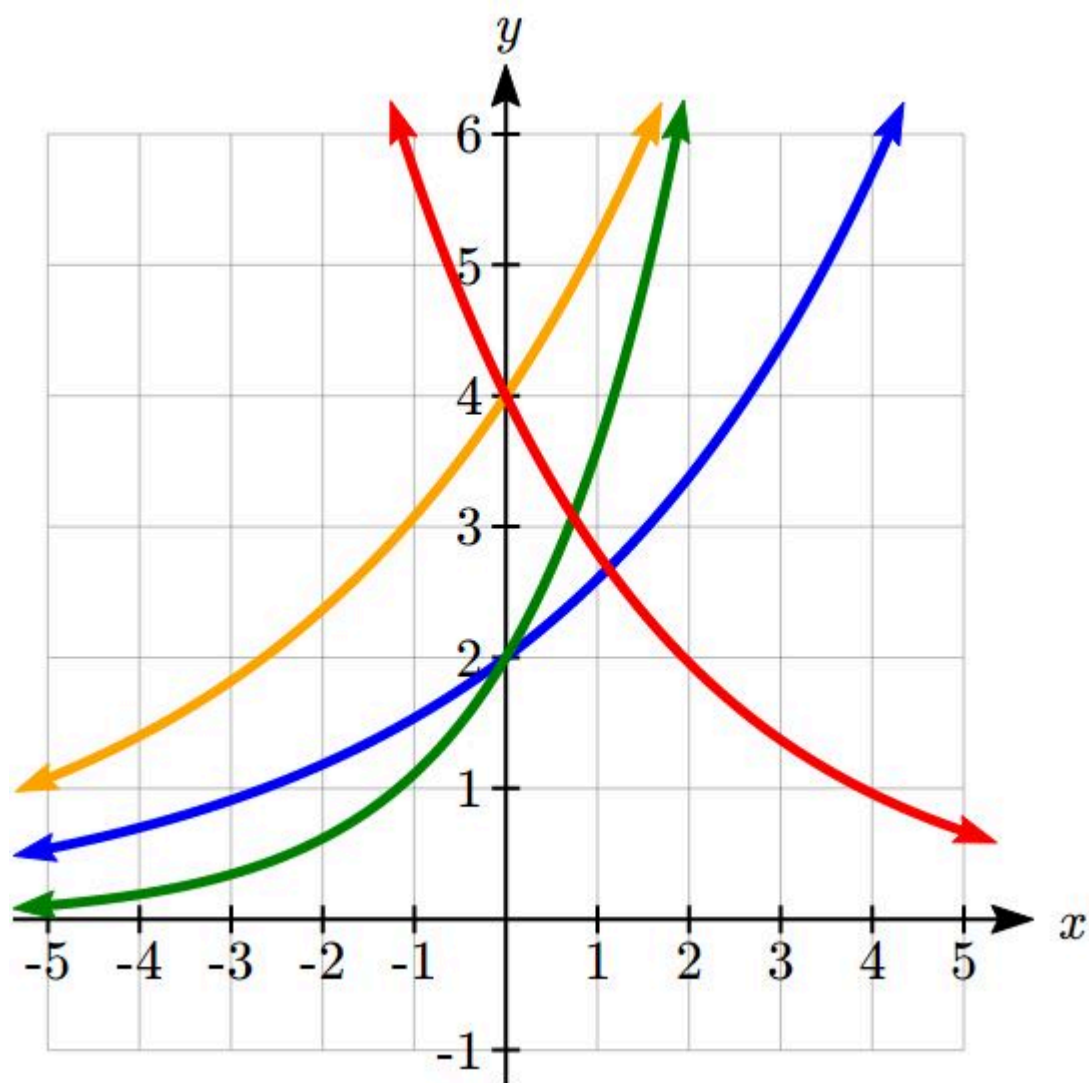
Match each equation with its graph.

$f(x) = 2(1.3)^x$
$g(x) = 2(1.8)^x$
$h(x) = 4(1.3)^x$
$k(x) = 4(0.7)^x </div>$

The graph of *k(x)* is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of *h(x)* can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of *f(x)* and *g(x)* both have a vertical intercept at (0,2), but since *g(x)* has a larger growth factor, we can identify it as the graph increasing faster.
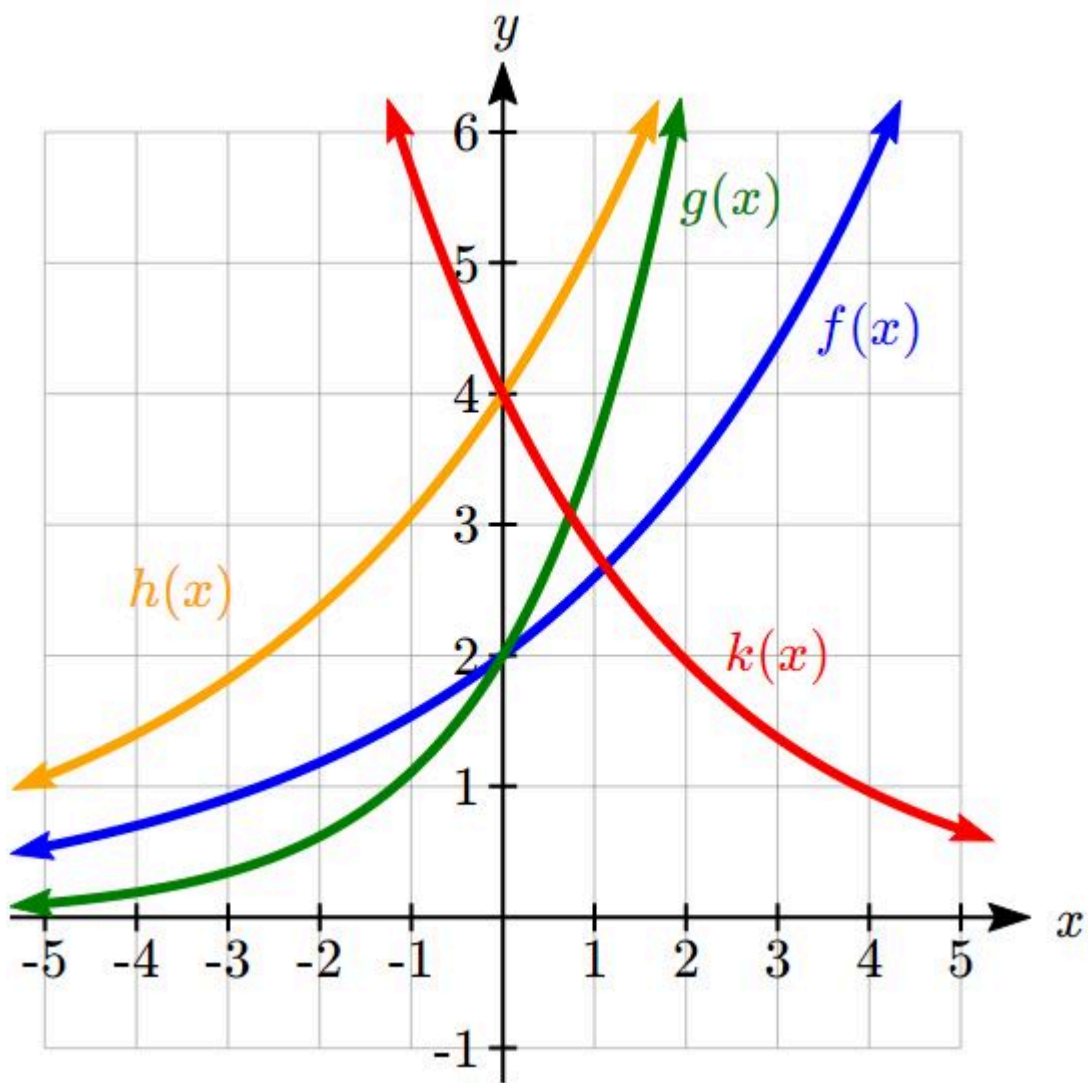
5. Graph the following functions on the same axis: $f(x) = (2)^x$; $g(x) = 2(2)^x$; $h(x) = 2(1/2)^x$.

Compound Interest

In the bank certificate of deposit (CD) example earlier in the section, we encountered compound interest. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the accumulated value of the account. We'll explore compound interest in greater depth in a later chapter, but will introduce basics here.

In the example from earlier, the interest was compounded monthly, so we took the annual interest rate, usually called the **nominal rate** or **annual percentage rate (APR)** and divided by 12, the number of compounds in a year, to find the monthly interest. The exponent was then measured in months.

Generalizing this, we can form a general formula for compound interest. If the APR is written in decimal form as r, and there are k compounding periods per year, then the interest per compounding period will be r/k. Likewise, if we are interested in the value after t years, then there will be kt compounding periods in that time.

Compound Interest Formula

Compound Interest can be calculated using the formula

$$A(t) = a\left(1 + \tfrac{r}{k}\right)^{kt}$$

Where

$A(t)$ is the account value

t is measured in years

a is the starting amount of the account, often called the principal

r is the annual percentage rate (APR), also called the nominal rate

k is the number of compounding periods in one year

Example 8

If you invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Since we are starting with $3000, $a = 3000$

Our interest rate is 3%, so $r = 0.03$

Since we are compounding quarterly, we are compounding 4 times per year, so $k = 4$

We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4(10)} = 4045.05$$

The account will be worth $4045.05 in 10 years.

Example 9

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to $40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Since the account is earning 6%, $r = 0.06$

Since interest is compounded twice a year, $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount needed. We do know we want the end amount to be $40,000, so we will be looking for the value of a so that $A(18) = 40,000$.

$$40,000 = A(18) = a\left(1 + \frac{0.06}{2}\right)^{2(18)}$$
$$40,000 = a(2.8983)$$
$$a = \frac{40,000}{2.8983} \approx 13,801 ¡/div¿$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Try it now

6. Recalculate example 8 from above with monthly compounding.

A Limit to Compounding

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If $1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$11010.16

The amount we earn increases as we increase the compounding frequency. Notice, though, that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of $1 invested at 100% interest for 1 year.

Frequency	Value
Annual	$2
Semiannually	$2.25
Quarterly	$2.441406
Monthly	$2.613035
Daily	$2.714567
Hourly	$2.718127
Once per minute	$2.718279
Once per second	$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how π represents a number.

Euler's Number: e

e is the letter used to represent the value that $\left(1 + \frac{1}{k}\right)^k$ approaches as k gets big.

$$e \approx 2.718282$$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function $exp(x)$, where $exp(x) = e^x$.

Because e arises when the time between compounds becomes very small, e allows us to define **continuous growth** and allows us to define a new toolkit function,

$$f(x) = e^x$$

.

Continuous Growth Formula

Continuous Growth can be calculated using the formula

$$f(x) = ae^{rx}$$

where

a is the starting amount

r is the continuous growth rate

This type of equation is commonly used when describing quantities that change more or less continuously, like chemical reactions, growth of large populations, and radioactive decay. It is also used extensively in calculus.

Example 10

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: *r* = -0.173

$$f(3) = 100e^{-0.173(3)} \approx 59.512 \text{mg of Radon-222 will remain.}$$

Try it Now

7. Interpret the following: if *S(t)* represents the growth of a substance in grams, and time is measured in days.

Continuous growth is also often applied to compound interest, allowing us to talk about continuous compounding.

Example 11

If $1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Here, the continuous growth rate is 10%, so *r* = 0.10. We start with $1000, so *a* = 1000.

To find the value after 1 year,

$$f(1) = 1000e^{0.10(1)} \approx 1105.17$$

Notice this is a $1010.17 increase for the year. As a percent increase, this is

$\frac{105.17}{1000} = 0.10517 = 10.517\%$ increase over the original $1000.

Notice that this value is slightly larger than the amount generated by daily compounding in the table computed earlier.

The continuous growth rate is like the nominal growth rate (or APR) – it reflects the growth rate before compounding takes effect. This is different than the annual growth rate used in the formula $f(x) = a(1 + r)^x$, which is like the annual percentage yield – it reflects the *actual* amount the output grows in a year.

While the continuous growth rate in the example above was 10%, the actual annual yield was 10.517%. This means we could write two different looking but equivalent formulas for this account's growth:

$$f(t) = 1000e^{0.10t}$$

using the 10% continuous growth rate

$$f(t) = 1000(1.10517)^t$$

using the 10.517% actual annual yield rate.

Try it Now Answers

1. A & C are exponential functions, they grow by a % not a constant number.

2. B(t) is growing faster, but after 3 years A(t) still has a higher account balance

3. , or about 401 people.

4.

$$f(x) = 10(0.8879)^x$$

.
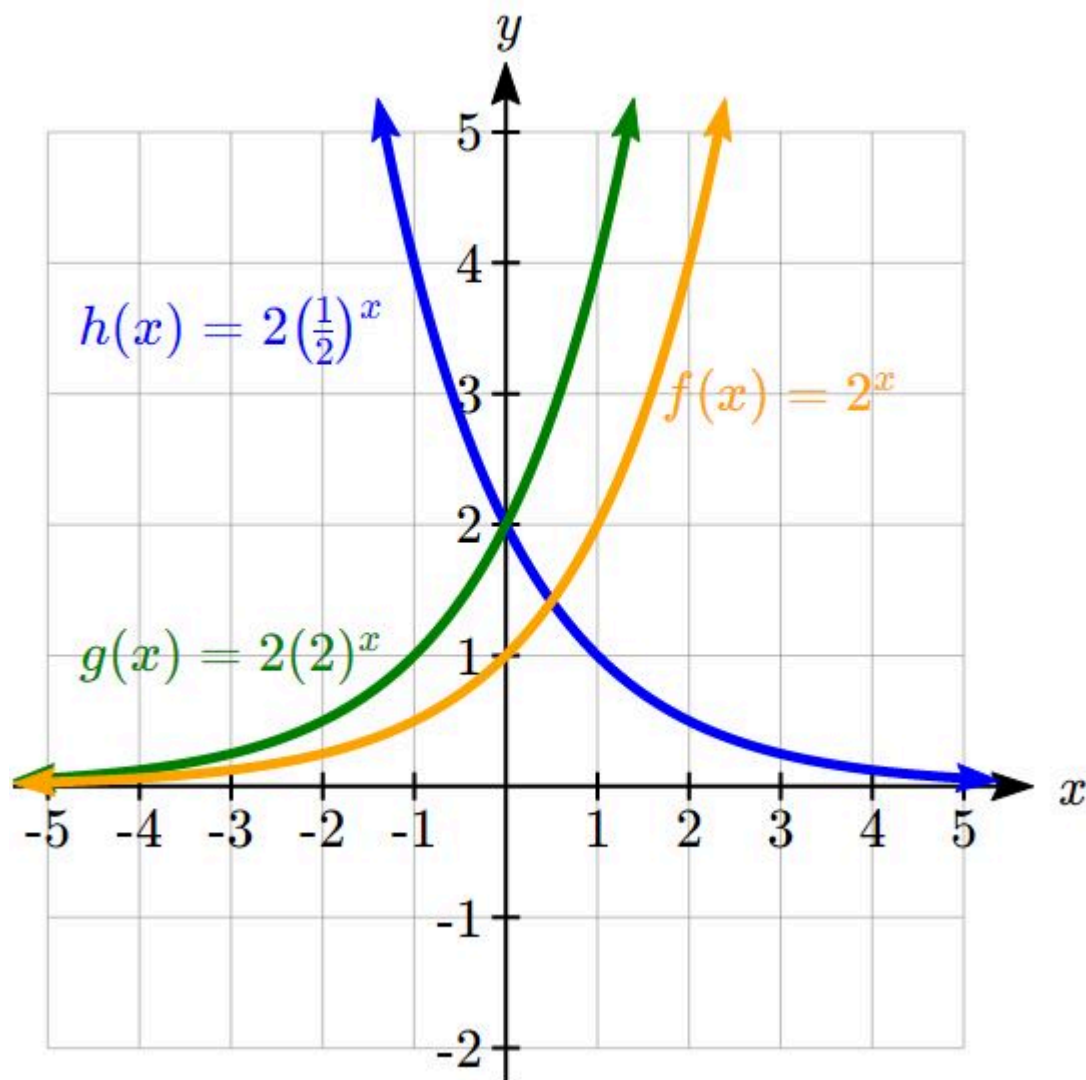
5.

6. $4048.06

7. An initial substance weighing 20g is growing at a continuous rate of 12% per day.

10.2 Logarithmic Functions

A population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where x represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to:

$$500 = 50(2)^x$$

Dividing both sides by 50 to isolate the exponential

$$10 = 2^x$$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$

above. We know that $2^3 = 8$ and $2^4 = 16\%$, so it is clear that x must be some value between 3 and 4 since

$$g(x) = 2^x$$

is increasing. We could use technology to create a table of values or graph to better estimate the solution.
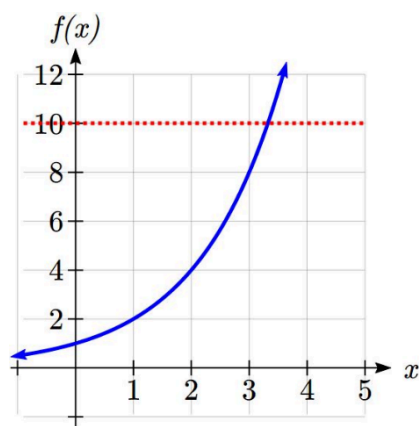
From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, since it would be nice to have a function that gives the answer. None of the functions we have already discussed would work, so we must introduce a new function, named **log**, as the function that "undoes" an exponential function, like how a square root "undoes" a square. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.

Logarithm

The **logarithm** (base b) function, written $\log_b (x)$, "undoes" exponential function

$$b^x$$

.

The statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Since the logarithm and exponential "undo" each other (in technical terms, they are inverses), it follows that:

Properties of Logs: Inverse Properties

$\log_b(b^x) = x$

$b^{\log_b x} = x$

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 1

Write these exponential equations as logarithmic equations:

$2^3 = 8$ $5^2 = 25$ $10^{-4} = \frac{1}{10000}$

$\log_2(8) = 3$ is equivalent to $2^3 = 8$

$5^2 = 25$ is equivalent to $\log_5(25) = 2$

$10^{-4} = \frac{1}{10000}$ is equivalent to $log_{10}\left(\frac{1}{10000}\right) = -4$

Example 2

Write these logarithmic equations as exponential equations:

$$\log_6\left(\sqrt{6}\right) = \tfrac{1}{2} \qquad \log_3\left(9\right) = 2$$

$\log_6\left(\sqrt{6}\right) = \tfrac{1}{2}$ is equivalent to $6^{1/2} = \sqrt{6}$

$\log_3\left(9\right) = 2$ is equivalent to $3^2 = 9$

Try it Now

1. Write the exponential equation $4^2 = 16$ as a logarithmic equation.

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 3

Solve $\log_4\left(x\right) = 2$ for x.

By rewriting this expression as an exponential, $4^2 = x$, so x = 16

Example 4

Solve $2^x = 10$ for x.

By rewriting this expression as a logarithm, we get $x = \log_2(10)$

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$.

The **natural log** is the logarithm with base e, and is typically written $\ln(x)$.

Example 5

Evaluate $\log(1000)$ using the definition of the common log.

Values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

To evaluate $\log(1000)$, we can say $x = \log(1000)$, then rewrite into exponential form using the common log base of 10.

$10^x = 1000$

From this, we might recognize that 1000 is the cube of 10, so x = 3.

We also can use the inverse property of logs to write $\log_{10}\left(10^3\right) = 3$

Try it Now

2. Evaluate $\log(1,000,000)$.

Example 6

Evaluate $\ln\left(\sqrt{e}\right)$.

We can rewrite $\ln\left(\sqrt{e}\right)$ as $\ln\left(e^{1/2}\right)$. Since ln is a log base e, we can use the inverse property for logs: $\ln\left(e^{1/2}\right) = \log_e\left(e^{1/2}\right) = \frac{1}{2}$.

Example 7

Evaluate log(500) using your calculator or computer.

Using a computer, we can evaluate $\log(500) \approx 2.69897$.

Graphs of Logarithms

Recall that the exponential function produces this table of values

| x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Since the logarithmic function "undoes" the exponential, produces the table of values

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

In this second table, notice that

1. As the input increases, the output increases.
2. As input increases, the output increases more slowly.
3. Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0, \infty)$.
4. Since the exponential function can accept all real numbers as inputs, the logarithm can output any real number, so the range is all real numbers or $(-\infty, \infty)$.
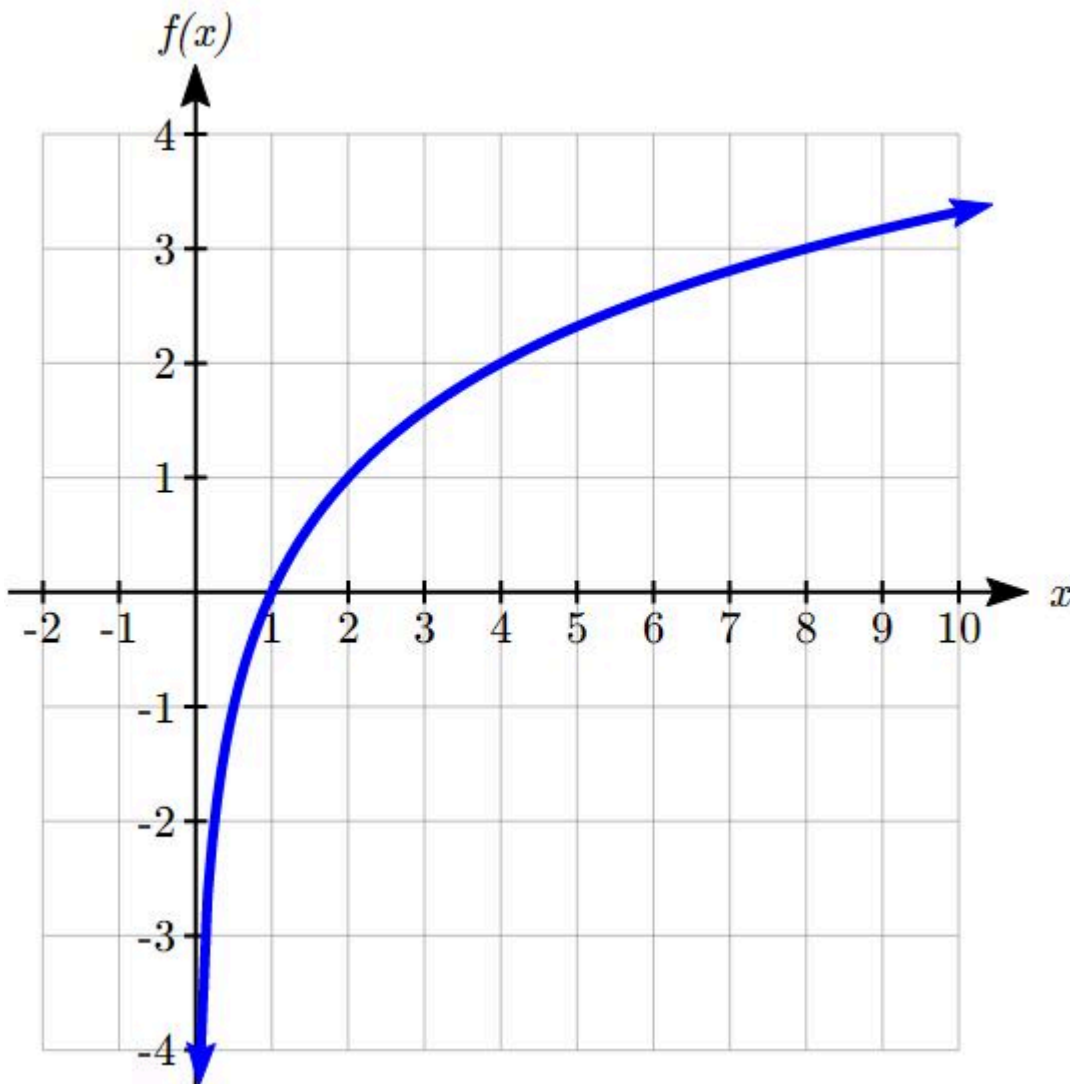
Sketching the graph, notice that as the input approaches zero from the right, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.

In symbolic notation, we write

as $x \to 0^+$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$

Graphical Features of the Logarithm

Graphically, in the function $g(x) = log_b(x)$

The graph has a horizontal intercept at (1, 0)

The graph has a vertical asymptote at $x = 0$
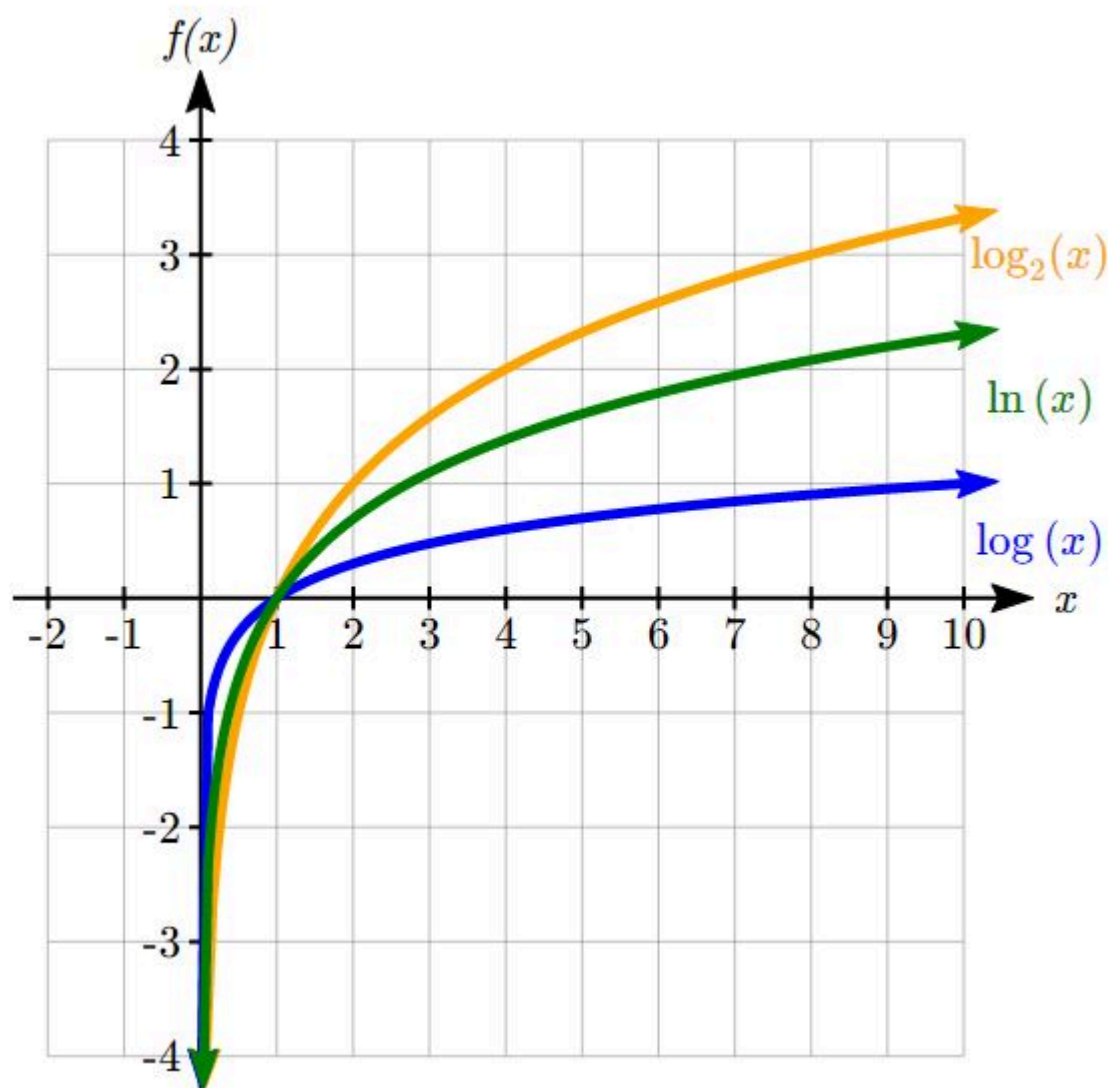
The graph is increasing and concave down

The domain of the function is $x > 0$, or $(0, \infty)$

The range of the function is all real numbers, or $(-\infty, \infty)$

When sketching a general logarithm with base b, it can be helpful to remember that the graph will pass through the points (1, 0) and (b, 1).

To get a feeling for how the base affects the shape of the graph, examine the graphs below.

Notice that the larger the base, the slower the graph grows. For example, the common log graph, while it grows without bound, it does so very slowly. For example, to reach an output of 8, the input must be 100,000,000.

Another important observation made was the domain of the logarithm. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

Example 8

Find the domain of the function $f(x) = \log(5 - 2x)$

The logarithm is only defined with the input is positive, so this function will only be defined when $5 - 2x > 0$.

$-2x > -5$
$x < \frac{5}{2} < /div >$ % Solving this inequality,

The domain of this function is $x < \frac{5}{2}$, or in interval notation, $\left(-\infty, \frac{5}{2}\right)$.

Logarithm Properties

To utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need to establish some additional properties.

Properties of Logs: Exponent Property

$$\log_b\left(A^r\right) = r\log_b\left(A\right)$$

To show why this is true, we offer a proof.

Since the logarithmic and exponential functions are inverses,

$b^{\log_b A} = A$ %.

So $A^r = \left(b^{\log_b A}\right)^r$

Utilizing the exponential rule that states $\left(x^p\right)^q = x^{pq}$, $A^r = \left(b^{\log_b A}\right)^r = b^{r\log_b A}$

So then $\log_b\left(A^r\right) = \log_b\left(b^{r\log_b A}\right)$

Again utilizing the inverse property on the right side yields the result $\log_b\left(A^r\right) = r\log_b A$

Example 9

Rewrite $\log_3 (25)$ using the exponent property for logs.

Since 25 = 5^2, $\log_3 (25) = \log_3 (5^2) = 2\log_3 5$

Example 10

Rewrite $4 \ln(x)$ using the exponent property for logs.

Using the property in reverse, $4 \ln(x) = \ln (x^4)$

Try it Now

3. Rewrite using the exponent property for logs: $\ln \left(\frac{1}{x^2} \right)$.

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base

$\log_b (A) = \frac{\log_c (A)}{\log_c (b)}$

Proof:

Let $\log_b (A) = x$. Rewriting as an exponential gives $b^x = A$. Taking the log base c of both sides of this equation gives $\log_c b^x = \log_c A$

Now utilizing the exponent property for logs on the left side, $x\log_c b = \log_c A$

Dividing, we obtain $x = \frac{\log_c A}{\log_c b}$

or replacing our expression for x, $\log_b A = \dfrac{\log_c A}{\log_c b}$

With this change of base formula, we can finally find a good decimal approximation to our question from the beginning of the section.

Example 11

Evaluate $\log_2(10)$ using the change of base formula.

According to the change of base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e:

$$\log_2 10 = \frac{\log_e 10}{\log_e 2} = \frac{\ln 10}{\ln 2}$$

Using our calculators to evaluate this, $\dfrac{\ln 10}{\ln 2} \approx \dfrac{2.30259}{0.69315} \approx 3.3219$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500

.

Example 12

Evaluate $\log_5(100)$ using the change of base formula.

We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10} 100}{\log_{10} 5} \approx \frac{2}{0.69897} = 2.861$$

While we were able to solve the basic exponential equation $2^x = 10$ by rewriting in logarithmic form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

Solving exponential equations:

Isolate the exponential expressions when possible

Take the logarithm of both sides

Utilize the exponent property for logarithms to pull the variable out of the exponent

Use algebra to solve for the variable.

Example 13

Solve $2^x = 10$ for x.

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$$\ln\left(2^x\right) = \ln(10)$$

Utilizing the exponent property for logs, $x \ln\left(2\right) = \ln(10)$

Now dividing by ln(2), $x = \frac{ln(10)}{ln(2)} \approx 2.861$

Notice that this result matches the result we found using the change of base formula.

Example 14

In the first section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1 + 0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

We need to solve for the t so that $f(t) = 2$

$$2 = 1.14(1.0134)^t$$

Divide by 1.14 to isolate the exponential expression

$$\frac{2}{1.14} = 1.0134^t$$

Take the logarithm of both sides of the equation

$$\ln\left(\frac{2}{1.14}\right) = \ln\left(1.0134^t\right)$$

Apply the exponent property on the right side

$$\ln\left(\frac{2}{1.14}\right) = t \ln\left(1.0134\right)$$

Divide both sides by ln(1.0134)

$$t = \frac{\ln\left(\frac{2}{1.14}\right)}{\ln(1.0134)} \approx 42.23$$

years

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Try it now

4. Solve $5(0.93)^x = 10$.

Additional Properties of Logarithms

Some situations cannot be addressed using the properties already discussed. For these, we need some additional properties.

Properties of Logs

Sum of Logs Property: $\log_b(A) + \log_b(C) = \log_b(AC)$

Difference of Logs Property: $\log_b(A) - \log_b(C) = \log_b\left(\frac{A}{C}\right)$

It's just as important to know what properties logarithms do *not* satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute: log(*A* + *B*) ≠ log(*A*) + log(*B*).

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 15

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Using the sum of logs property on the first two terms,

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to $\log_3(40) - \log_3(2)$

Then using the difference of logs property,

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Example 16

Evaluate $2 \log (5) + \log (4)$ without a calculator by first rewriting as a single logarithm.

On the first term, we can use the exponent property of logs to write $2 \log (5) = \log (5^2) = \log (25)$

With the expression reduced to a sum of two logs, $\log (25) + \log (4)$, we can utilize the sum of logs property $\log (25) + \log (4) = \log(4 \cdot 25) = \log(100)$

Since 100 = 10², we can evaluate this log without a calculator:

$$\log(100) = \log (10^2) = 2$$

Example 17

Rewrite $\ln \left(\frac{x^4 y}{7} \right)$ as a sum or difference of logs

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write
$\ln \left(\frac{x^4 y}{7} \right) = \ln (x^4 y) - \ln(7)$

Then seeing the product in the first term, we use the sum property
$\ln (x^4 y) - \ln(7) = \ln (x^4) + \ln(y) - \ln(7)$

Finally, we could use the exponent property on the first term
$\ln (x^4) + \ln(y) - \ln(7) = 4 \ln(x) + \ln(y) - \ln(7)$

Try it Now

5. Without a calculator evaluate by first rewriting as a single logarithm $\log_2 (8) + \log_2 (4)$:

Log properties in solving equations

The logarithm properties often arise when solving problems involving logarithms.

Example 18

Solve $\log(50x + 25) - \log(x) = 2$.

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$\log\left(\frac{50x+25}{x}\right) = 2$

Rewriting in exponential form reduces this to an algebraic equation:

$\frac{50x+25}{x} = 10^2 = 100$

Solving,

$50x + 25 = 100x$
$25 = 50x$
$x = \frac{25}{50} = \frac{1}{2} < /div >$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

Try it Now

6. Solve $\log(x^2 - 4) = 1 + \log(x + 2)$.

Try it Now Answers

1.

2. 6

3.

Key Takeaways

4.

5. 5

6. 12

10.3 Exponential and Logarithmic Models

While we have explored some basic applications of exponential and logarithmic functions, in this section we explore some important applications in more depth.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example 1a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year[2]. If these trends continue, when will the population of Kenya match that of Sudan?

We start by writing an equation for each population in terms of t, the number of years after 2008.

$$Kenya(t) = 38.8(1 + 0.0264)^t$$
$$Sudan(t) = 41.3(1 + 0.0224)^t </div>$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t = 41.3(1.0224)^t$$

For our first approach, we take the log of both sides of the equation

$$log\left(38.8(1.0264)^t\right) = log\left(41.3(1.0224)^t\right)$$

Utilizing the sum property of logs, we can rewrite each side,

$$log(38.8) + log\left(1.0264^t\right) = log(41.3) + log\left(1.0224^t\right)$$

Then utilizing the exponent property, we can pull the variables out of the exponent
$$log(38.8) + tlog\left(1.0264\right) = log(41.3) + tlog\left(1.0224\right)$$

Moving all the terms involving t to one side of the equation and the rest of the terms to the other side,
$$tlog\left(1.0264\right) - tlog\left(1.0224\right) = log(41.3) - log(38.8)$$

Factoring out the t on the left,

$$t\left(log\left(1.0264\right) - log\left(1.0224\right)\right) = log(41.3) - log(38.8)$$

Dividing to solve for t

$$t = \frac{log(41.3) - log(38.8)}{log(1.0264) - log(1.0224)} \approx 15.991$$

years until the populations will be equal.

Example 1b

Solve the problem above by rewriting before taking the log.

Starting at the equation $38.8(1.0264)^t = 41.3(1.0224)^t$

Divide to move the exponential terms to one side of the equation and the constants to the other side $\frac{1.0264^t}{1.0224^t} = \frac{41.3}{38.8}$

Using exponent rules to group on the left, $\left(\frac{1.0264}{1.0224}\right)^t = \frac{41.3}{38.8}$

Taking the log of both sides $log\left(\left(\frac{1.0264}{1.0224}\right)^t\right) = log\left(\frac{41.3}{38.8}\right)$

Utilizing the exponent property on the left, $t\,log\left(\frac{1.0264}{1.0224}\right) = log\left(\frac{41.3}{38.8}\right)$

Dividing gives $t = \frac{log\left(\frac{41.3}{38.8}\right)}{log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991$

years

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{log(41.3/38.8)}{log\left(\frac{1.0264}{1.0224}\right)} =$$

$$\frac{log(41.3) - log(38.8)}{log(1.0264) - log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

Try it Now

1. Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Radioactive Decay

In an earlier section, we discussed radioactive decay – the idea that radioactive isotopes change over time. One of the common terms associated with radioactive decay is half-life.

Half Life

The **half-life** of a radioactive isotope is the time it takes for half the substance to decay.

Given the basic exponential growth/decay equation $h(t) = ab^t$, the half-life can be found by solving for when half the original amount remains; by solving $\frac{1}{2}a = a(b)^t$, or more simply $\frac{1}{2} = b^t$. Notice how the initial amount is irrelevant when solving for half-life.

Example 2

Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

We were not given a starting quantity, so we could either make up a value or use an unknown constant to represent the starting amount. To show that starting quantity does not affect the result, let us denote the initial quantity by the constant *a*. Then the decay of Bismuth-210 can be described by the equation $Q(d) = a(0.87)^d$.

To find the half-life, we want to determine when the remaining quantity is half the original:

$$\frac{1}{2}a$$

Solving, $\frac{1}{2}a = a(0.87)^d$

Dividing by *a*, $\frac{1}{2} = 0.87^d$

Take the log of both sides $\log\left(\frac{1}{2}\right) = \log\left(0.87^d\right)$

Use the exponent property of logs

$$\log\left(\tfrac{1}{2}\right) = d\log\left(0.87\right)$$

Divide to solve for *d*

$$d = \frac{log\left(\frac{1}{2}\right)}{log(0.87)} \approx 4.977$$

days

This tells us that the half-life of Bismuth-210 is approximately 5 days.

Example 3

Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how much will remain after 30 years? 60 years? 90 years?

Since the half-life is 30 years, after 30 years, half the original amount, 100mg, will remain.

After 60 years, another 30 years have passed, so during that second 30 years, another half of the substance will decay, leaving 50mg.

After 90 years, another 30 years have passed, so another half of the substance will decay, leaving 25mg.

Example 4

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

To find how old the bone is, we first will need to find an equation for the decay of the carbon-14. We could either use a continuous or annual decay formula, but opt to use the continuous decay formula since it is more common in scientific texts. The half life tells us that after 5730 years, half the original substance remains. Solving for the rate, $\frac{1}{2}a = ae^{r5730}$

Dividing by a $\frac{1}{2} = e^{r5730}$

Taking the natural log of both sides $\ln\left(\frac{1}{2}\right) = \ln\left(e^{r5730}\right)$

Use the inverse property of logs on the right side $\ln\left(\frac{1}{2}\right) = 5730r$

Divide by 5730

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

Now we know the decay will follow the equation $Q(t) = ae^{-0.000121t}$. To find how old the bone fragment is that contains 20% of the original amount, we solve for t so that $Q(t) = 0.20a$. $0.20a = ae^{-0.000121t}$

$$0.20 = e^{-0.000121t}$$
$$\ln(0.20) = \ln\left(e^{-0.000121t}\right)$$
$$\ln(0.20) = -0.000121t$$
$$t = \frac{\ln(0.20)}{-0.000121} \approx 13301$$

years

The bone fragment is about 13,300 years old.

Try it Now

2. In Example 3, we learned that Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how long will it take until only 1 milligram remains?

For decaying quantities, we asked how long it takes for half the substance to decay. For growing quantities we might ask how long it takes for the quantity to double.

Doubling Time

The **doubling time** of a growing quantity is the time it takes for the quantity to double.

Given the basic exponential growth equation $h(t) = ab^t$, doubling time can be found by solving for when the original quantity has doubled; by solving $2a = a(b)^x$, or more simply $2 = b^x$. Again notice how the initial amount is irrelevant when solving for doubling time.

Example 5

If you invest money at 8% compounded quarterly, how long will it take your money to double?

Using the compound interest equation, we can write $A(t) = P\left(1 + \frac{0.08}{4}\right)^{4t} = P(1.02)^{4t}$.

To find the doubling time, we look for the time until we have twice the original amount, so when $A(t) = 2P$. Notice we don't need to know how much was invested.

$$2P = P(1.02)^{4t}$$
$$2 = (1.02)^{4t}$$
$$\log(2) = \log\left(1.02^{4t}\right)$$
$$\log(2) = 4t \log(1.02)$$
$$t = \frac{\log(2)}{4\log(1.02)} \approx 8.751$$

years.

It will take about 8.75 years for the investment double in value.

Example 6

Use of a new social networking website has been growing exponentially, with the number of new members doubling every 5 months. If the site currently has 120,000 users and this trend continues, how many users will the site have in 1 year?

We can use the doubling time to find a function that models the number of site users, and then use that equation to answer the question. While we could use an arbitrary *a* as we have before for the initial amount, in this case, we know the initial amount was 120,000.

If we use a continuous growth equation, it would look like $N(t) = 120e^{rt}$, measured in thousands of users after t months. Based on the doubling time, there would be 240 thousand users after 5 months. This allows us to solve for the continuous growth rate:

$$240 = 120e^{r5}$$
$$2 = e^{r5}$$
$$\ln 2 = 5r$$
$$r = \frac{\ln 2}{5} \approx 0.1386$$

Now that we have an equation, $N(t) = 120e^{0.1386t}$, we can predict the number of users after 12 months:

$$N(12) = 120e^{0.1386(12)} = 633.140$$

thousand users.

So after 1 year, we would expect the site to have around 633,140 users.

Try it Now

3. If tuition at a college is increasing by 6.6% each year, how many years will it take for tuition to double?

Logarithmic Scales

For quantities that vary greatly in magnitude, a standard linear scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale – one with equally spaced values – these values get bunched up.
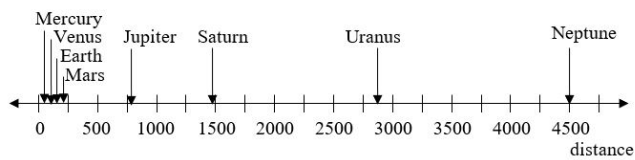
However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent.

Planet	Distance (millions of km)	log(distance)
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65
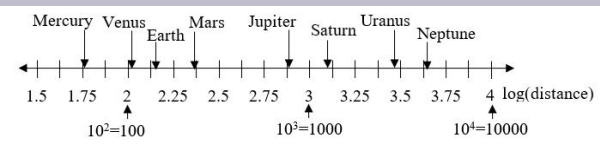
Sometimes, as shown above, the scale on a logarithmic number line will show the log values, but more commonly the original values are listed as powers of 10, as shown below.
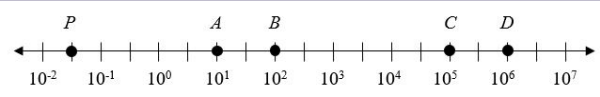
Example 7

Estimate the value of point *P* on the log scale above

The point *P* appears to be half way between -2 and -1 in log value, so if *V* is the value of this point, $\log(V) \approx -1.5$
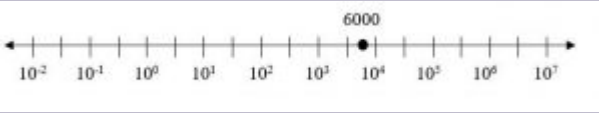
Rewriting in exponential form,

$$V \approx 10^{-1.5} = 0.0316$$

Example 8

Place the number 6000 on a logarithmic scale.

Since $\log(6000) \approx 3.8$, this point would belong on the log scale about here:

Try it Now

4. Plot the data in the table below on a logarithmic scale[3].

Source of Sound/Noise	Approximate Sound Pressure in μPa (micro Pascals)
Launching of the Space Shuttle	2,000,000,000
Full Symphony Orchestra	2,000,000
Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio	200
Softest Sound a human can hear	20

It is also common to see two dimensional graphs with one or both axes using a logarithmic scale. One common use of a logarithmic scale on the vertical axis is to graph quantities that are changing exponentially, since it helps reveal relative differences. This is commonly used in stock charts, since values historically have grown exponentially over time. Both stock charts below show the Dow Jones Industrial Average, from 1928 to 2010.

Linear scale:

Jul 1, 1929 : ^DJI 343.45

Logarithmic scale:

Oct 1, 1928 : ^DJI 300

Both charts have a linear horizontal scale, but the first graph has a linear vertical scale, while the second has a logarithmic vertical scale. The first scale is the one we are more familiar with, and shows what appears to be a strong exponential trend, at least up until the year 2000.

Example 9

There were stock market drops in 1929 and 2008. Which was larger?

In the first graph, the stock market drop around 2008 looks very large, and in terms of dollar values, it was indeed a large drop. However the second graph shows relative changes, and the drop in 2009 seems less major on this graph, and in fact the drop starting in 1929 was, percentage-wise, much more significant.

Specifically, in 2008, the Dow value dropped from about 14,000 to 8,000, a drop of 6,000. This is obviously a large value drop, and amounts to about a 43% drop. In 1929, the Dow value dropped from a high of around 380 to a low of 42 by July of 1932. While value-wise this drop of 338 is much smaller than the 2008 drop, it corresponds to a 89% drop, a much larger relative drop than in 2008. The logarithmic scale shows these relative changes

.

Try it Now Answers

1. 4.1874 weeks

2. 229.3157 , or approximately 229 years

3. It will take 10.845 years, or approximately 11 years, for tuition to double.

4.
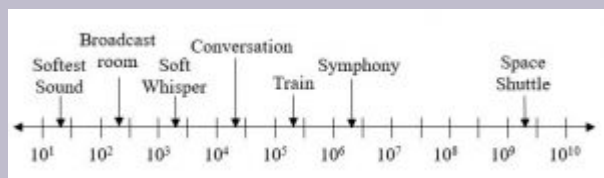
1 World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 20, 2010

2 World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010

3 From http://www.epd.gov.hk/epd/noise_education/web/ENG_EPD_HTML/m1/intro_5.html, retrieved Oct 2, 2010

Key Takeaway

Radioactive decay

Half life

Doubling time

Logarithmic Scales

11.

FINANCE

Section 11.1 Simple and Compound Interest

Discussing interest starts with the **principal**, or amount your account starts with. This could be a starting investment, or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest

$$I = Pr$$

$$A = P + I = P + Pr = P\left(1 + r\right)$$

I is the interest

A is the end amount: principal plus interest

P is the principal (starting amount)

r is the interest rate in decimal form. Example: 5% = 0.05

Example 1

A friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

P = $300 the principal

r = 0.03 3% rate

I = $300(0.03) = $9. You will earn $9 interest.

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$$I = Prt$$

$$A = P + I = P + Prt = P(1 + rt)$$

I is the interest

A is the end amount: principal plus interest

P is the principal (starting amount)

r is the interest rate in decimal form

t is time

The units of measurement (years, months, etc.) for the time should match the time period for the interest rate.

APR – Annual Percentage Rate

Interest rates are usually given as an **annual percentage rate (APR)** – the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.

A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1,000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

$P = \$1000$	the principal
$r = 0.02$	2% rate per half-year
$t = 8$	4 years = 8 half-years
$I = \$1000(0.02)(8) = \$160.$	You will earn $160 interest total over the four years.

Try it Now

1. A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called **compounding**. We looked at this situation earlier, in the chapter on exponential growth.

Compound Interest

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

A is the balance in the account after t years.

P is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account once and let it sit there earning interest.

Example 4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

In this example,

P = $3000   the initial deposit

r = 0.06   6% annual rate

k = 12   12 months in 1 year
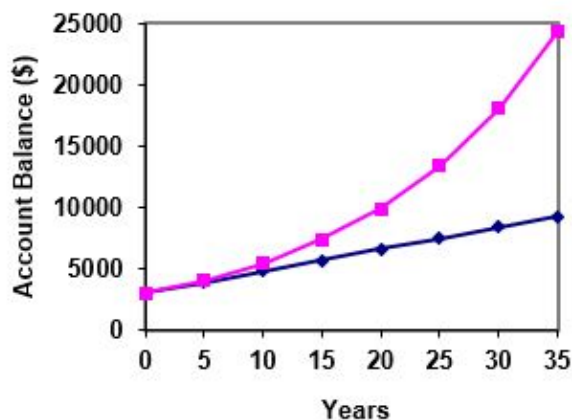
t = 20   since we're looking for how much we'll have after 20 years

So

$$A = 3000 \left(1 + \frac{0.06}{12} \right)^{20 \cdot 12} = \$9930.61$$

(round your answer to the nearest penny)

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65

As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$. But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:

$\boxed{\wedge}$, $\boxed{y^x}$, or $\boxed{x^y}$.

To evaluate 1.005^{240} we'd typ

e 1.005 $\boxed{\wedge}$ 240, or 1.005 $\boxed{y^x}$ 240. Try it out – you should get something around 3.3102044758.

Example 5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

In this example,

We're looking for P.

$r = 0.04$	4%
$k = 4$	4 quarters in 1 year
$t = 18$	Since we know the balance in 18 years
$A = \$40,000$	The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P.

$$40000 = P\left(1 + \tfrac{0.04}{4}\right)^{18 \times 4}$$
$$40000 = P(2.0471)$$
$$P = \tfrac{40000}{2.0471} = \$19539.84 </div>$$

So you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Rounding

It is important to be very careful about rounding when calculating things with exponents. In general, you want to keep as many decimals during calculations as you can. Be sure to **keep at least 3 significant digits** (numbers after any leading zeros). Rounding 0.00012345 to 0.000123 will usually give you a "close enough" answer, but keeping more digits is always better. If your calculator allows it, do all your calculations without rounding in the calculator and only round the final answer.

Example 6

To see why not over-rounding is so important, suppose you were investing $1000 at 5% interest compounded monthly for 30 years.

P = $1000 the initial deposit

r = 0.05   5%

k = 12   12 months in 1 year

t = 30   since we're looking for the amount after 30 years

If we first compute r/k, we find 0.05/12 = 0.00416666666667

Here is the effect of rounding this to different values:

r/k rounded to:	Gives A to be:	Error
0.004	$4208.59	$259.15
0.0042	$ 4521.45	$53.71
0.00417	$ 4473.09	$5.35
0.004167	$ 4468.28	$0.54
0.0041667	$ 4467.80	$0.06
no rounding	$ 4467.74	

If you're working in a bank, of course you wouldn't round at all. For our purposes, the answer we got by rounding to 0.00417, three significant digits, is close enough – $5 off of $4500 isn't too bad. Certainly keeping that fourth decimal place wouldn't have hurt.

Using your calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate $A = 1000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 30}$

We can quickly calculate 12×30 = 360, giving $A = 1000\left(1 + \frac{0.05}{12}\right)^{360}$.

Now we can use the calculator.

Type this	Calculator Shows
0.05 ÷ 12 =	0.00416666666667
+ 1 =	1.00416666666667
y^x 360 =	4.46774431400613
× 1000 =	4467.74431400613

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated. If you have a calculator like this, you will probably just need to enter:

1000 × (1 + 0.05 ÷ 12) y^x 360 = .

Or

1000 × (1 + 0.05 ÷ 12) ^ 360 = .

Try it Now

2. If $70,000 are invested at 7% compounded monthly for 25 years, find the end balance.

Because of compounding throughout the year, with compound interest the actual increase in a year is *more* than the annual percentage rate. If $1,000 were invested at 10%, the table below shows the value after 1 year at different compounding frequencies:

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

If we were to compute the actual percentage increase for the daily compounding, there was an increase of $105.16 from an original amount of $1,000$, for a percentage increase of $\frac{105.16}{1000} = 0.10516 = 10.516\%$ increase. This quantity is called the **annual percentage yield (APY)**.

Notice that given any starting amount, the amount after 1 year would be

$A = P\left(1 + \frac{r}{k}\right)^k$. To find the total change, we would subtract the original amount, then to find the percentage change we would divide that by the original amount:

$$\frac{P\left(1+\frac{r}{k}\right)^k - P}{P} = \left(1 + \frac{r}{k}\right)^k - 1$$

Annual Percentage Yield

The **annual percentage yield** is the actual percent a quantity increases in one year. It can be calculated as

$$APY = \left(1 + \frac{r}{k}\right)^k - 1$$

Notice this is equivalent to finding the value of $1 after 1 year, and subtracting the original dollar.

Example 7

Bank A offers an account paying 1.2% compounded quarterly. Bank B offers an account paying 1.1% compounded monthly. Which is offering a better rate?

We can compare these rates using the annual percentage yield – the actual percent increase in a year.

Bank A: $\quad APY = \left(1 + \frac{0.012}{4}\right)^4 - 1 = 0.012054$= 1.2054%

Bank B: $\quad APY = \left(1 + \frac{0.011}{12}\right)^{12} - 1 = 0.011056$= 1.1056%

Bank B's monthly compounding is not enough to catch up with Bank A's better APR. Bank A offers a better rate.

Example 8

If you invest $2000 at 6% compounded monthly, how long will it take the account to double in value?

This is a compound interest problem, since we are depositing money once and allowing it to grow. In this problem,

P = $2000 the initial deposit

r = 0.06 6% annual rate

k = 12 12 months in 1 year

So our general equation is

$$A = 2000\left(1 + \frac{0.06}{12}\right)^{12t}$$

. We also know that we want our ending amount to be double of $2000, which is $4000, so we're looking for t so that $A = 4000$. To solve this, we set our equation for A equal to 4000.

$$4000 = 2000\left(1 + \frac{0.06}{12}\right)^{12t}$$

Divide both sides by 2000

$$2 = (1.005)^{12t}$$

To solve for the exponent, take the log of both sides

$$\log(2) = \log\left((1.005)^{12t}\right)$$

Use the exponent property of logs on the right side

$$\log(2) = 12t \log(1.005)$$

Now we can divide both sides by 12log(1.005)

$$\frac{\log(2)}{12\log(1.005)} = t$$

Approximating this to a decimal

$t = 11.581$

It will take about 11.581 years for the account to double in value. Note that your answer may come out slightly differently if you had evaluated the logs to decimals and rounded during your calculations, but your answer should be close. For example if you rounded log(2) to 0.301 and log(1.005) to 0.00217, then your final answer would have been about 11.577 years.

Key Takeaways

Try it Now Answers

1.

$I = \$30$ of interest

$P = \$500$ principal

r = unknown

$t = 1$ month

Using $I = Prt$, we get $30 = 500 \cdot r \cdot 1$. Solving, we get $r = 0.06$, or 6%. Since the time was monthly, this is the monthly interest. The annual rate would be 12 times this: 72% interest.

2.

$$A = 70,000\left(1 + \frac{0.07}{12}\right)^{12(25)} = 400,779.27$$

APR

Simple interest

Compound interest

Compounding frequency

APY

Evaluating on a calculator

Section 11.2 Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a **savings annuity**. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

Suppose we will deposit $100 each month into an account paying 6% interest. How much will we have after a year? We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

$r = 0.06$ (6%)

$k = 12$ (12 compounds/deposits per year)

$d = \$100$ (our deposit per month)

$t = 1$ year

With **ordinary** annuities we assume the payment is made at the end of the period. The $100 we deposit at the end of the first month will earn interest for 11 months and at the end of the year will be worth.

$$A = 100\left(1 + \frac{0.06}{12}\right)^{11} = 100(1.005)^{11}$$

The $100 deposited at the end of the second month will have 10 months to grow, and will be worth

$$A = 100(1.005)^{10}$$

at the end of the year. This pattern continues down to the last deposit, which has no time to compound, and will be worth $A = 100$.

In total, we will have accumulated:

$$A = 100(1.005)^{11} + 100(1.005)^{10} + \cdots + 100(1.005)^{2} + 100(1.005)^{1} + 100$$

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! To simplify things, multiply both sides of the equation by 1.005:

$$1.005A = 1.005 \left(100(1.005)^{11} + 100(1.005)^{10} + \cdots + 100(1.005)^{2} + 100(1.005) + 100 \right)$$

Distributing on the right side of the equation gives

$$1.005A = 100(1.005)^{12} + 100(1.005)^{11} + \cdots + 100(1.005)^{3} + 100(1.005)^{2} + 100(1.005)$$

Now we'll line this up with like terms from our original equation, and subtract each side

$$\begin{aligned} 1.005A &= 100(1.005)^{12} + &100(1.005)^{11} + \cdots + &100(1.005) \\ A &= &100(1.005)^{11} + \cdots + &100(1.005) &+100 \end{aligned}$$

Almost all the terms cancel on the right hand side when we subtract, leaving

$$1.005A - A = 100(1.005)^{12} - 100$$

Now we solve this equation for A.

$$0.005A = 100 \left((1.005)^{12} - 1 \right)$$
$$A = \frac{100\left((1.005)^{12}-1\right)}{0.005}$$

Recall 0.005 was r/k, 100 was the deposit d, and 12 was the number of months, kt. Generalizing this result, we get the saving annuity formula.

Annuity Formula

$$A = \frac{d\left(\left(1+\frac{r}{k}\right)^{kt}-1\right)}{\left(\frac{r}{k}\right)}$$

A is the balance in the account after *t* years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, $k = 12$.

If you make your deposits every year, use yearly compounding, $k = 1$.

If you make your deposits every quarter, use quarterly compounding, $k = 4$.

Etc.

When do you use this

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

Compound interest assumes that you put money in the account once and let it sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Example 1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

In this example,

d = $100   the monthly deposit

r = 0.06   6% annual rate

k = 12   since we're doing monthly deposits, we'll compound monthly

t = 20   we want the amount after 20 years

Putting this into the equation:

$$A = \frac{100\left(\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}$$
$$A = \frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$
$$A = \frac{100(3.310-1)}{(0.005)}$$
$$A = \frac{100(2.310)}{(0.005)} = \$46200 </div>$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $46,200 – $24,000 = $22,200.

Example 2

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

In this example,

We're looking for d.

$r = 0.08$ 8% annual rate

$k = 12$ since we're depositing monthly

$t = 30$ 30 years

$A = \$200{,}000$ The amount we want to have in 30 years

In this case, we're going to have to set up the equation, and solve for d.

$$200{,}000 = \frac{d\left(\left(1+\frac{0.08}{12}\right)^{30(12)}-1\right)}{\left(\frac{0.08}{12}\right)}$$

$$200{,}000 = \frac{d\left((1.00667)^{360}-1\right)}{(0.00667)}$$

$$200{,}000 = d(1491.57)$$

$$d = \frac{200{,}000}{1491.57} = \$134.09 </div>$$

So you would need to deposit $134.09 each month to have $200,000 in 30 years if your account earns 8% interest

Try it Now

1. A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

Example 3

If you invest $100 each month into an account earning 3% compounded monthly, how long will it take the account to grow to $10,000?

This is a savings annuity problem since we are making regular deposits into the account.

d = $100	the monthly deposit
r = 0.03	3% annual rate
k = 12	since we're doing monthly deposits, we'll compound monthly

We don't know t, but we want A to be $10,000.

Putting this into the equation:

$$10,000 = \frac{100 \left(\left(1 + \frac{0.03}{12}\right)^{12t} - 1 \right)}{\left(\frac{0.03}{12}\right)}$$

Simplifying the fractions a bit

$$10,000 = \frac{100 \left((1.0025)^{12t} - 1 \right)}{0.0025}$$

We want to isolate the exponential term, 1.0025^{12t}, so multiply both sides by 0.0025

$$25 = 100 \left((1.0025)^{12t} - 1 \right)$$

Divide both sides by 100

$$0.25 = (1.0025)^{12t} - 1$$

Add 1 to both sides

$$1.25 = (1.0025)^{12t}$$

Now take the log of both sides

$$\log(1.25) = \log\left((1.0025)^{12t} \right)$$

Use the exponent property of logs

$$\log(1.25) = 12t \log(1.0025)$$

Divide by 12log(1.0025)

$$\frac{\log\left(1.25\right)}{12\log\left(1.0025\right)} = t$$

Approximating to a decimal

t = 7.447 years

It will take about 7.447 years to grow the account to $10,000.

Try it Now Answers

1.

d = $5   the daily deposit

r = 0.03 3%   annual rate

k = 365   since we're doing daily deposits, we'll compound daily

t = 10   we want the amount after 10 years

$$A = \frac{5\left(\left(1+\frac{0.03}{365}\right)^{365\times 10}-1\right)}{\frac{0.03}{365}} =$$

$21,282.07

We would have deposited a total of $5·365·10 = $18,250, so $3,032.07 is from interest

Section 11.3 Payout Annuities

In the last section you learned about annuities. In an annuity, you start with nothing, put money into an account on a regular basis, and end up with money in your account.

In this section, we will learn about a variation called a **Payout Annuity**. With a payout annuity, you start with money in the account, and pull money out of the account on a regular basis. Any remaining money in the account earns interest. After a fixed amount of time, the account will end up empty.

Payout annuities are typically used after retirement. Perhaps you have saved $500,000 for retirement, and want to take money out of the account each month to live on. You want the money to last you 20 years. This is a payout annuity. The formula is derived in a similar way as we did for savings annuities. The details are omitted here.

Payout Annuity Formula

$$P = \frac{w\left(1-\left(1+\frac{r}{k}\right)^{-kt}\right)}{\left(\frac{r}{k}\right)}$$

P is the balance in the account at the beginning (starting amount, or principal).

w is the regular withdrawal (the amount you take out each year, each month, etc.)

r is the annual interest rate (in decimal form. Example: 5% = 0.05)

k is the number of compounding periods in one year.

t is the number of years we plan to take withdrawals

Like with annuities, the compounding frequency is not always explicitly given, but is determined by how often you take the withdrawals.

When do you use this

Payout annuities assume that you take money from the account on a regular schedule (every month, year, quarter, etc.) and let the rest sit there earning interest.

Compound interest: One deposit

Annuity: Many deposits.

Payout Annuity: Many withdrawals

Example 1

After retiring, you want to be able to take $1000 every month for a total of 20 years from your retirement account. The account earns 6% interest. How much will you need in your account when you retire?

In this example,

w = $1000   the monthly withdrawal

r = 0.06   6% annual rate

k = 12   since we're doing monthly withdrawals, we'll compound monthly

t = 20   since were taking withdrawals for 20 years

We're looking for *P*; how much money needs to be in the account at the beginning.

Putting this into the equation:

$$P = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-20(12)}\right)}{\left(\frac{0.06}{12}\right)}$$

$$P = \frac{1000\times\left(1-(1.005)^{-240}\right)}{(0.005)}$$

$$P = \frac{1000\times(1-0.302)}{(0.005)} = \$139,600 </div>$$

You will need to have $139,600 in your account when you retire.

Notice that you withdrew a total of $240,000 ($1000 a month for 240 months). The difference between what you pulled out and what you started with is the interest earned. In this case it is $240,000 – $139,600 = $100,400 in interest.

Evaluating negative exponents on your calculator

With these problems, you need to raise numbers to negative powers. Most calculators have a separate button for negating a number that is different than the subtraction button. Some calculators label this $(-)$, some with $+/-$. The button is often near the = key or the decimal point.

If your calculator displays operations on it (typically a calculator with multiline display), to calculate 1.005^{-240} you'd type something like: 1.005 $^$ $(-)$ 240

If your calculator only shows one value at a time, then usually you hit the (-) key after a number to negate it, so you'd hit: 1.005 y^x 240 $(-)$ =

Give it a try – you should get 1.005^{-240} = 0.302096

Example 2

You know you will have $500,000 in your account when you retire. You want to be able to take monthly

withdrawals from the account for a total of 30 years. Your retirement account earns 8% interest. How much will you be able to withdraw each month?

In this example,

We're looking for w.

$r = 0.08$	8% annual rate
$k = 12$	since we're withdrawing monthly
$t = 30$	30 years
$P = \$500,000$	we are beginning with $500,000

In this case, we're going to have to set up the equation, and solve for w.

$$500,000 = \frac{w\left(1-\left(1+\frac{0.08}{12}\right)^{-30(12)}\right)}{\left(\frac{0.08}{12}\right)}$$

$$500,000 = \frac{w\left(1-(1.00667)^{-360}\right)}{(0.00667)}$$

$$500,000 = w(136.232)$$

$$w = \frac{500,000}{136.232} = \$3670.21 </div>$$

You would be able to withdraw $3,670.21 each month for 30 years.

1. A donor gives $100,000 to a university, and specifies that it is to be used to give annual scholarships for the next 20 years. If the university can earn 4% interest, how much can they give in scholarships each year?

Try it Now Answers

1.

w = unknown

$r = 0.04$ 4% annual rate

$k = 1$ since we're doing annual scholarships

$t = 20$ 20 years

$P = 100,000$ we're starting with $100,000

Solving for w gives $7,358.18 each year that they can give in scholarships.

It is worth noting that usually donors instead specify that only interest is to be used for scholarship, which makes the original donation last indefinitely. If this donor had specified that, $100,000(0.04) = $4,000 a year would have been available.