

## NSCC Business Math

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# *NSCC Business Math*

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## **MATH 2500**

ANDRÉ DAVEY; UTE FIEDLER; SUZANNE MILNER; PAUL RUDDERHAM; MANAL SALEH; AND JEAN-PAUL OLIVIER

SANJA KRAJISNIK; CAROL LEPPINEN; JELENA LONCAR-VINES; AND BETTINA SCHNEIDER

Nova Scotia Community College  
Nova Scotia



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Business Mathematics was created by remixing chapters from NSCC Math for Bookkeeping and Business Math: A Step-by-Step Handbook Abridged and adding new content.

Both books are Pressbooks hosted adapted versions of *Business Math: A Step-by-Step Handbook* by J. Olivier published via Lyryx Learning.

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# Contents

Welcome to Business Mathematics	ix
Instructor Resources	xi
<b>1. BASIC MATH REVIEW</b>	
1.1 Order of Operations	3
1.2 Fractions and Decimals	13
1.3 Rules for Exponents	35
1.4 Solving Linear Equations	64
1.5 Solving Word Problems	82
1.6 Break-Even Analysis	90
1.7 Symbols and Formulas	101
<b>2 MATHEMATICS OF MERCHANDISING</b>	
2.1 Figuring Out the Cost: Discounts	104
2.2 Markup: Setting the Regular Price	126
2.3 Markdown: Setting the Sale Price	146
2.4 Sales Taxes	159
2.5 Invoicing: Terms of Payment and Cash Discounts	173
2.6 Symbols and Formulas	196
<b>3 EVALUATING CHOICES: TIME, RISK, AND VALUE</b>	
3.1 The Time Value of Money	199
3.2 Valuing a Series of Cash Flows	202
3.3 Calculating the Relationship of Time and Value	210

## **4 SIMPLE INTEREST**

4.1 Principal, Rate, Time	217
4.2 Maturity Value (or Future Value) Moving Money Involving Simple Interest	225
4.3 Application of Future Value: Savings Accounts And Short-Term GICs	235
4.4 Application of Present Value: Treasury Bills and Commercial Paper	258
4.5 Simple Interest Terminology (Interactive Activity)	266
4.6 Review, Symbols, and Formulas	267

## **5 COMPOUND INTEREST**

5.1 Compound Interest Fundamentals	270
5.2 Determining the Future (Maturity) Value	274
5.3 Determining the Present Value	282
5.4 Equivalent Payments	288
5.5 Determining the Interest Rate	296
5.6 Compound Interest Terminology (Interactive Activity)	304
5.7 Review, Symbols, and Formulas	305

## **6 COMPOUND INTEREST - APPLICATIONS & SINGLE PAYMENTS**

6.1 Application: Long Term GICs	308
6.2 Application: Long Term Promissory Notes	317
6.3 Application: Strip Bonds	323
6.4 Review, Symbols, and Formulas	330

## **7 COMPOUND INTEREST - ANNUITIES**

7.1 Fundamentals of Annuities	333
7.2 Future Value of Annuities	337
7.3 Present Value of Annuities	350
7.4 Annuity Payment Amounts	365
7.5 Number of Annuity Payments	373

7.6 Annuity Interest Rate	382
7.7 Annuities Terminology (Interactive Activity)	388
7.8 Review, Symbols, and Formulas	389

## **8 COMPOUND INTEREST - SPECIAL APPLICATIONS**

8.1 Deferred Annuities	393
8.2 Review, Symbols, and Formulas	405
1. Basic Math Review	406
2. Mathematics of Merchandising: Exercise Solutions	407
3. Evaluating Choices: Exercise Answers	413
4. Simple Interest: Solutions to Exercises	414
5. Compound Interest: Solution to Exercises	435
6. GIC: Solutions to Exercises	463
7. Annuities: Solution to Exercises	479

## **VIDEOS**

Videos: Definitions, Ratios and Proportions	496
Videos: Markups and Markdowns	498
Videos: Currency Conversions	500
Videos: Trade and Cash Discounts	502
Videos: Linear Functions and Costs	505
Videos: System of Equations	508
Videos: Breakeven Analysis	509
Videos: Simple Interest	512
Videos: Payment Plans and Making Choices	515
Videos: Equations of Value	517
Videos: Compound Interest	519
Videos: Payment Plans and Making Choices, Compound	522
Videos: Equations of Value and Compound Interest	524

Videos: Average and Effective Rates	526
Videos: Introduction to Annuities	530
Videos: 5 fingers of finance	532
Videos: General Annuities	534
Videos: Annuities Due	535
Videos: Back-to-Back Annuities	537
Videos: Bonds	540
Videos: Perpetuities and Stocks	541
Videos: Mortgages and Amortization, part 1	543
Videos: Mortgages and Amortization, part 2	546
Videos: Investment Decisions	549
Glossary	554
Rounding Rules	565
Version History	566

# Welcome to Business Mathematics

## ABOUT THIS TEXTBOOK

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Business Mathematics was created by combining chapters from NSCC Math for Bookkeeping and Business Math: A Step-by-Step Handbook Abridged by Sanja Krajisnik, Carol Leppinen, and Jelena Loncar-Vines, Conestoga College.

Both books are Pressbooks hosted condensed adapted versions of *Business Math: A Step-by-Step Handbook* by J. Olivier published via Lyryx Learning.

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## WHAT IS BUSINESS MATH?

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Business math is the study of mathematics required by the field of business. By the fact that you are reading this textbook, you must be interested in a business field such as accounting, marketing, human resources, or economics.

Regardless of your path, you cannot avoid dealing with money and numbers. Both personally and in your career you certainly use elementary arithmetic such as addition, subtraction, multiplication, and division. However, there is a whole field of mathematics that deals specifically with money. You will be offered loans, lines of credit, mortgages, leases, savings bonds, and other financial tools. Do you know what these are and how these financial tools can maximize your earnings and minimize your costs? Do you have what it takes to execute smart monetary decisions both personally and for your business? Do you know how interest works and how it gets calculated? If you can answer “yes” to these questions, then you are already off to a great start. If not, by the end of this textbook you will have a better understanding of all of these topics and more.

## HOW DO I LEARN ABOUT BUSINESS MATH?

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Let's be realistic. In some areas of life and business, you can achieve a reasonable degree of understanding just by reading. However, reading about business mathematics without doing it would be disastrous. To succeed, you must follow a structured approach:

1. Always read the content prior to your professor covering the topic in class.

2. Attend class, ask questions, and explore the topic to advance your understanding.
3. Do the homework and assignments—you absolutely must practice, practice, and practice!
4. Seek help immediately when you need it. Learning mathematics is like constructing a building. Each floor of the building requires the floor below it to be completed first. In mathematics, each section of a textbook requires the concepts and techniques from the sections that preceded it. If you have trouble with a concept, you must fix it NOW before it causes a large ripple effect on your ability to succeed in subsequent topics. So the bottom line is that you absolutely cannot replace this approach—you must follow it.

## MathJax

This resource uses LaTeX and a MathJax plugin to render math formulas. Please note that “some screen readers support MathML, MathJax’s internal format. Screen readers like ChromeVox, JAWS (on IE), and TextHelp support MathJax directly (most only version 2); other screen readers are supported by the `assistive-mml` extension as of version 3.0.1.” (MathJax Consortium, 2021). It is important to also note that the quality of screen reader support varies greatly with the software you are using and the various settings enabled. For more information on MathJax and screen reader support please visit the *MathJax Consortium Accessibility Features page*.

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## *Instructor Resources*

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# 1. BASIC MATH REVIEW

## Learning Objectives

- Order of Operations
- Fractions and Decimals
- Rules and Properties of Exponents
- Solving Linear Equations
- Solving Word Problems
- Break-Even Analysis

Where can you go in life and not be exposed to numbers and mathematics? Whether you are figuring out the price of a product (including shipping) on eBay, or balancing your bank accounts, you use your elementary mathematical skills from both your primary and secondary education.

## THINK ABOUT THE MATH YOU PERFORM EVERY DAY:

- At the grocery store, you compare products to calculate the best value. One brand of potato chips retails for \$3.99 for 300 g, while the equally satisfying brand beside it is priced at \$3.49 for 250 g. Which offers the better value?
- As the host for a large gathering, you are preparing a homemade lasagna and need to triple the original recipe, which calls for  $1\frac{2}{3}$  cups of tomato sauce. In the expanded recipe, how many cups of tomato sauce do you need?
- Many employers pay out bonuses. Perhaps in your company managers get twice as large a bonus as employees. Your company has five managers and 25 employees. If it announces a \$35,000 total bonus, what is your share as an employee?

Mathematics and numbers surround you in the business world, where you must read many numerical reports, interpret how the numbers fit together, and create your own reports showing such metrics as sales and profit projections.

Away from work, you must manage your income and pay your bills. This is a mathematical problem you likely solve on a daily basis, ensuring that the money flowing out of your bank account does not exceed the money flowing in. To purchase groceries, vacations, or entertainment, you need numbers.

This chapter gives you a refresher on your basic mathematical skills, which are essential for success in later chapters.

## ATTRIBUTION

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## 1.1 Order of Operations

### INTRODUCTION

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#### Try It

1) You have just won \$50,000 in a contest. Congratulations! But before you can claim it, you are required to answer a mathematical skill-testing question, and no calculators are permitted. After you hand over your winning ticket to the redemption agent, she hands you your time-limited skill-testing question:

$2 \times 5 + 30 \div 5$ . What does this question equal?

#### Solution

If you figured out the solution is **16**, you are on the right path. If you thought it was something else, this is a great time to review order of operations.

### THE SYMBOLS

---

While some mathematical operations such as addition use a singular symbol (+), there are other operations, like multiplication, for which multiple representations are acceptable. With the advent of computers, even more new symbols have crept into mathematical symbology. The table below lists the various mathematical operations and the corresponding mathematical symbols you can use for them.

Table 1.1.1

Mathematical Operation	Symbol or Appearance	Comments
Brackets	$()$ or $[]$ or $\{\}$	In order, these are known as round, square, and curly brackets.
Exponents	$2^3$ or $2^3$	In $2^3$ , <b>3</b> is the exponent; an exponent is always written as a superscript. On a computer, the exponent is recognized with the ^ symbol; $2^3$ is represented by $2^3$ .
Multiplication	$\times$ or $*$ or $\cdot$ or $2(2)$ or $(2)(2)$	In order, these are known as times, star, or bullet. The last two involve implied multiplication, which is when two terms are written next to each other joined only by brackets.
Division	$/$ or $\div$ or $\frac{4}{2}$	In order, these are known as the slash, divisor, and divisor line. Note in the last example that the division is represented by the horizontal line.
Addition	$+$	There are no other symbols.
Subtraction	$-$	There are no other symbols.

You may wonder if the different types of brackets mean different things. Although mathematical fields like calculus use specialized interpretations for the different brackets, business math uses all the brackets to help the reader visually pair up the brackets. Consider the following two examples:

**Example 1:**  $3 \times (4/(6 - (2 + 2)) + 2)$

**Example 2:**  $3 \times [4/\{6 - (2 + 2)\} + 2]$

Notice that in the second example you can pair up the brackets much more easily, but changing the shape of the brackets did not change the mathematical expression. This is important to understand when using a calculator, which usually has only round brackets. Since the shape of the bracket has no mathematical impact, solving example 1 or example 2 would involve the repeated usage of the round brackets.

## BEDMAS

In the section opener, your skill-testing question was  $2 \times 5 + 30 \div 5$ . Do you just solve this expression from left to right, or should you start somewhere else? To prevent any confusion over how to resolve these mathematical operations, there is an agreed-upon sequence of mathematical steps commonly referred to as **BEDMAS**.

## HOW TO USE BEDMAS

**BEDMAS** is an acronym for **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and **S**ubtraction.

**Step 1:** Brackets must be resolved first. As brackets can be nested inside of each other, you must resolve the innermost set of brackets first before proceeding outwards to the next set of brackets. When resolving a set of brackets, you must perform the mathematical operations within the brackets by following the remaining steps in this model (EDMAS). If there is more than one set of brackets but the sets are not nested, work from left to right and top to bottom.

**Step 2:** If the expression has any exponents, you must resolve these next. Remember that an exponent indicates how many times you need to multiply the base against itself. For example,  $2^3 = 2 \times 2 \times 2$ . More review of exponents is found in Section 1.3.

**Step 3:** The order of appearance for multiplication and division does not matter. However, you must resolve these operations in order from left to right and top to bottom as they appear in the expression.

**Step 4:** The last operations to be completed are addition and subtraction. The order of appearance doesn't matter; however, you must complete the operations working left to right through the expression.

Also note that on most calculators you have two ways to key in an exponent:

- If the exponent is squaring the base (e.g.,  $3^2$ ), press  $3 x^2$ . It calculates the solution of 9.
- If the exponent is anything other than a 2, you must use the  $x^y$  button. For  $2^3$ , you press  $2 x^y 3 =$ . It calculates the solution of 8.

## THINGS TO WATCH FOR

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### Negative Signs

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Remember that mathematics use both positive numbers (such as  $+3$ ) and negative numbers (such as  $-3$ ). Positive numbers do not need to have the  $+$  sign placed in front of them since it is implied. Thus  $+3$  is written as just  $3$ . Negative numbers, though, must have the negative sign placed in front of them. Be careful not to confuse the terminology of a negative number with a subtraction or minus sign. For example,  $4 + (-3)$  is read as "four plus negative three" and not "four plus minus three." A common method to key a negative number on a calculator is to enter the number first followed by the  $\pm$  button, which switches the sign of the number.

### Horizontal Divisor Line

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One of the areas in which people make the most mistakes involves the "hidden brackets." This problem almost always occurs when the horizontal line is used to represent division. Consider the following mathematical expression:

$$(4 + 6) \div (2 + 3)$$

If you rewrite this expression using the horizontal line to represent the divisor, it looks like this:

$$\frac{4 + 6}{2 + 3}$$

Notice that the brackets disappear from the expression when you write it with the horizontal divisor line because they are implied by the manner in which the expression appears. Your best approach when working with a horizontal divisor line is to reinsert the brackets around the terms on both the top and bottom. Thus, the expression looks like this:

$$\frac{(4 + 6)}{(2 + 3)}$$

Employing this technique will ensure that you arrive at the correct solution, especially when using calculators.

## PATHS TO SUCCESS

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### Hidden and Implied Symbols

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If there are hidden or implied symbols in the expressions, your first step is to reinsert those hidden symbols in their correct locations. In the example below, note how the hidden multiplication and brackets are reinserted into the expression:

$$4 \left[ \frac{3 + 2^2 \times 3}{(2 + 8) \div 2} \right] \text{ transforms into } 4 \times \left[ \frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right].$$

Once you have reinserted the symbols, you are ready to follow the **BEDMAS** model.

Calculators are not programmed to be capable of recognizing implied symbols. If you key in “3(4 + 2)” on your calculator, failing to input the multiplication sign between the “3” and the “(4 + 2),” you get a solution of 6. Some calculators ignore the “3” since they don’t know what mathematical operation to perform on it. To have your calculator solve the expression correctly, you must punch the equation through as “3 × (4 + 2) =”. This produces the correct answer of 18.

## SIMPLIFYING NEGATIVES

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If your question involves positive and negative numbers, it is sometimes confusing to know what symbol to put when simplifying or solving. Remember these two rules:

**Rule #1:** A pair of the same symbols is always positive.

Thus “4 + (+3)” and “4 – (–3)” both become “4 + 3.”

**Rule #2:** A pair of the opposite symbols is always negative.

Thus " $4 + (-3)$ " and " $4 - (+3)$ " both become " $4 - 3$ ."

A simple way to remember these rules is to count the total sticks involved, where a "+" sign has two sticks and a "-" sign has one stick. If you have an odd number of total sticks, the outcome is a negative sign. If you have an even number of total sticks, the outcome is a positive sign.

Note the following examples:

- $4 + (-3) = 3$  total sticks is odd and therefore simplifies to negative  $4 - 3 = 1$
- $(-2) \times (-2) = 2$  total sticks is even and therefore simplifies to positive  
 $(-2) \times (-2) = +4$

### Examples

#### EXAMPLE 1.1.1

Evaluate each of the following expressions:

A.  $2 \times 5 + 30 \div 5$

B.  $(6 + 3)^2 + 18 \div 2$

C.  $4 \times \left[ \frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right]$

#### Question A

##### Solution

*Step 1: B – Are there brackets?* No.

*Step 2: E – Are there exponents?* No.

*Step 3: D – Resolve the division.*

$$30 \div 5 = 6$$

The expression now looks like this:

$$2 \times 5 + 6$$

*Step 4: M – Resolve the multiplication.*

$$2 \times 5 = 10$$

The expression becomes:

$$10 + 6$$

**Step 5: A – Perform the remaining addition.**

$$10 + 6 = 16$$

**Step 6: S – There is no subtraction, so we state the final solution.**

$$16$$

### Question B

---

#### Solution

**Step 1: B – Are there brackets?** Yes, start with the brackets.

$$(6 + 3)^2 = 9$$

The expression now looks like this:

$$(9)^2 + 18 \div 2$$

**Step 2: E – Are there exponents?** Yes, resolve exponents next.

$$(9)^2 = 81$$

The expression now looks like this:

$$81 + 18 \div 2$$

**Step 3: D – Perform the division.**

$$81 + 18 \div 2 = 9$$

The expression now looks like this:

$$81 + 9$$

**Step 4: M – Is there multiplication?** No.

**Step 5: A – Perform the remaining addition.**

$$81 + 9 = 90$$

**Step 6: S – There is no subtraction, so we state the final solution.**

90

### Question C

#### Solution

**Step 1: B – Are there brackets?**

Yes, start with the innermost set of brackets and perform EDMAS.

$$4 \times \left[ \frac{\{3 + 2^2 \times 3\}}{\{(2 + 8) \div 2\}} \right] = 4 \times \left[ \frac{\{3 + 2^2 \times 3\}}{\{10 \div 2\}} \right]$$

Now solve the curly brackets, starting with the top. Perform EDMAS.

**Step 2: E – Are there exponents?** Yes, resolve exponents next.

$$4 \times \left[ \frac{\{3 + 2^2 = 4 \times 3\}}{\{10 \div 2\}} \right]$$

The expression now looks like this:

$$4 \times \left[ \frac{\{3 + 4 \times 3\}}{\{10 \div 2\}} \right]$$

**Step 3: D – Perform the division.**

$$4 \times \left[ \frac{\{3 + 4 \times 3\}}{\{10 \div 2 = 5\}} \right]$$

We no longer need the curly brackets on the bottom, so they are dropped.

The expression now looks like this:

$$4 \times \left[ \frac{\{3 + 4 \times 3\}}{5} \right]$$

**Step 4: M – Perform multiplication on the top.**

$$4 \times \left[ \frac{\{3 + 12\}}{5} \right]$$

The expression now looks like this:

$$4 \times \left[ \frac{\{3 + 4 \times 3 = 12\}}{5} \right]$$

**Step 5: A – Perform the remaining addition.**

We no longer need the curly brackets on the top, so they are dropped.

$$4 \times \left[ \frac{\{3 + 12 = 15\}}{5} \right]$$

The expression now looks like this:

$$4 \times \left[ \frac{15}{5} \right]$$

**Step 6: Repeating the division, we solve the fraction.**

$$4 \times \left[ \frac{15}{5} \right] \\ = 4 \times 3$$

We no longer need the square brackets so they are dropped here.

**Step 7: The last step is to perform multiplication.**

$$4 \times 3 = 12$$

**Step 6: S – There is no subtraction, so we state the final solution.**

$$12$$

## SECTION EXERCISES

Solve the following. *Hint: If a question involves money, round the answers to the nearest cent.*

### Mechanics

1.  $81 \div 27 + 3 \times 4$
2.  $100 \div (5 \times 4 - 5 \times 2)$

3.  $3^3 - 9 + (1 + 7 \times 3)$
4.  $(6 + 3)^2 - 17 \times 3 + 70$
5.  $100 - (4^2 + 3) - (3 + 9 \times 3 - 4)$

### Applications

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6.  $[(7^2 - \{-41\}) - 5 \times 2] \div (80 \div 10)$
7.  $\$1,000 \left( 1 + 0.09 \times \frac{88}{365} \right)$
8.  $3[\$2,000(1 + 0.003)^8] + \$1,500$
9.  $\frac{\$20,000}{1 + 0.07 \times \frac{7}{12}}$
10.  $4 \times [(5^2 + 15)^2 \div (13^2 - 9)]^2$
11.  $\$500 \left[ \frac{(1 + 0.00875)^{43} - 1}{0.00875} \right]$
12.  $\$1,000 \left( 1 + \frac{0.12}{6} \right)^{15}$

### Challenge, Critical Thinking, & Other Applications

13.  $\left( \frac{\$2,500}{1 + 0.10} \right) + \left( \frac{\$7,500}{(1 + 0.10)^2} \right) + \left( \frac{-\$1,500}{(1 + 0.10)^3} \right) + \left( \frac{-\$2,000}{(1 + 0.10)^4} \right)$
14.  $\$175,000(1 + 0.07)^{15} + \$14,000 \left[ \frac{(1 + 0.07)^{20} - 1}{0.07} \right]$
15.  $\$5,000 \left[ \frac{\left\{ 1 + [(1 + 0.08)^{0.5} - 1] \right\}^{75} - 1}{(1 + 0.08)^{0.5} - 1} \right]$
16.  $\$800 \left[ \frac{(1 + 0.07)^{20} - (1 + 0.03)^{20}}{0.07 - 0.03} \right]$

$$17. \$60,000(1 + 0.0058)^{80} - \$450 \left[ \frac{(1 + 0.0058)^{25} - 1}{0.0058} \right]$$

$$18. \left( \frac{0.08}{2} \right) \$1,000 \left[ \frac{1 - \frac{1}{\left\{ 1 + \left( \frac{0.08}{2} \right) \right\}^{16}}}{\left( \frac{0.08}{2} \right)} \right] + \$1,000 \left[ 1 + \left( \frac{0.08}{2} \right) \right]^{16}$$

$$19. \$1,475 \left[ \frac{\left( 1 + \frac{0.06}{4} \right)^{16} - 1}{\frac{0.06}{4}} \right]$$

$$20. \$6,250(1 + 0.0525)^{10} + \$325 \left[ \frac{(1 + 0.0525)^{10} - 1}{0.0525} \right]$$

**Solutions:** See Solutions to Exercises – Basic Math Review 1.1

## ATTRIBUTION

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## 1.2 Fractions and Decimals

### INTRODUCTION

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Imagine your local newspaper quotes a political candidate as saying, “The top half of the students are well-educated, the bottom half receive extra help, but the middle half we are leaving out”. You stare at the sentence for a moment and then laugh. To halve something means to split it into two. However, there are three halves here! You conclude that the speaker was not thinking carefully.

In coming to this conclusion, you are applying your knowledge of fractions. In this section, you will review fraction types, convert fractions into decimals, perform operations on fractions, and also address rounding issues in business mathematics.

### TYPES OF FRACTIONS

---

To understand the characteristics, rules, and procedures for working with fractions, you must become familiar with fraction terminology. First of all, what is a fraction? A **fraction** is a part of a whole. It is written in one of three formats:

$$1/2 \text{ or } \frac{1}{2} \text{ or } \frac{1}{2}$$

Each of these formats means exactly the same thing. The number on the top, side, or to the left of the line is known as the **numerator**. The number on the bottom, side, or to the right of the line is known as the **denominator**. The slash or line in the middle is the **divisor line**. In the above example, the numerator is **1** and the denominator is **2**. There are five different types of fractions, as explained in the table below.

Table 1.2.1

Fraction	Terminology	Characteristics	Result of Division*
$\frac{2}{5}$	Proper	The numerator is smaller than the denominator.	Answer is between 0 and 1
$\frac{5}{2}$	Improper	The numerator is larger than the denominator.	Answer is greater than 1
$3\frac{2}{5}$	Mixed Number	A fraction that combines an integer with a proper fraction. When the division is performed, the proper fraction is <i>added</i> to the integer.	Answer is greater than the integer
$3\frac{\frac{3}{4}}{7}$	Complex	A fraction that has fractions within fractions, combining elements of compound, proper, or improper fractions together. It is important to follow <b>BEDMAS</b> in resolving these fractions.	Answer varies depending on the fractions involved
$\frac{1}{2}$ and $\frac{2}{4}$	Equivalent	Two or more fractions of any type that have the same numerical value upon completion of the division. Note that both of these examples work out to 0.5.	Answers are equal

\*Assuming all numbers are positive.

## HOW IT WORKS

First, focus on the correct identification of proper, improper, compound, equivalent, and complex fractions. In the next section, you will work through how to accurately convert these fractions into their decimal equivalents.

Equivalent fractions require you to either solve for an unknown term or express the fraction in larger or smaller terms.

## HOW TO SOLVE FOR AN UNKNOWN TERM

These situations involve two fractions where only one of the numerators or denominators is missing. Follow this four-step procedure to solve for the unknown:

**Step 1:** Set up the two fractions.

**Step 2:** Note that your equation contains two numerators and two denominators. Pick the pair for which you know both values.

**Step 3:** Determine the multiplication or division relationship between the two numbers.

**Step 4:** Apply the same relationship to the pair of numerators or denominators containing the unknown.

### Example 1.2.1

Assume you are having a party and one of your friends says he would like to eat one-third of the pizza. You notice the pizza has been cut into nine slices. How many slices would you give to your friend?

#### Solution

**Step 1: Assign a meaningful variable to represent unknown.**

$s$  = the number of slices to give out

Your friend wants one out of three pieces. This is one-third. You want to know how many pieces out of nine to give him. There are a total of 9 pieces, so we are looking for  $s/9$ .

$$\frac{1}{3} = \frac{s}{9}$$

**Step 2: Work with the denominators 3 and 9 since you know both of them.**

**Step 3: Take the larger number and divide it by the smaller number.**

$$9 \div 3 = 3$$

The denominator on the right is three times larger than the denominator on the left.

**Step 4: Take the 1 and multiply it by 3 to get the  $s$ .**

$$\frac{1 \times 3}{3 \times 3} = \frac{3}{9}$$

$$s = 3$$

**Step 5: Write as a statement.**

You should give your friend three slices of pizza.

## EXPRESSING THE FRACTION IN LARGER OR SMALLER TERMS

When you need to make a fraction easier to understand or you need to express it in a certain format, it helps to try to express it in larger or smaller terms:

To express a fraction in larger terms, multiply both the numerator and denominator by the same number.

- **Larger terms:**  $\frac{2}{12}$  expressed with terms twice as large would be  $\frac{2 \times 2}{12 \times 2} = \frac{4}{24}$

To express a fraction in smaller terms, divide both the numerator and denominator by the same number.

- **Smaller terms:**  $\frac{2}{12}$  expressed with terms half as large would be  $\frac{2 \div 2}{12 \div 2} = \frac{1}{6}$

When expressing fractions in higher or lower terms, you do not want to introduce decimals into the fraction unless there would be a specific reason for doing so. For example, if you divided 4 into both the numerator and denominator of  $\frac{2}{12}$ , you would have  $\frac{0.5}{3}$ , which is not a typical format.

## HOW TO

**Find numbers that divide evenly into the numerator or denominator (called factoring).**

- **Step 1:** Pick the smallest number in the fraction.
- **Step 2:** Use your multiplication tables and start with  $1 \times$  before proceeding to  $2 \times$ ,  $3 \times$ , and so on.
- **Step 3:** When you find a number that works, check to see if it also divides evenly into the other number.

### Example 1.2.2a

Reduce the following fraction:  $\frac{12}{18}$ .

**Solution**

**Step 1: Factor the numerator.**

$$1 \times 12 = 12$$

**Step 2: Does it divide evenly?**

12 does not divide evenly into the denominator.

**Step 3: Try another factor.**

$$2 \times 6 = 12$$

6 does divide evenly into the denominator.

**Step 4: Reduce the fraction into smaller terms.**

Reduce the fraction to smaller terms by dividing by 6.

$$\frac{12 \div 6}{18 \div 6} = \frac{2}{3}$$

**Step 5: Write as a statement.**

The reduced fraction is  $\frac{2}{3}$ .

## THINGS TO WATCH OUT FOR

With complex fractions, it is critical to obey the rules of **BEDMAS**.

Note in the following example that an addition sign and two sets of brackets were hidden:

You should rewrite  $3\frac{3}{7}$  as  $3 + \left[ \frac{\left( \frac{3}{4} \right)}{7} \right]$  before you attempt to solve with **BEDMAS**.

## PATHS TO SUCCESS

What do you do when there is a negative sign in front of a fraction, such as  $-\frac{1}{2}$ ? Do you put the negative with the numerator or the denominator? The common solution is to multiply the numerator by negative 1, resulting in  $\frac{(-1) \times 1}{2} = \frac{-1}{2}$ .

In the special case of a compound fraction, multiply the entire fraction by  $-1$ . Thus:

$$\begin{aligned}
 &= -1\frac{1}{2} \\
 &= (-1) \times \left(1 + \frac{1}{2}\right) \\
 &= -1 - \frac{1}{2}
 \end{aligned}$$

**Example 1.2.2b**

Identify the type of fraction represented by each of the following:

a.  $\frac{2}{3}$

b.  $6\frac{7}{8}$

c.  $12\frac{\frac{4}{3}}{6\frac{4}{5}}$

d.  $\frac{15}{11}$

e.  $\frac{5}{6}$

f.  $\frac{3}{4}$  &  $\frac{9}{12}$

**Step 1: Identify what we are looking for.**

For each of these six fractions, identify the type of fraction.

**Step 2: State what we know.**

There are five types of fractions, including proper, improper, compound, complex, or equivalent.

**Step 3: Use the definition from the Types of Fractions table at the beginning of this section to identify the type.**

**Solutions**

a.

---

$\frac{2}{3}$  The numerator is smaller than the denominator. This matches the characteristics of a **proper fraction**.

b.

---

$6\frac{7}{8}$  This fraction combines an integer with a proper fraction (since the numerator is smaller than the denominator). This matches the characteristics of a **mixed number**.

c.

---

$12\frac{\frac{4}{3}}{6\frac{4}{5}}$  There are lots of fractions involving fractions nested inside other fractions. The fraction as a whole is a mixed number, containing an integer with a proper fraction (since the numerator is smaller than the denominator). Within the proper fraction, the numerator is an improper fraction  $\left(\frac{4}{3}\right)$  and the denominator is a mixed number containing an integer and a proper fraction  $\left(6\frac{4}{5}\right)$ . This all matches the definition of a **complex fraction**: nested fractions combining elements of mixed, proper, and improper fractions together.

d.

---

$\frac{15}{11}$  The numerator is larger than the denominator. This matches the characteristics of an **improper fraction**.

e.

---

$\frac{5}{6}$  The numerator is smaller than the denominator. This matches the characteristics of a **proper fraction**.

f.

---

$\frac{3}{4}$  &  $\frac{9}{12}$  There are two proper fractions here that are equal to each other. If you were to complete the division, both fractions calculate to 0.75. These are **equivalent fractions**.

**Example 1.2.2c**

- a. Solve for the unknown term:  $x : \frac{7}{12} = \frac{49}{x}$
- b. Express this fraction in lower terms:  $\frac{5}{50}$

**Solution****Question a.**

**Step 1: Is the fraction in a format I can solve in?**

Yes. You have both of the numerators, so work with that pair.

**Step 2: Take the larger number and divide by the smaller number.**

$$49 \div 7 = 7$$

**Step 3: Multiple the fraction on the left by 7 to get the fraction on the right. Applying the same relationship:**

$$12 \times 7 = 84$$

**Step 4: Write as a statement.**

The unknown denominator on the right is 84, and therefore  $\frac{7}{12} = \frac{49}{84}$ .

**Solution****Question b.**

**Step 1: Find a common divisor that divides into the numerator and denominator evenly.**

As only 1 and 5 go into the number 5, it makes sense that you should choose 5 to divide into both the numerator and denominator.

Note that 5 factors evenly into the denominator, 50, meaning that no remainder or decimals are left over.

$$\frac{5 \div 5}{50 \div 5} = \frac{1}{10}$$

**Step 2: Write as a statement.**

In lower terms,  $\frac{5}{50}$ , is expressed as  $\frac{1}{10}$ .

## CONVERTING TO DECIMALS

Although fractions are common, many people have trouble interpreting them. For example, in comparing  $\frac{27}{37}$  to  $\frac{57}{73}$ , which is the larger number? The solution is not immediately apparent. As well, imagine a retail world where your local Walmart was having a  $\frac{3}{20}$ th off sale! It's not that easy to realize that this equates to 15% off. In other words, fractions are converted into decimals by performing the division to make them easier to understand and compare.

## CONVERTING FRACTIONS

Convert fractions into decimals based on the fraction types and fraction rules

### *Proper and Improper Fractions*

Resolve the division. For example,  $\frac{3}{4}$  is the same as  $3 \div 4 = 0.75$ . As well:

$$\begin{aligned} &\frac{5}{4} \\ &= 5 \div 4 \\ &= 1.25 \end{aligned}$$

### *Mixed Numbers*

The decimal number and the fraction are joined by a hidden addition symbol. Therefore, to convert to a decimal you need to reinsert the addition symbol and apply **BEDMAS**:

$$\begin{aligned} &3\frac{4}{5} \\ &= 3 + 4 \div 5 \\ &= 3 + 0.8 \\ &= 3.8 \end{aligned}$$

### Complex Fractions

The critical skill here is to reinsert all of the hidden symbols and then apply the rules of **BEDMAS**:

$$\begin{aligned}
 & 2 \frac{\frac{11}{4}}{1 \frac{1}{4}} \\
 &= 2 + \left[ \frac{(11 \div 4)}{(1 + 1 \div 4)} \right] \\
 &= 2 + \left[ \frac{(11 \div 4)}{(1 + 0.25)} \right] \\
 &= 2 + \left[ \frac{2.75}{1.25} \right] \\
 &= 2 + 2.2 \\
 &= 4.2
 \end{aligned}$$

#### Example 1.2.3

Convert the following fractions into decimals:

a.  $\frac{2}{5}$

b.  $6\frac{7}{8}$

c.  $12\frac{\frac{9}{2}}{1\frac{2}{10}}$

#### Solutions

##### Question a.

*Step 1:* This is a proper fraction requiring you to complete the division.

$$\frac{2}{5}$$
$$= 2 \div 5$$
$$= 0.4$$

**Step 2: Write as a statement.**

The decimal form is 0.4.

#### Question b.

---

**Step 1: This is a mixed number requiring you to reinsert the hidden addition symbol and then apply BEDMAS.**

$$6\frac{7}{8}$$
$$= 6 + 7 \div 8$$
$$= 6 + 0.875$$
$$= 6.875$$

**Step 2: Write as a statement.**

The decimal form is 6.875.

#### Question c.

---

**Step 1: This is a complex fraction requiring you to reinsert all hidden symbols and apply BEDMAS.**

$$\begin{aligned}
 & 12\frac{\frac{9}{2}}{1\frac{2}{10}} \\
 &= 12 + \left[ \frac{(9 \div 2)}{(1 + 2 \div 10)} \right] \\
 &= 12 + \left[ \frac{(9 \div 2)}{(1 + 0.2)} \right] \\
 &= 12 + \left[ \frac{4.5}{1.2} \right] \\
 &= 12 + 3.75 \\
 &= 15.75
 \end{aligned}$$

**Step 2: Write as a statement.**

The decimal form is 15.75.

## ROUNDING PRINCIPLE

Your company needs to take out a loan to cover some short-term debt. The bank has a posted rate of 6.825%. Your bank officer tells you that, for simplicity, she will just round off your interest rate to 6.9%. Is that all right with you? It shouldn't be!

What this example illustrates is the importance of rounding. This is a slightly tricky concept that confuses most students to some degree. In business math, sometimes you should round your calculations off and sometimes you need to retain all of the digits to maintain accuracy.

## HOW TO APPLY THE ROUNDING PRINCIPLE

To round a number off, you always look at the number to the right of the digit being rounded. If that number is 5 or higher, you add one to your digit; this is called **rounding up**. If that number is 4 or less, you leave your digit alone; this is called **rounding down**.

- For example, if you are rounding 8.345 to two decimals, you need to examine the number in the third decimal place (the one to the right). It is a 5, so you add one to the second digit and the number becomes 8.35.
- For a second example, let's round 3.6543 to the third decimal place. Therefore, you look at

the fourth decimal position, which is a 3. As the rule says, you would leave the digit alone and the number becomes 3.654

## NONTERMINATING DECIMALS

---

What happens when you perform a calculation and the decimal doesn't terminate?

1. You need to assess if there is a pattern in the decimals:

- **The Nonterminating Decimal without a Pattern:**

For example,  $\frac{6}{17} = 0.352941176$  with no apparent ending decimal and no pattern to the decimals.

- **The Nonterminating Decimal with a Pattern:**

For example,  $\frac{2}{11} = 0.18181818$  endlessly. You can see that the numbers 1 and 8 repeat. A shorthand way of expressing this is to place a horizontal line above the digits that repeat. Thus, you can rewrite 0.18181818 as  $0.\overline{18}$ .

2. You need to know if the number represents an interim or final solution to a problem:

- **Interim Solution**

You must carry forward all of the decimals in your calculations, as the number should not be rounded until you arrive at a final answer. If you are completing the question by hand, write out as many decimals as possible; to save space and time, you can use the shorthand horizontal bar for repeating decimals. If you are completing the question by calculator, store the entire number in a memory cell.

- **Final Solution**

To round this number off, an industry protocol or other clear instruction must apply. If these do not exist, then you would make an arbitrary rounding choice, subject to the condition that you must maintain enough precision to allow for reasonable interpretation of the information.

### Example 1.2.5

Convert the following to decimals. Round each to four decimals or use the repeating decimal notation.

a.  $\frac{6}{13}$

- b.  $\frac{4}{9}$
- c.  $\frac{4}{11}$
- d.  $\frac{3}{22}$
- e.  $5\frac{1}{7}$   
 $\frac{10}{27}$

**Solutions****Question a.**

**Step 1: Divide to convert to decimal.**

$$\frac{6}{13} = 0.461538$$

**Step 2: Round and write as a statement.**

The fifth decimal is a 3, so round down.

**Step 3: Write as a statement.**

The answer is 0.4615.

**Question b.**

**Step 1: Divide to convert to decimal.**

$$\frac{4}{9} = 0.444444$$

**Step 2: Round and write as a statement.**

Note the repeating decimal of 4.

**Step 3: Write as a statement.**

Using the horizontal bar, write  $0.\overline{4}$ .

**Question c.**

---

**Step 1: Divide to convert to decimal.**

$$\frac{4}{11} = 0.363636$$

**Step 2: Round and write as a statement.**

Note the repeating decimals of **3** and **6**.

**Step 3: Write as a statement.**

Using the horizontal bar, write  $0.\overline{36}$ .

**Question d.**

---

**Step 1: Divide to convert to decimal.**

$$\frac{3}{22} = 0.136363$$

**Step 2: Round and write as a statement.**

Note the repeating decimals of **3** and **6** after the **1**.

**Step 3: Write as a statement.**

Using the horizontal bar, write  $0.1\overline{36}$ .

**Question e.**

---

**Step 1: Divide to convert to decimal.**

$$\begin{aligned}
 & 5\frac{\frac{1}{7}}{\frac{10}{27}} \\
 &= 5 + \frac{(1 \div 7)}{(10 \div 27)} \\
 &= 5 + \frac{0.142857}{0.\overline{370}} \\
 &= 5 + 0.385714 \\
 &= 5.385714
 \end{aligned}$$

**Step 2: Round and write as a statement.**

Since the fifth digit is a 1, round down.

**Step 3: Write as a statement.**

The answer is 5.3857.

## SECTION EXERCISES

### Mechanics Exercises

1. For each of the following, identify the type of fraction presented.

a.  $\frac{1}{8}$

b.  $3\frac{3}{4}$

c.  $\frac{10}{9}$

d.  $\frac{34}{49}$

e.  $1\frac{\frac{2}{3}}{9\frac{5}{3}}$

f.  $\frac{56}{27}$

g.  $\frac{10\frac{1}{5}}{9}$

h.  $\frac{6}{11}$

2. In each of the following equations, identify the value of the unknown term.

a.  $\frac{3}{4} = \frac{x}{36}$

b.  $\frac{y}{8} = \frac{16}{64}$

c.  $\frac{2}{z} = \frac{18}{45}$

d.  $\frac{5}{6} = \frac{75}{p}$

3. Take each of the following fractions and provide one example of the fraction expressed in both higher and lower terms.

a.  $\frac{5}{10}$

b.  $\frac{6}{8}$

4. Convert each of the following fractions into decimal format.

a.  $\frac{7}{8}$

b.  $15\frac{5}{4}$

c.  $\frac{13}{5}$

d.  $133\frac{\frac{17}{2}}{3\frac{2}{5}}$

5. Convert each of the following fractions into decimal format and round to three decimals.

a.  $\frac{7}{8}$

b.  $15\frac{3}{4}$

c.  $\frac{10}{9}$

d.  $\frac{15}{32}$

6. Convert each of the following fractions into decimal format and express in repeating decimal notation.

a.  $\frac{1}{12}$

b.  $5\frac{8}{33}$

c.  $\frac{4}{3}$

d.  $\frac{-34}{110}$

**Solutions**

1a. proper

1b. mixed number

1c. improper

1d. proper

1e. complex

1f. improper

1g. complex

1h. proper

2a.  $\frac{3}{4} = \frac{27}{36}$

2b.  $\frac{2}{8} = \frac{16}{64}$

2c.  $\frac{2}{5} = \frac{18}{45}$

$$2d. \frac{5}{6} = \frac{75}{90}$$

$$3a. \frac{5 \times 2}{10 \times 2} = \frac{10}{20} \quad \frac{5 \div 5}{10 \div 5} = \frac{1}{2}$$

$$3b. \frac{6 \times 5}{8 \times 5} = \frac{30}{40} \quad \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

$$4a. 0.875$$

$$4b. 16.25$$

$$4c. 2.6$$

$$4d. 137.\overline{72}$$

$$5a. 0.875$$

$$5b. 15.750$$

$$5c. 1.111$$

$$5d. 0.469$$

$$6a. 0.08\overline{3}$$

$$6b. 5.\overline{24}$$

$$6c. 1.\overline{3}$$

$$6d. -0.30\overline{9}$$

### Applications Exercises

---

7. Calculate the solution to each of the following expressions. Express your answer in decimal format.

$$a. \frac{1}{5} + 3\frac{1}{4} + \frac{5}{2}$$

$$b. 1\frac{\frac{3}{8}}{2} - \frac{11}{40} + 19\frac{1}{2} \times \frac{3}{4}$$

8. Calculate the solution to each of the following expressions. Express your answer in decimal format with two decimals.

a.  $\left(1 + \frac{0.11}{12}\right)^4$

b.  $1 - 0.05 \times \frac{263}{365}$

c.  $200 \left[ 1 - \frac{1}{\left(1 + \frac{0.10}{4}\right)^2} \right]$

9. Calculate the solution to each of the following expressions. Express your answer in repeating decimal notation as needed.

a.  $\frac{1}{11} + 3\frac{1}{9}$

b.  $\frac{5}{3} - \frac{7}{6}$

Questions 10–14 involve fractions. For each, evaluate the expression and round your answer to the nearest cent.

10.  $\$134,000(1 + 0.14 \times 23/365)$

11.  $\$10,000 \left(1 + \frac{0.0525}{2}\right)^{13}$

12.  $\frac{\$535,000}{\left(1 + \frac{0.07}{12}\right)^3}$

13.  $\$2,995 \left(1 + 0.13 \times \frac{90}{365}\right) - \frac{\$400}{1 + 0.13 \times \frac{15}{365}}$

14.  $\frac{\$155,600}{\left(1 + \frac{0.06}{12}\right)^8}$

### Solutions

7a. 5.95

7b. 15.0375

8a. 1.04

8b. 0.96

8c. 9.64

9a.  $3.\overline{20}$

9b. 0.5

10. \$135,182.14

11. \$14,005.26

12. \$525,745.68

13. \$2,693.13

14. \$149,513.74

### Challenge, Critical Thinking, & Other Applications Exercises

---

Questions 15–20 involve more complex fractions and reflect business math equations encountered later in this textbook. For each, evaluate the expression and round your answer to the nearest cent.

$$15. \frac{\$648}{0.0575/12} \left[ 1 - \frac{1}{\left(1 + \frac{0.0575}{12}\right)^7} \right]$$

$$16. \frac{\$10,000}{\left(1 + \frac{0.115}{4}\right)^2} + \$68 \frac{\left[ 1 - \frac{1}{\left(1 + \frac{0.115}{4}\right)^2} \right]}{\frac{0.115}{4}}$$

$$17. \frac{\$2,000,000}{\left[ \frac{\left(1 + \frac{0.065}{2}\right)^{12} - 1}{\frac{0.065}{2}} \right]}$$

$$18. \$8,500 \left[ \frac{1 - \left(\frac{1}{1.08}\right)^4}{1.08} \right] + \$19,750 \left(\frac{1}{1.08}\right)^4 - \$4,350$$

$$19. \quad \$15,000 \left[ \frac{\left(1 + \frac{0.058}{4}\right)^{16} - 1}{\frac{0.058}{4}} \right]$$

$$20. \quad \frac{0.08}{2}(\$1,000) \left[ \frac{1 - \frac{1}{\left(1 + \frac{0.07}{2}\right)^{10}}}{\frac{0.07}{2}} \right] + \$1,000 \frac{1}{\left(1 + \frac{0.07}{2}\right)^{10}}$$

### Solutions

- 15. \$4,450.29
- 16. \$9,579.23
- 17. \$138,934.38
- 18. \$12,252.25
- 19. \$267,952.30
- 20. \$1,041.58

### ATTRIBUTION

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## 1. 3 Rules for Exponents

### Learning Objectives

- Product and Quotient Rules
  - Use the product rule to multiply exponential expressions
  - Use the quotient rule to divide exponential expressions
- The Power Rule for Exponents
  - Use the power rule to simplify expressions involving products, quotients, and exponents
- Negative and Zero Exponents
  - Define and use the zero exponent rule
  - Define and use the negative exponent rule
- Simplify Expressions Using the Exponent Rules
  - Simplify expressions using a combination of the exponent rules
  - Simplify compound exponential expressions with negative exponents



Repeated Image

## ANATOMY OF EXPONENTIAL TERMS

We use exponential notation to write repeated multiplication. For example  $10 \cdot 10 \cdot 10$  can be written more succinctly as  $10^3$ . The 10 in  $10^3$  is called the **base**. The 3 in  $10^3$  is called the **exponent**. The expression  $10^3$  is called the exponential expression. Knowing the names for the parts of an exponential expression or term will help you learn how to perform mathematical operations on them.

$$\text{base} \rightarrow 10^{\text{3} \leftarrow \text{exponent}}$$

$10^3$  is read as “10 to the third power” or “10 cubed.” It means  $10 \cdot 10 \cdot 10$ , or 1,000.

$8^2$  is read as “8 to the second power” or “8 squared.” It means  $8 \cdot 8$ , or 64.

$5^4$  is read as “5 to the fourth power.” It means  $5 \cdot 5 \cdot 5 \cdot 5$ , or 625.

$b^5$  is read as “b to the fifth power.” It means  $b \cdot b \cdot b \cdot b \cdot b$ . Its value will depend on the value of  $b$ .

The exponent applies only to the number that it is next to. Therefore, in the expression  $xy^4$ , only the  $y$  is affected by the 4.  $xy^4$  means  $x \cdot y \cdot y \cdot y \cdot y$ . The  $x$  in this term is a **coefficient** of  $y$ .

If the exponential expression is negative, such as  $-3^4$ , it means  $-(3 \cdot 3 \cdot 3 \cdot 3)$  or  $-81$ .

If  $-3$  is to be the base, it must be written as  $(-3)^4$ , which means  $-3 \cdot -3 \cdot -3 \cdot -3$ , or 81.

Likewise,  $(-x)^4 = (-x) \cdot (-x) \cdot (-x) \cdot (-x) = x^4$ , while  $-x^4 = -(x \cdot x \cdot x \cdot x)$ .

You can see that there is quite a difference, so you have to be very careful! The following examples show how to identify the base and the exponent, as well as how to identify the expanded and exponential format of writing repeated multiplication.

### Examples

Identify the exponent and the base in the following terms, then simplify:

1.  $7^2$

2.  $\left(\frac{1}{2}\right)^3$

3.  $2x^3$

4.  $(-5)^2$

### Solution

1)  $7^2$

The exponent in this term is 2 and the base is 7. To simplify, expand the term:  $7^2 = 7 \cdot 7 = 49$

2)  $\left(\frac{1}{2}\right)^3$

The exponent on this term is 3, and the base is  $\frac{1}{2}$ . To simplify, expand the multiplication and remember how

to multiply fractions:  $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

3)  $2x^3$

The exponent on this term is 3, and the base is x, the 2 is not getting the exponent because there are no parentheses that tell us it is. This term is in its most simplified form.

4)  $(-5)^2$

The exponent on this terms is 2 and the base is  $-5$ . To simplify, expand the multiplication:

$(-5)^2 = -5 \cdot -5 = 25$

In the following video you are provided more examples of applying exponents to various bases.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-1>

## Evaluate expressions

Evaluating expressions containing exponents is the same as evaluating the linear expressions from earlier in the course. You substitute the value of the variable into the expression and simplify.

You can use the order of operations to evaluate the expressions containing exponents. First, evaluate anything in Parentheses or grouping symbols. Next, look for Exponents, followed by Multiplication and Division (reading from left to right), and lastly, Addition and Subtraction (again, reading from left to right).

So, when you evaluate the expression  $5x^3$  if  $x = 4$ , first substitute the value 4 for the variable  $x$ . Then evaluate, using order of operations.

### Examples

Evaluate  $5x^3$  if  $x = 4$ .

#### **Solution**

Substitute 4 for the variable  $x$ .

$$5 \cdot 4^3$$

Evaluate  $4^3$ . Multiply.

$$5(4 \cdot 4 \cdot 4) = 5 \cdot 64 = 320$$

#### *Answer*

$$5x^3 = 320 \text{ when } x = 4$$

In the example below, notice the how adding parentheses can change the outcome when you are simplifying terms with exponents.

## Examples

Evaluate  $(5x)^3$  if  $x = 4$ .

**Solution**

$$(5 \cdot 4)^3$$

Multiply inside the parentheses, then apply the exponent—following the rules of PEMDAS.

$$20^3$$

Evaluate  $20^3$ .

$$20 \cdot 20 \cdot 20 = 8,000$$

*Answer*

$$(5x)^3 = 8,000 \text{ when } x = 4$$

The addition of parentheses made quite a difference! Parentheses allow you to apply an exponent to variables or numbers that are multiplied, divided, added, or subtracted to each other.

## Examples

Evaluate  $x^3$  if  $x = -4$ .

**Solution**

Substitute  $-4$  for the variable  $x$ .

$$(-4)^3$$

Evaluate. Note how placing parentheses around the  $-4$  means the negative sign also gets multiplied.

$$-4 \cdot -4 \cdot -4$$

Multiply.

$$-4 \cdot -4 \cdot -4 = -64$$

*Answer*

$$x^3 = -64 \text{ when } x = -4$$

**CAUTION!**

Whether to include a negative sign as part of a base or not often leads to confusion. To clarify whether a negative sign is applied before or after the exponent, here is an example.

What is the difference in the way you would evaluate these two terms?

1.  $-3^2$

2.  $(-3)^2$

To evaluate 1), you would apply the exponent to the three first, then apply the negative sign last, like this:

$$\begin{aligned} & -(3^2) \\ &= -(9) = -9 \end{aligned}$$

To evaluate 2), you would apply the exponent to the 3 and the negative sign:

$$\begin{aligned} & (-3)^2 \\ &= (-3) \cdot (-3) \\ &= 9 \end{aligned}$$

The key to remembering this is to follow the order of operations. The first expression does not include parentheses so you would apply the exponent to the integer 3 first, then apply the negative sign. The second expression includes parentheses, so hopefully you will remember that the negative sign also gets squared.

In the next sections, you will learn how to simplify expressions that contain exponents. Come back to this page if you forget how to apply the order of operations to a term with exponents, or forget which is the base and which is the exponent!

In the following video you are provided with examples of evaluating exponential expressions for a given number.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-2>

## USE THE PRODUCT RULE TO MULTIPLY EXPONENTIAL EXPRESSIONS

**Exponential notation** was developed to write repeated multiplication more efficiently. There are times when it is easier or faster to leave the expressions in exponential notation when multiplying or dividing. Let's look at rules that will allow you to do this.

For example, the notation  $5^4$  can be expanded and written as  $5 \cdot 5 \cdot 5 \cdot 5$ , or 625. And don't forget, the exponent only applies to the number immediately to its left, unless there are parentheses.

What happens if you multiply two numbers in exponential form with the same base? Consider the expression  $2^3 2^4$ . Expanding each exponent, this can be rewritten as  $(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)$  or  $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ . In exponential form, you would write the product as  $2^7$ . Notice that 7 is the sum of the original two exponents, 3 and 4.

What about  $x^2 x^6$ ? This can be written as  $(x \cdot x)(x \cdot x \cdot x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$  or  $x^8$ . And, once again, 8 is the sum of the original two exponents. This concept can be generalized in the following way:

### The Product Rule for Exponents

For any number  $x$  and any integers  $a$  and  $b$ ,  $(x^a)(x^b) = x^{a+b}$ .

To multiply exponential terms with the same base, add the exponents.



**CAUTION!**

When you are reading mathematical rules, it is important to pay attention to the conditions on the rule. For example, when using the product rule, you may only apply it when the terms being multiplied have the same base and the exponents are integers. Conditions on mathematical rules are often given before the rule is stated, as in this example it says "For any number  $x$ , and any integers  $a$  and  $b$ ."

#### Examples

Simplify.

$$(a^3)(a^7)$$

#### **Solution**

The base of both exponents is  $a$ , so the product rule applies.

$$(a^3)(a^7)$$

Add the exponents with a common base.

$$a^{3+7}$$

*Answer*

$$(a^3)(a^7) = a^{10}$$

When multiplying more complicated terms, multiply the coefficients and then multiply the variables.

#### Examples

Simplify.

$$5a^4 \cdot 7a^6$$

**Solution**

Multiply the coefficients.

$$35 \cdot a^4 \cdot a^6$$

The base of both exponents is  $a$ , so the product rule applies. Add the exponents.

$$35 \cdot a^{4+6}$$

Add the exponents with a common base.

$$35 \cdot a^{10}$$

*Answer*

$$5a^4 \cdot 7a^6 = 35a^{10}$$



#### CAUTION!

Do not try to apply this rule to sums.

Think about the expression  $(2 + 3)^2$

Does  $(2 + 3)^2$  equal  $2^2 + 3^2$ ?

No, it does not because of the order of operations!

$$(2 + 3)^2 = 5^2 = 25$$

and

$$2^2 + 3^2 = 4 + 9 = 13$$

Therefore, you can only use this rule when the numbers inside the parentheses are being multiplied (or divided, as we will see next).



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-3>

## USE THE QUOTIENT RULE TO DIVIDE EXPONENTIAL EXPRESSIONS

Let's look at dividing terms containing exponential expressions. What happens if you divide two numbers in exponential form with the same base? Consider the following expression.

$$\frac{4^5}{4^2}$$

You can rewrite the expression as:  $\frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4}$ . Then you can cancel the common factors of 4 in the numerator and denominator: **Formula does not parse**

Finally, this expression can be rewritten as  $4^3$  using exponential notation. Notice that the exponent, 3, is the difference between the two exponents in the original expression, 5 and 2.

$$\text{So, } \frac{4^5}{4^2} = 4^{5-2} = 4^3.$$

Be careful that you subtract the exponent in the denominator from the exponent in the numerator.

So, to divide two exponential terms with the same base, subtract the exponents.

## THE QUOTIENT (DIVISION) RULE FOR EXPONENTS

For any non-zero number  $x$  and any integers  $a$  and  $b$ :  $\frac{x^a}{x^b} = x^{a-b}$

## Examples

Evaluate.  $\frac{4^9}{4^4}$

**Solution**

These two exponents have the same base, 4. According to the Quotient Rule, you can subtract the power in the denominator from the power in the numerator.

$$4^{9-4}$$

*Answer*

$$\frac{4^9}{4^4} = 4^5$$

[/hidden-answer]

When dividing terms that also contain coefficients, divide the coefficients and then divide variable powers with the same base by subtracting the exponents.

## Examples

Simplify.  $\frac{12x^4}{2x}$

**Solution**

Separate into numerical and variable factors.

$$\left(\frac{12}{2}\right) \left(\frac{x^4}{x}\right)$$

Since the bases of the exponents are the same, you can apply the Quotient Rule. Divide the coefficients and subtract the exponents of matching variables.

$$6(x^{4-1})$$

*Answer*

$$\frac{12x^4}{2x} = 6x^3$$

In the following video we show another example of how to use the quotient rule to divide exponential expressions



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nscc.ca/businessmath2500/?p=1905#oembed-4>

## RAISE POWERS TO POWERS

Another word for exponent is power. You have likely seen or heard an example such as  $3^5$  can be described as 3 raised to the 5th power. In this section we will further expand our capabilities with exponents. We will learn what to do when a term with a power is raised to another power, and what to do when two numbers or variables are multiplied and both are raised to an exponent. We will also learn what to do when numbers or variables that are divided are raised to a power. We will begin by raising powers to powers.

Let's simplify  $(5^2)^4$ . In this case, the base is  $5^2$  and the exponent is 4, so you multiply  $5^2$  four times:

$$(5^2)^4 = 5^2 \cdot 5^2 \cdot 5^2 \cdot 5^2 = 5^8 \text{ (using the Product Rule—add the exponents).}$$

$(5^2)^4$  is a power of a power. It is the fourth power of 5 to the second power. And we saw above that the answer is  $5^8$ . Notice that the new exponent is the same as the product of the original exponents:  $2 \cdot 4 = 8$ .

So,  $(5^2)^4 = 5^{2 \cdot 4} = 5^8$  (which equals 390,625, if you do the multiplication).

$$\text{Likewise, } (x^4)^3 = x^{4 \cdot 3} = x^{12}$$

This leads to another rule for exponents—the **Power Rule for Exponents**. To simplify a power of a power, you multiply the exponents, keeping the base the same. For example,  $(2^3)^5 = 2^{15}$ .

## THE POWER RULE FOR EXPONENTS

---

For any positive number  $x$  and integers  $a$  and  $b$ :  $(x^a)^b = x^{a \cdot b}$ .

Take a moment to contrast how this is different from the product rule for exponents found on the previous page.

### Examples

Simplify  $6(c^4)^2$ .

#### Solution

Since you are raising a power to a power, apply the Power Rule and multiply exponents to simplify. The coefficient remains unchanged because it is outside of the parentheses.

$$6(c^4)^2$$

#### Answer

---

$$6(c^4)^2 = 6c^8$$

## Raise a product to a power

---

Simplify this expression.

$$(2a)^4 = (2a)(2a)(2a)(2a) = (2 \cdot 2 \cdot 2 \cdot 2)(a \cdot a \cdot a \cdot a) = (2^4)(a^4) = 16a^4$$

Notice that the exponent is applied to each factor of  $2a$ . So, we can eliminate the middle steps.

$$\begin{aligned}(2a)^4 &= (2^4)(a^4), \text{ applying the 4 to each factor, 2 and } a \\ &= 16a^4\end{aligned}$$

The product of two or more numbers raised to a power is equal to the product of each number raised to the same power.

## A Product Raised to a Power

---

For any nonzero numbers  $a$  and  $b$  and any integer  $x$ ,  $(ab)^x = a^x \cdot b^x$ .

How is this rule different from the power raised to a power rule? How is it different from the product rule for exponents on the previous page?

#### Examples

Simplify.  $(2yz)^6$

#### Solution

Apply the exponent to each number in the product.

$$2^6 y^6 z^6$$

#### Answer

$$(2yz)^6 = 64y^6 z^6$$

If the variable has an exponent with it, use the Power Rule: multiply the exponents.

#### Examples

Simplify.  $(-7a^4b)^2$

#### Solution

Apply the exponent 2 to each factor within the parentheses.

$$(-7)^2 (a^4)^2 (b)^2$$

Square the coefficient and use the Power Rule to square  $(a^4)^2$ .

$$49a^{4 \cdot 2} b^2$$

Simplify.

$$49a^8 b^2$$

#### Answer

$$(-7a^4b)^2 = 49a^8 b^2$$



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-5>

## Raise a quotient to a power

Now let's look at what happens if you raise a quotient to a power. Remember that quotient means divide. Suppose you have  $\frac{3}{4}$  and raise it to the 3<sup>rd</sup> power.

$$\left(\frac{3}{4}\right)^3 = \left(\frac{3}{4}\right) \left(\frac{3}{4}\right) \left(\frac{3}{4}\right) = \frac{3 \cdot 3 \cdot 3}{4 \cdot 4 \cdot 4} = \frac{3^3}{4^3}$$

You can see that raising the quotient to the power of 3 can also be written as the numerator (3) to the power of 3, and the denominator (4) to the power of 3.

Similarly, if you are using variables, the quotient raised to a power is equal to the numerator raised to the power over the denominator raised to power.

$$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right) \left(\frac{a}{b}\right) \left(\frac{a}{b}\right) \left(\frac{a}{b}\right) = \frac{a \cdot a \cdot a \cdot a}{b \cdot b \cdot b \cdot b} = \frac{a^4}{b^4}$$

When a quotient is raised to a power, you can apply the power to the numerator and denominator individually, as shown below.

$$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$$

## A Quotient Raised to a Power

For any number  $a$ , any non-zero number  $b$ , and any integer  $x$ ,  $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ .

Examples

Simplify.  $\left(\frac{2x^2y}{x}\right)^3$

**Solution**

Apply the power to each factor individually.

$$\frac{2^3 (x^2)^3 y^3}{x^3}$$

Separate into numerical and variable factors.

$$2^3 \cdot \frac{x^{3 \cdot 2}}{x^3} \cdot \frac{y^3}{1}$$

Simplify by taking 2 to the third power and applying the Power and Quotient Rules for exponents—multiply and subtract the exponents of matching variables.

$$8 \cdot x^{(6-3)} \cdot y^3$$

Simplify.

$$8x^3y^3$$

**Answer**

$$\left(\frac{2x^2y}{x}\right)^3 = 8x^3y^3$$

In the following video you will be shown examples of simplifying quotients that are raised to a power.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-6>

## DEFINE AND USE THE ZERO EXPONENT RULE

---

When we defined the quotient rule, we only worked with expressions like the following:  $\frac{4^9}{4^4}$ , where the exponent in the numerator (up) was greater than the one in the denominator (down), so the final exponent after simplifying was always a positive number, and greater than zero. In this section, we will explore what happens when we apply the quotient rule for exponents and get a negative or zero exponent.

### What if the exponent is zero?

---

To see how this is defined, let us begin with an example. We will use the idea that dividing any number by itself gives a result of 1.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is  $t^0 = 1$ . This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression  $0^0$ . This appears later in more advanced courses, but for now, we will consider the value to be undefined, or DNE (Does Not Exist).

### Exponents of 0 or 1

---

Any number or variable raised to a power of 1 is the number itself.

$$n^1 = n$$

Any non-zero number or variable raised to a power of 0 is equal to 1

$$n^0 = 1$$

The quantity  $0^0$  is undefined.

As done previously, to evaluate expressions containing exponents of 0 or 1, substitute the value of the variable into the expression and simplify.

## Examples

Evaluate  $2x^0$  if  $x = 9$

**Solution**

Substitute 9 for the variable  $x$ .

$$2 \cdot 9^0$$

Evaluate  $9^0$ . Multiply.

$$2 \cdot 1 = 2$$

*Answer*

$$2x^0 = 2, \text{ if } x = 9$$

## Examples

Simplify  $\frac{c^3}{c^3}$ .

**Solution**

Use the quotient and zero exponent rules to simplify the expression.

$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3} \\ &= c^0 \\ &= 1\end{aligned}$$

*Answer*

$$1$$

In the following video there is an example of evaluating an expression with an exponent of zero, as well as simplifying when you get a result of a zero exponent.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-7>

## DEFINE AND USE THE NEGATIVE EXPONENT RULE

We proposed another question at the beginning of this section. Given a quotient like  $\frac{2^m}{2^n}$  what happens when  $n$  is larger than  $m$ ? We will need to use the *negative rule of exponents* to simplify the expression so that it is easier to understand.

Let's look at an example to clarify this idea. Given the expression:

$$\frac{h^3}{h^5}$$

Expand the numerator and denominator, all the terms in the numerator will cancel to 1, leaving two  $h$ s multiplied in the denominator, and a numerator of 1.

$$\begin{aligned}\frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\ &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} \\ &= \frac{1}{h \cdot h} \\ &= \frac{1}{h^2}\end{aligned}$$

We could have also applied the quotient rule from the last section, to obtain the following result:

$$\begin{aligned}\frac{h^3}{h^5} &= h^{3-5} \\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have  $h^{-2} = \frac{1}{h^2}$ . This is true when  $h$ , or any variable, is a real number and is not zero.

## The Negative Rule of Exponents

For any nonzero real number  $a$  and natural number  $n$ , the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

Let's look at some examples of how this rule applies under different circumstances.

### Examples

Evaluate the expression  $4^{-3}$ .

#### Solution

First, write the expression with positive exponents by putting the term with the negative exponent in the denominator.

$$4^{-3} = \frac{1}{4^3} = \frac{1}{4 \cdot 4 \cdot 4}$$

Now that we have an expression that looks somewhat familiar.

$$\frac{1}{4 \cdot 4 \cdot 4} = \frac{1}{64}$$

#### Answer

$$\frac{1}{64}$$

### Examples

Write  $\frac{(t^3)}{(t^8)}$  with positive exponents.

#### Solution

Use the quotient rule to subtract the exponents of terms with like bases.

$$\begin{aligned}\frac{(t^3)}{(t^8)} &= t^{3-8} \\ &= t^{-5}\end{aligned}$$

Write the expression with positive exponents by putting the term with the negative exponent in the denominator.

$$= \frac{1}{t^5}$$

*Answer*

---

$$\frac{1}{t^5}$$

Examples

Simplify  $\left(\frac{1}{3}\right)^{-2}$ .

**Solution**

Apply the power property of exponents.

$$\frac{1^{-2}}{3^{-2}}$$

Write each term with a positive exponent, the numerator will go to the denominator and the denominator will go to the numerator.

$$\frac{3^2}{1^2} = \frac{3 \cdot 3}{1 \cdot 1}$$

Simplify.

$$\frac{3 \cdot 3}{1 \cdot 1} = \frac{9}{1} = 9$$

*Answer*

---

$$9$$

Examples

Simplify.  $\frac{1}{4^{-2}}$  Write your answer using positive exponents.

**Solution**

Write each term with a positive exponent, the denominator will go to the numerator.

$$\frac{1}{4^{-2}} = 1 \cdot \frac{4^2}{1} = \frac{16}{1} = 16$$

**Answer**

16

In the following video you will see examples of simplifying expressions with negative exponents.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1905#oembed-8>

## SIMPLIFY EXPRESSIONS USING A COMBINATION OF EXPONENT RULES

Once the rules of exponents are understood, you can begin simplifying more complicated expressions. There are many applications and formulas that make use of exponents, and sometimes expressions can get pretty cluttered. Simplifying an expression before evaluating can often make the computation easier, as you will see in the following example which makes use of the quotient rule to simplify before substituting 4 for  $x$ .

**Examples**

Evaluate  $\frac{24x^8}{2x^5}$  when  $x = 4$ .

**Solution**

Separate into numerical and variable factors.

$$\left(\frac{24}{2}\right) \left(\frac{x^8}{x^5}\right)$$

Divide coefficients, and subtract the exponents of the variables.

$$12(x^{8-5})$$

Simplify.

$$12x^3$$

Substitute the value 4 for the variable  $x$ .

$$(12)(4^3) = 12 \cdot 64$$

*Answer*

$$\frac{24x^8}{2x^5} = 768$$

### Examples

Evaluate  $\frac{24x^8y^2}{(2x^3y)^2}$  when  $x = 4$  and  $y = -2$ .

**Solution**

**In the denominator, notice that a product is being raised to a power.**

Use the rules of exponents to simplify the denominator.

$$(2x^3y)^2 = 2^2x^{3 \cdot 2}y^2 = 2^2x^6y^2 = 4x^6y^2$$

Here is the fraction with a simplified denominator:

$$\frac{24x^8y^2}{4x^6y^2}$$

Separate into numerical and variable factors to simplify further.

$$\left(\frac{24}{4}\right) \left(\frac{x^8}{x^6}\right) \left(\frac{y^2}{y^2}\right)$$

Divide coefficients, use the Quotient Rule to divide the variables—subtract the exponents.

$$6(x^{8-6})(y^{2-2})$$

Simplify. Remember that  $y^0$  is 1.

$$6x^2y^0 = 6x^2$$

Substitute the value 4 for the variable  $x$ .

$$(6)(4^2) = 6 \cdot 16$$

*Answer*

$$\frac{24x^8y^2}{(2x^3y)^2} = 96 \text{ when } x = 4 \text{ and } y = -2$$

Notice that you could have worked this problem by substituting 4 for  $x$  and 2 for  $y$  in the original expression. You would still get the answer of 96, but the computation would be much more complex. Notice that you didn't even need to use the value of  $y$  to evaluate the above expression.

In the following video you are shown examples of evaluating an exponential expression for given numbers.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nscc.ca/businessmath2500/?p=1905#oembed-9>

Usually, it is easier to simplify the expression before substituting any values for your variables, but you will get the same answer either way. In the next examples, you will see how to simplify expressions using different combinations of the rules for exponents.

Examples

Simplify.  $a^2 (a^5)^3$

**Solution**

**Raise  $a^5$  to the power of 3 by multiplying the exponents together (the Power Rule).**

$$a^2 a^{5 \cdot 3}$$

Since the exponents share the same base,  $a$ , they can be combined (the Product Rule).

$$\begin{aligned} a^2 a^{15} \\ a^{2+15} \end{aligned}$$

*Answer*

$$a^2 (a^5)^3 = a^{17}$$

The following examples require the use of all the exponent rules we have learned so far. Remember that the product, power, and quotient rules apply when your terms have the same base.

Examples

Simplify.  $\frac{a^2 (a^5)^3}{8a^8}$

**Solution**

Use the order of operations.

Parentheses, Exponents, Multiply/ Divide, Add/ Subtract

There is nothing inside parentheses or brackets that we can simplify further, so we will evaluate exponents first.

Use the Power Rule to simplify  $(a^5)^3$ .

$$(a^5)^3 = a^{5 \cdot 3} = a^{15}$$

The expression now looks like this:

$$\frac{a^2 a^{15}}{8a^8}$$

Now we can multiply, using the Product Rule to simplify the numerator because the bases are the same.

$a^2 a^{15} = a^{17}$ , and the expression looks like this:

$$\frac{a^{17}}{8a^8}$$

Now we can divide using the Quotient Rule.

$$\frac{a^{17-8}}{8}$$

*Answer*

$$\frac{a^2 (a^5)^3}{8a^8} = \frac{a^9}{8}$$

## SIMPLIFY EXPRESSIONS WITH NEGATIVE EXPONENTS

Now we will add the last layer to our exponent simplifying skills and practice simplifying compound expressions that have negative exponents in them. It is standard convention to write exponents as positive because it is easier for the user to understand the value associated with positive exponents, rather than negative exponents.

Use the following summary of negative exponents to help you simplify expressions with negative exponents.

### Rules for Negative Exponents

With  $a$ ,  $b$ ,  $m$ , and  $n$  not equal to zero, and  $m$  and  $n$  as integers, the following rules apply:

$$a^{-m} = \frac{1}{a^m}$$

$$\frac{1}{a^{-m}} = a^m$$

$$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$$

When you are simplifying expressions that have many layers of exponents, it is often hard to know where to start. It is common to start in one of two ways:

- Rewrite negative exponents as positive exponents

- Apply the product rule to eliminate any “outer” layer exponents such as in the following term:  
 $(5y^3)^2$

We will explore this idea with the following example:

Simplify.  $(4x^3)^5 \cdot (2x^2)^{-4}$

Write your answer with positive exponents. The table below shows how to simplify the same expression in two different ways, rewriting negative exponents as positive first, and applying the product rule for exponents first. You will see that there is a column for each method that describes the exponent rule or other steps taken to simplify the expression.

Rewrite with positive Exponents First	Description of Steps Taken	Apply the Product Rule for Exponents First	Description of Steps Taken
$\frac{(4x^3)^5}{(2x^2)^4}$	move the term $(2x^2)^{-4}$ to the denominator with a positive exponent	$(4^5 x^{15}) (2^{-4} x^{-8})$	Apply the exponent of 5 to each term in expression on the left, and the exponent of -4 to each term in the expression on the right.
$\frac{(4^5 x^{15})}{(2^4 x^8)}$	Use the product rule to apply the outer exponents to the terms inside each set of parentheses.	$(4^5) (2^{-4}) (x^{15} \cdot x^{-8})$	Regroup the numerical terms and the variables to make combining like terms easier
$\left(\frac{4^5}{2^4}\right) \left(\frac{x^{15}}{x^8}\right)$	Regroup the numerical terms and the variables to make combining like terms easier	$(4^5) (2^{-4}) (x^{15-8})$	Use the rule for multiplying terms with exponents to simplify the x terms
$\left(\frac{4^5}{2^4}\right) (x^{15-8})$	Use the quotient rule to simplify the x terms	$\left(\frac{4^5}{2^4}\right) (x^7)$	Rewrite all the negative exponents with positive exponents
$\left(\frac{1,024}{16}\right) (x^7)$	Expand the numerical terms	$\left(\frac{1,024}{16}\right) (x^7)$	Expand the numerical terms
$64x^7$	Divide the numerical terms	$64x^7$	Divide the numerical terms

If you compare the two columns that describe the steps that were taken to simplify the expression, you will see that they are all nearly the same, except the order is changed slightly. Neither way is better or more correct than the other, it truly is a matter of preference.

#### Examples

Simplify  $\frac{(t^3)^2}{(t^2)^{-8}}$

Write your answer with positive exponents.

**Solution**

We can either rewrite this expression with positive exponents first or use the Product Raised to a Power Rule first.

Let's start by simplifying the numerator and denominator using the Product Raised to a Power Rule.

$$\text{Numerator: } (t^3)^2 = t^{3 \cdot 2} = t^6$$

$$\text{Denominator: } (t^2)^{-8} = t^{2 \cdot -8} = t^{-16}$$

Now the expression looks like this:

$$\frac{t^6}{t^{-16}}$$

We can use the quotient rule because we have the same base.

$$\text{Quotient Rule: } \frac{t^6}{t^{-16}} = t^{6 - (-16)} = t^{6+16} = t^{22}$$

**Answer**

$$\frac{(t^3)^2}{(t^2)^{-8}} = t^{22}$$

**Examples**

$$\text{Simplify } \frac{(5x)^{-2}y}{x^3y^{-1}}$$

Write your answer with positive exponents.

**Solution**

This time, let's start by rewriting the terms in the expression so they have positive exponents. The terms with negative exponents in the top will go to the bottom of the fraction, and the terms with negative exponents in the bottom will go to the top.

$$\frac{(5x)^{-2}y}{x^3y^{-1}}$$

$$= \frac{(y^1)y}{x^3(5x)^2}$$

Note how we left the single  $y$  term in the top because it did not have a negative exponent on it, and we left the  $x^3$  term in the bottom because it did not have a negative exponent on it.

Now we can apply the Product Raised to a Power Rule:

$$\frac{yy^1}{5^2x^3x^2}$$

Use the product rule to simplify further:

$$\frac{yy^1}{5^2x^3x^2} = \frac{y^2}{25x^{3+2}} = \frac{y^2}{25x^5}$$

We can't simplify any further, so our answer is

*Answer*

$$\frac{(5x)^{-2}y}{x^3y^{-1}} = \frac{y^2}{25x^5}$$

In the next section, you will learn how to write very large and very small numbers using exponents. This practice is widely used in science and engineering.

## SUMMARY

- Evaluating expressions containing exponents is the same as evaluating any expression. You substitute the value of the variable into the expression and simplify.
- The product rule for exponents: For any number  $x$  and any integers  $a$  and  $b$ ,  

$$(x^a)(x^b) = x^{a+b}.$$
- The quotient rule for exponents: For any non-zero number  $x$  and any integers  $a$  and  $b$ :  

$$\frac{x^a}{x^b} = x^{a-b}$$
- The power rule for exponents:

1. For any nonzero numbers  $a$  and  $b$  and any integer  $x$ ,  $(ab)^x = a^x \cdot b^x$ .
2. For any number  $a$ , any non-zero number  $b$ , and any integer  $x$ ,  $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

## CHAPTER ATTRIBUTION

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Rules for Exponents in *Mathematics for the Liberal Arts* by Lumen Learning. CC BY License.

## 1.4 Solving Linear Equations

### INTRODUCTION

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You are shopping at Old Navy for seven new outfits. The price points are \$10 and \$30. You really like the \$30 outfits; however, your total budget can't exceed \$110. How do you spend \$110 to acquire all the needed outfits without exceeding your budget while getting as many \$30 items as possible?

This is a problem of linear equations, and it illustrates how you can use them to make an optimal decision. Let  $L$  represent the quantity of clothing at the low price point of \$10, and  $H$  represent the quantity of clothing at the high price point of \$30. This results in the following algebraic equations:

$$\begin{aligned}L + H &= 7 \text{ (the total number of outfits you need)} \\ \$10L + \$30H &= \$110 \text{ (your total budget)}\end{aligned}$$

By simultaneously solving these equations you can determine how many outfits at each price point you can purchase.

You will encounter many situations like this in your business career, for example, in making the best use of a manufacturer's production capacity. Assume your company makes two products on the same production line and sells all its output. Each product contributes differently to your profitability, and each product takes a different amount of time to manufacture. What combination of each of these products should you make such that you operate your production line at capacity while also maximizing the profits earned? This section explores how to solve linear equations for unknown variables.

### UNDERSTANDING EQUATIONS

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To manipulate algebraic equations and solve for unknown variables, you must first become familiar with some important language, including linear versus nonlinear equations and sides of the equation.

$$4x + 3 = -2x - 3$$

$4x + 3$  is the Left side: Every equation has two sides. Everything to the left side of the equal sign is known as the left side of the equation.

$-2x - 3$  is the Right side: Everything to the right side of the equal sign is known as the right side of the

equation.

$x$  is the Variable  $x$ : Observe two important features of this variable:

1. Note that the variable has an exponent of **1**. So if you were to plot each algebraic expression onto a Cartesian coordinate system, the expressions would form straight lines (see Figure 1.4.1a). Such equations are called **linear equations**. Because each equation has only one variable, there is only one solution that makes the equation true. In business mathematics, this is the most common situation you will encounter.
2. If the exponent is anything but a one, the equation is a nonlinear equation since the graph of the expression does not form a straight line. Figure 1.4.1b illustrates two algebraic expressions where the exponent is other than a 1.

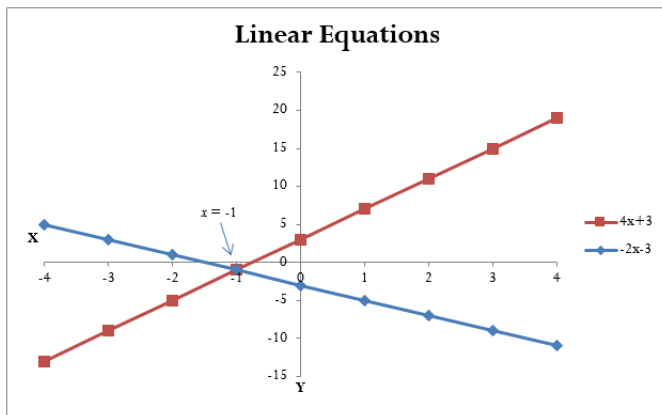


Figure 1.4.1a

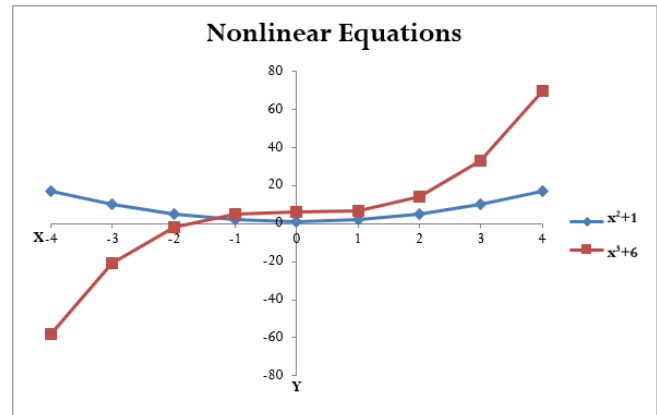


Figure 1.4.1b

The goal in manipulating and solving a linear equation is to find a value for the unknown variable that makes the equation true. If you substitute a value of  $x = -1$  into the above example, the left-hand side of the equation equals the right-hand side of the equation (see Figure 1.4.1a). The value of  $x = -1$  is known as the **root**, or solution, to the linear equation.

## SOLVING ONE LINEAR EQUATION WITH ONE UNKNOWN VARIABLE

In your study of solving linear equations, you need to start by manipulating a single equation to solve for a single unknown variable. Later in this section you will extend from this foundation to the solution of two linear equations with two unknowns.

### HOW TO Determine the root of a linear equation with only one unknown variable

Apply the following steps:

**Step 1:** Your first goal is to separate the terms containing the **literal coefficient** from the terms that only have numerical coefficients. Collect all of the terms with literal coefficients on only one side of the equation and collect all of the terms with only numerical coefficients on the other side of the equation. It does not matter which terms go on which side of the equation, so long as you separate them.

**Step 2:** To move a term from one side of an equation to another, take the mathematical opposite of the term being moved and add it to both sides.

For example, if you want to move the  $+3$  in  $4x + 3 = -2x - 3$  from the left-hand side to the right-hand side, the mathematical opposite of  $+3$  is  $-3$ . When you move a term, remember the cardinal rule: *What you do to one side of an equation you must also do to the other side of the equation. Breaking this rule breaks the equality in the equation.*

**Step 3:** Combine all like terms on each side and simplify the equation according to the rules of algebra.

**Step 4:** In the term containing the literal coefficient, reduce the numerical coefficient to a  $1$  by dividing both sides of the equation by the numerical coefficient.

#### Key Takeaway

When you are unsure whether your calculated root is accurate, an easy way to verify your answer is to take the original equation and substitute your root in place of the variable. If you have the correct root, the left-hand side of the equation equals the right-hand side of the equation. If you have an incorrect root, the two sides will be unequal. The inequality typically results from one of the three most common errors in algebraic manipulation:

1. The rules of **BEDMAS** have been broken.
2. The rules of algebra have been violated.
3. What was done to one side of the equation was not done to the other side of the equation.

## THINGS TO WATCH OUT FOR

When you move a term from one side of the equation to another using multiplication or division, remember that this affects *each and every term on both sides of the equation*. To remove the  $x$  from the denominator in the following equation, multiply both sides of the equation by  $x$ :

$$\frac{5}{x} + \frac{1}{x} = \frac{2}{x} + 2$$

becomes:

$$x \left( \frac{5}{x} + \frac{1}{x} \right) = \left( \frac{2}{x} + 2 \right) x$$

which then becomes:

$$5 + 1 = 2 + 2x$$

Multiplying every term on both sides by  $x$  maintains the equality.

## PATHS TO SUCCESS

Negative numbers can cause some people a lot of grief. In moving terms from a particular side of the equation, many people prefer to avoid negative numerical coefficients in front of literal coefficients. Revisiting  $4x + 3 = -2x - 3$ , you could move the  $4x$  from the left side to the right side by subtracting  $4x$  from both sides. However, on the right side this results in  $-6x$ . The negative is easily overlooked or accidentally dropped in future steps. Instead, move the variable to the left side of the equation, yielding a positive coefficient of  $6x$ .

### Example 1.4.1

Take the ongoing example in this section and solve it for  $x$ :

$$4x + 3 = -2x - 3.$$

#### Solution

**Step 1: Move terms with literal coefficients to one side and terms with only numerical coefficients to the other side.** Let's collect the literal coefficient on the left-hand side of the equation. Move  $-2x$  to the left-hand side by placing  $+2x$  on both sides.

On the right-hand side, the  $-2x$  and  $+2x$  cancel out to zero.

$$4x + 3 + 2x = -2x - 3 + 2x.$$

**Step 2: All of the terms with the literal coefficient are now on the left.** Let's move all of the terms containing only numerical coefficients to the right-hand side. Move the  $+3$  to the right-hand side by placing  $-3$  on both sides.

On the left-hand side, the  $+3$  and  $-3$  cancel out to zero.

$$4x + 3 + 2x - 3 = -3 - 3.$$

**Step 3: The terms are now separated. Combine like terms according to the rules of algebra.**

$$6x = -6$$

**Step 4: The term with the literal coefficient is being multiplied by the numerical coefficient of 6.** Therefore, divide both sides by 6.

The left-hand side numerical coefficients will divide to 1. Resolve the numerical coefficients on the right-hand side.

$$\frac{\cancel{6}x}{\cancel{6}} = \frac{-\cancel{6}}{\cancel{6}}$$

**Step 5: Write a statement.**

The root of the equation is  $x = -1$ .

**Step 6: Substitute back into the equation to check.**

$$\begin{aligned} 4(-1) + 3 &= -2 - 1) - 3 \\ -4 + 3 &= 2 - 3 \\ -1 &= -1 \checkmark \end{aligned}$$

#### Example 1.4.2

Solve the following equation for  $m$ :

$$\frac{3m}{4} + 2m = 4m - 15.$$

**Solution**

**Step 1: Simplify all fractions to make the equation easier to work with.**

Still simplifying, collect like terms where possible.

$$0.75m + 2 = 4m - 15.$$

**Step 2: Collect all terms with the literal coefficient on one side of the equation.** Move all terms with literal coefficients to the right-hand side.

$$2.75m = 4m - 15.$$

**Step 3: Combine like terms and move all terms with only numerical coefficients to the left-hand side.**

$$2.75m - 2.75m = 4m - 15 - 2.75m.$$

On the left-hand side, the  $+2.75m$  and  $-2.75m$  cancel each other out. Now move the numerical coefficients to the left-hand side.

$$0 = 4m - 15 - 2.75m$$

On the right-hand side, the  $-15$  and  $+15$  cancel each other out.

$$0 + 5 = 4m - 15 - 2.75m + 15$$

**Step 4: Combine like terms on each side.**

$$0 + 5 = 4m - 2.75m$$

**Step 5: Divide both sides by the numerical coefficient that accompanies the literal coefficient.**

Simplify.

$$\begin{aligned} 15 &= 1.25m \\ \frac{15}{1.25} &= \frac{1.25m}{1.25} \\ 12 &= m \end{aligned}$$

**Step 6: Write a statement.**

The root of the equation is  $m = 12$ .

This makes both sides of the equation ( $\frac{3m}{4} + 2m$  and  $4m - 15$ ) equal 33.

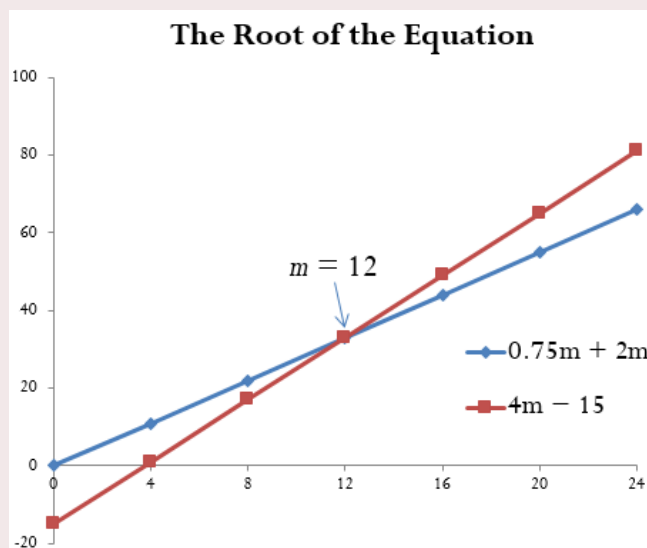


Figure 2.6.2

**Example 1.4.3**

Solve the following equation for  $b$  and round your answer to four decimals:  $\frac{5}{8}b + \frac{2}{5} = \frac{17}{20} - \frac{b}{4}$ .

**Solution**

**Step 1: Simplify the fractions and convert to decimals.**

$$0.625b + 0.4 = 0.85 - 0.25b$$

**Step 2: Move the literal coefficient terms to the left-hand side.**

The literal coefficients on the right-hand side cancel each other out.

$$0.625b + 0.4 + 0.25b = 0.85 - \cancel{0.25b} = 0.25b$$

**Step 3: Move the numerical coefficient terms to the right-hand side.**

The numerical coefficients on the left-hand side cancel each other out.

$$0.625b + 0.4 + 0.25b = 0.85$$

**Step 4: Combine like terms on each side.**

$$0.875b = 0.45$$

**Step 5: Divide both sides by the numerical coefficient that accompanies the literal coefficient.**

$$\frac{0.875b}{0.875} = \frac{0.45}{0.875}$$

**Step 6: Simplify.**

$$b = 0.514285$$

**Step 7: Round to four decimals as instructed and write a statement.**

$$b = 0.5143$$

**Step 8: Write as a statement.**

The root is  $b = 0.5143$ .

## SOLVING TWO LINEAR EQUATIONS WITH TWO UNKNOWN VARIABLES

The manipulation process you have just practiced works well for solving one linear equation with one variable. But what happens if you need to solve two linear equations with two variables simultaneously? Remember when you were at Old Navy purchasing seven outfits earlier in this chapter? You needed to stay within a pricing budget. Each equation had two unknown variables representing the number of lower-priced outfits ( $L$ ) and higher-priced outfits ( $H$ ).

$$\begin{aligned} L + H &= 7 \text{ (the total number of outfits you need)} \\ \$10L + \$30H &= \$110 \text{ (your total budget)} \end{aligned}$$

The goal is to reduce two equations with two unknowns into a single linear equation with one unknown. Once this transformation is complete, you then identify the unknown variable by applying the three-step procedure for solving one linear equation, as just discussed.

When you work with two linear equations with two unknowns, the rules of algebra permit the following two manipulations:

- What you do to one side of the equation must be done to the other side of the equation to maintain the equality. Therefore, you can multiply or divide any equation by any number without changing the root of the equation.
  - For example, if you multiply all terms of  $x + y = 2$  by 2 on both sides, resulting in  $2x + 2y = 4$ , the equality of the equation remains unchanged and the same roots exist.
- Terms that are on the same side of an equation can be added and subtracted between equations by combining like terms. Each of the two equations has a left side and right side. This rule permits taking the left side of the first equation and either adding or subtracting like

terms on the left side of the second equation. When you perform this action, remember the first rule above. If you add the left sides of the equations together, you then must add the right side of both equations together to maintain equality.

## HOW TO Solve two linear equations with two unknown variables

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- **Step 1:** Write the two equations one above the other, vertically lining up terms that have the same literal coefficients and terms that have only the numerical coefficient. If necessary, the equations may need to be manipulated such that all of the literal coefficients are on one side with the numerical coefficients on the other side.
- **Step 2:** Examine your two equations. Through multiplication or division, make the numerical coefficient on one of the terms containing a literal coefficient exactly equal to its counterpart in the other equation.
- **Step 3:** Add or subtract the two equations as needed so as to eliminate the identical term from both equations.
- **Step 4:** In the new equation, solve for the last literal coefficient.
- **Step 5:** Substitute the root of the known literal coefficient into either of the two original equations. If one of the equations takes on a simpler structure, pick that equation.
- **Step 6:** Solve your chosen equation for the other literal coefficient.

## PATHS TO SUCCESS

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Sometimes it is unclear exactly how you need to multiply or divide the equations to make two of the terms identical. For example, assume the following two equations:

$$4.9x + 1.5y = 38.3$$

$$2.7x - 8.6y = 17.8$$

If the goal is to make the terms containing the literal coefficient  $x$  identical, there are two alternative solutions:

- Take the larger numerical coefficient for  $x$  and divide it by the smaller numerical coefficient. The resulting number is the factor for multiplying the equation containing the smaller numerical coefficient. In this case,  $4.9 \div 2.7 = 1.\overline{814}$ . Multiply all terms in the second equation by  $1.\overline{814}$  to make the numerical coefficients for  $x$  equal to each other, resulting in this pair of equations:

$$4.9x + 1.5y = 38.3$$

and

$$4.9x - 15.\overline{6074}y = 32.\overline{3037} \text{ (every term multiplied by } 1.\overline{814}\text{)}$$

- Take the first equation and multiply it by the numerical coefficient in the second equation. Then take the second equation and multiply it by the numerical coefficient in the first equation. In this case, multiply all terms in the first equation by 2.7. Then multiply all terms in the second equation by 4.9.

$$13.23x + 4.05y = 103.41 \text{ (every term multiplied by 2.7)}$$

$$13.23x - 42.14y = 87.22 \text{ (every term multiplied by 4.9)}$$

Note that both approaches successfully result in both equations having the same numerical coefficient in front of the literal coefficient  $x$ .

## PATHS TO SUCCESS

Ultimately, every pairing of linear equations with two unknowns can be converted into a single equation through substitution. To make the conversion, do the following:

1. Solve either equation for one of the unknown variables.
2. Take the resulting algebraic expression and substitute it into the other equation. This new equation is solvable for one of the unknown variables.
3. Substitute your newfound variable into one of the original equations to determine the value for the other unknown variable.

Take the following two equations:

$$a + b = 4 \text{ and } 2a + b = 6$$

1. Solving the first equation for  $a$  results in  $a = 4 - b$ .
2. Substituting the expression for  $a$  into the second equation and solving for  $b$  results in  $2(4 - b) + b = 6$ , which solves as  $b = 2$ .
3. Finally, substituting the root of  $b$  into the first equation to calculate  $a$  gives  $a + 2 = 4$  resulting in  $a = 2$ . Therefore, the roots of these two equations are  $a = 2$  and  $b = 2$ .

### Example 1.4.4

Recall from the section opener that in shopping for outfits there are two price points of \$10 and \$30, your budget is \$110, and that you need seven articles of clothing. The equations below represent these conditions. Identify how many low-priced outfits ( $L$ ) and high-priced outfits ( $H$ ) you can purchase.

$$\begin{aligned}L + H &= 7 \\ \$10L + \$30H &= \$110\end{aligned}$$

**Solution**

*Step 1: Write the equations one above the other and line them up.*

$$\begin{aligned}L + H &= 7 \\ \$10L + \$30H &= \$110\end{aligned}$$

*Step 2: Multiply all terms in the first equation by 10 so that  $L$  has the same numerical coefficient in both equations.*

$$L + H = 7$$

*Step 3: Subtract the equations by subtracting all terms on both sides.*

$$\begin{aligned}10L + 10H &= 70 \\ \$10L + 30H &= \$110 \\ -\$20H &= \$40\end{aligned}$$

*Step 4: Solve for  $H$  by dividing both sides by  $-20$ .*

$$\begin{aligned}\frac{\$20H}{-\$20} &= \frac{-\$40}{-\$20} \\ H &= 2\end{aligned}$$

*Step 5: Substitute the known value for  $H$  into one of the original equations. The first equation is simple, so choose that one.*

$$\begin{aligned}L + H &= 7 \\ L + 2 &= 5\end{aligned}$$

*Step 6: Solve for  $L$  by subtracting 2 from both sides. You now have the roots for  $L$  and  $H$ .*

$$\begin{aligned}L + 2 - 2 &= 5 - 2 \\ L &= 3\end{aligned}$$

## PATHS TO SUCCESS

One of the most difficult areas of mathematics involves translating words into mathematical symbols and operations. To assist in this translation, the table below lists some common language and the mathematical symbol that is typically associated with the word or phrase.

**Table 1.4.1**

Language			Math Symbol	Language			Math Symbol
Sum Addition	In addition to In excess	Increased by Plus	+	Subtract Decreased by Diminished by	Less Minus	Difference Reduced by	-
Divide Division	Divisible Quotient	Per	÷	Multiplied by Times	Percentage of	Product of Of	×
Becomes Is/Was/ Were	Will be	Results in Totals	=	More than Greater than			>
Less than	Lower than		<	Greater than or equal to			≥
Less than or equal to			≤	Not equal to			≠

### Example 1.4.5

Tinkertown Family Fun Park charges \$15 for a child wrist band and \$10.50 for an adult wrist band. On a warm summer day, the amusement park had total wrist band revenue of \$15,783 from sales of 1,279 wrist bands. How many adult and child wrist bands did the park sell that day?

#### Solution

##### Step 1: Write out what you know.

- The price of the wrist bands, total quantity, and sales are known.
- Child wrist band price = \$15
- Adult wrist band price = \$10.50
- Total revenue = \$15,783
- Total unit sales = 1,279
- The quantity of adult wrist bands sold and the quantity of child wrist bands sold are unknown:
- Adult wrist bands quantity =  $a$
- Child wrist bands quantity =  $c$

##### Step 2: Write out how you will get to the solution.

1. Work with the quantities first. Calculate the total unit sales by adding the number of adult wrist bands to the number of child wrist bands: # of adult wrist bands + # of child wrist bands = total unit sales  
 $a + c = 1,279$
2. Now consider the dollar figures. Total revenue for any company is calculated as unit price multiplied by units sold. In this case, you must sum the revenue from two products to get the total revenue.

$$\begin{aligned} \text{Total adult revenue} + \text{Total child revenue} &= \text{Total revenue} \\ (\text{Adult price} \times \text{Adult quantity}) + (\text{Child price} \times \text{Child quantity}) &= \text{Total revenue} \\ \$10.50a + \$15c &= \$15,783 \end{aligned}$$

3. Apply the six-step procedure for solving two linear equations with two unknowns

**Step 3: Write the equations one above the other and line them up.**

$$\begin{aligned} a + c &= 1,279 \\ \$10.50a + \$15c &= \$15,783 \end{aligned}$$

**Step 4: Multiply all terms in the first equation by 10.5, resulting in  $a$  having the same numerical coefficient in both equations.**

$$\begin{aligned} 10.50a + 10.50c &= 13,429.50 \\ \$10.50a + \$15c &= \$15,783 \end{aligned}$$

**Step 5: Subtract the equations by subtracting all terms on both sides.**

$$\begin{aligned} 10.50a + 10.50c &= 13,429.50 \\ \$10.50a + \$15c &= \$15,783 \text{ Subtract} \\ -4.5c &= -2,353.50 \end{aligned}$$

**Step 6: Solve for  $c$  by dividing both sides by  $-4.5$ .**

$$\begin{aligned} \frac{-4.5c}{-4.5} &= \frac{-2,353.50}{-4.5} \\ c &= 523 \end{aligned}$$

**Step 7: Substitute the known value for  $c$  into one of the original equations. The first equation is simple, so choose that one.**

Solve for  $a$  by subtracting 523 from both sides. You now have the roots for  $a$  and  $c$ .

$$\begin{aligned}
 a + c &= 1,279 \\
 a + rgb[1.0, 0.0, 1.0]523 &= 1,279 \\
 a + rgb[1.0, 0.0, 1.0]523 - 523 &= 1,279 - rgb[1.0, 0.0, 1.0]523 \\
 a &= 756
 \end{aligned}$$

**Step 8: Write as a statement.**

Tinkertown Family Fun Park sold 523 child wrist bands and 756 adult wrist bands.

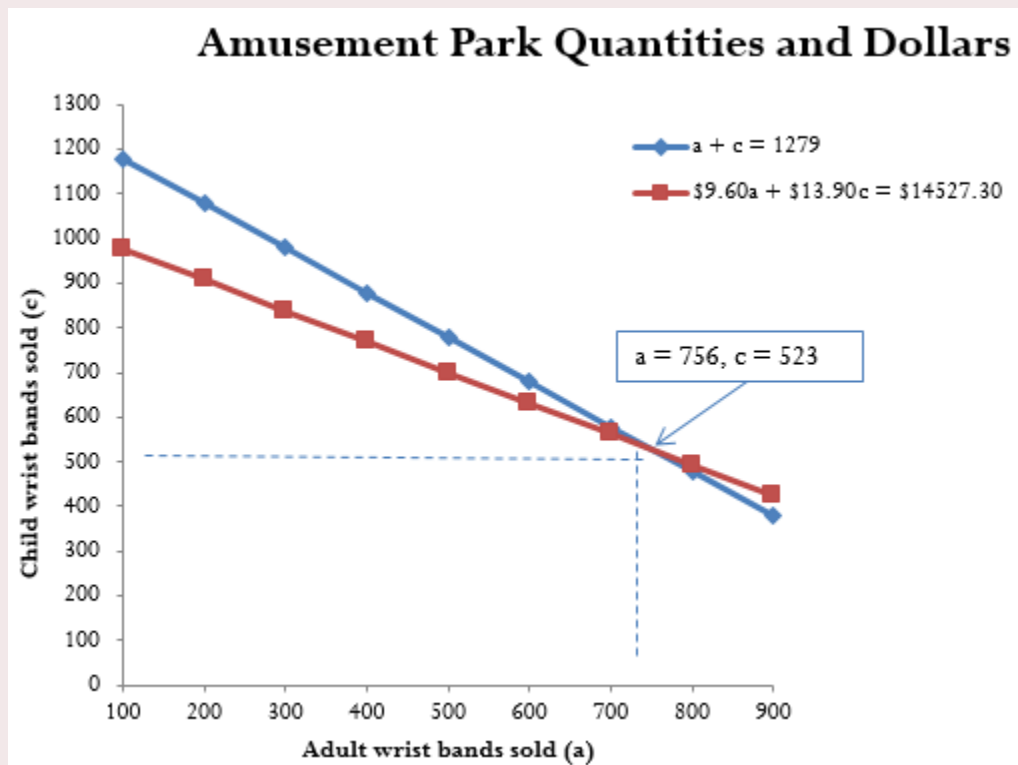


Figure 2.6.3

## 1.4.6 EXERCISES

### Exercises – Mechanics

Solve the following equations for the unknown variable.

1.  $3(x - 5) = 15$
2.  $12b - 3 = 4 + 5b$

$$3. \quad 0.75(4m + 12) + 15 - 3(2m + 6) = 5(-3m + 1) + 25$$

Solve each of the following pairs of equations for both unknown variables.

$$4. \quad \begin{aligned} x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$$

$$5. \quad \begin{aligned} 4h - 7q &= 13 \\ 6h + 3q &= 33 \end{aligned}$$

$$6. \quad \begin{aligned} 0.25a + \frac{5b}{2} &= 3.5 \\ \frac{3a}{4} - \frac{b}{0.2} &= 3 \end{aligned}$$

### Solutions

1.  $x = 1.0$
2.  $b = 1$
3.  $m = 2$
4.  $x = 4, y = 2$
5.  $h = 5, q = 1$
6.  $a = 8, b = 0.6$

### Exercises – Applications

In questions 7 and 8, solve the equation for the unknown variable.

$$7. \quad \frac{4y}{1.025^4} + y - 2y(1.05)^2 = \$1,500$$

$$8. \quad \$2,500(1 + 0.06t) + \$1,000(1 + 0.04t) = \$3,553.62$$

For exercises 9–14, read each question carefully and solve for the unknown variable(s).

9. Pamela is cooking a roast for a 5:30 p.m. dinner tonight. She needs to set a delay timer on her oven. The roast takes **1** hour and **40** minutes to cook. The time right now is 2:20 p.m. How long of a delay must she set the oven for (before it automatically turns on and starts to cook the roast)?
10. In 2010, **266 million** North Americans were using the Internet, which represented a **146.3%**

increase in Internet users over the year 2000. How many North American Internet users were there in 2000?

11. A human resource manager is trying to estimate the cost of a workforce accident. These costs usually consist of direct costs (such as medical bills, equipment damage, and legal expenses) and indirect costs (such as decreased output, production delays, and fines). From past experience, she knows that indirect costs average six times as much as direct costs. If she estimates the cost of an accident to be \$21,000, what is the direct cost of the accident?
12. In 2011, Canadian federal tax rates were 0% on the first \$10,527 of gross income earned, 15% on the next \$31,017, 22% on the next \$41,544, 26% on the next \$45,712, and 29% on anything more. If a taxpayer paid \$28,925.35 in federal tax, what was her gross annual income for 2011?
13. St. Boniface Hospital raises funds for research through its Mega Lottery program. In this program, 16,000 tickets are available for purchase at a price of one for \$100 or three for \$250. This year, the lottery sold out with sales of \$1,506,050. To better plan next year's lottery, the marketing manager wants to know how many tickets were purchased under each option this year.
14. An accountant is trying to allocate production costs from two different products to their appropriate ledgers. Unfortunately, the production log sheet for last week has gone missing. However, from other documents he was able to figure out that 1,250 units in total were produced last week. The production machinery was run for 2,562.5 minutes, and he knows that Product A takes 1.5 minutes to manufacture while Product B takes 2.75 minutes to manufacture. How many units of each product were produced last week?

#### Solutions

7.  $y = \$620.14$
8.  $t = 0.282211$
9. Delay = 1.5 hours
10. Users in 2000 = 107,998,376
11. Direct cost = \$3,000
12. Gross income = \$140,000
13. Single tickets = 10,363. 3-pack Tickets = 1,879
14. A = 700 units. B = 550 units

#### Exercises – Challenge, Critical Thinking, & Other Applications

15. Jacob owns 15,000 shares in a corporation, which represents 2% of all issued shares for the company. He sold 25 of his shares to another investor for \$7,800. What is the total value for all of

the shares issued by the company?

16. Two cellphone companies are offering different rate plans. Rogers is offering \$19.99 per month, which includes a maximum of 200 weekday minutes plus \$0.35 for every minute above the maximum. TELUS is offering \$39.99 for a maximum 300 weekday minutes, but it charges \$0.10 for every minute above the maximum. Above how many minutes would TELUS be the better choice?
17. Marianne, William, Hendrick, and Charlotte have all decided to go into business together. They need \$175,000 in initial capital funding. William was able to contribute 20% less than Marianne, Hendrick contributed 62.5% more than William, and Charlotte contributed \$5,000 less than half as much as Marianne. How much did each partner contribute to the initial funds?
18. A mall is being constructed and needs to meet the legal requirements for parking availability. Parking laws require one parking stall for every 100 square feet of retail space. The mall is designed to have 1,200,000 square feet of retail space. Of the total parking stalls available, 2% need to be handicap accessible, there need to be three times as many small car spaces as handicap spaces, RV spaces need to be one-quarter of the number of small car spaces, and the rest of the spaces are for regular stalls. How many of each type of parking space does the mall require?
19. Simplify the following equation into the format of " $z =$ " and find the root. Verify the solution through substitution.

$$z \left( 1 + 0.073 \times \frac{280}{365} \right) - \frac{z}{1 + 0.073 \times \frac{74}{365}} + \$1,000 = \$2,764.60$$

20. Find the roots for the following pair of equations. Verify the solution through substitution into both equations.

$$3\frac{4}{5}q + 0.18r = 12.2398$$

$$-5.13q - \frac{13r}{5} = -38.4674$$

### Solutions

15. Company shares = \$975,000
16. 240 minutes
17. Marianne = \$50,000;      William = \$40,000;      Hendrick = \$65,000;  
Charlotte = \$20,000
18. Regular stalls = 10,860;      Handicap stalls = 240;      Small car stalls = 720;  
RV stalls = 180
19.  $z = \$25,000$

$$20. \quad q = 2.78; r = 9.31$$

## ATTRIBUTION

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## 1.5 Solving Word Problems

### Learning Objectives

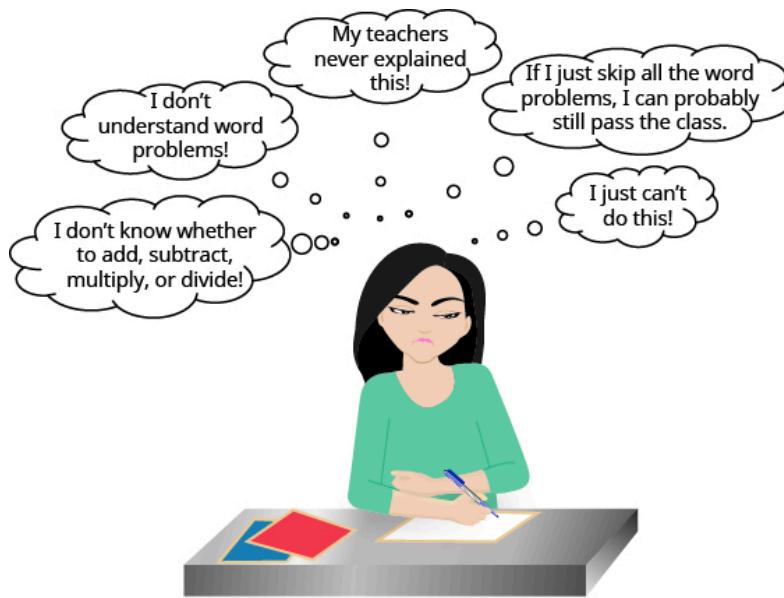
- Practice mindfulness with your attitude about word problems
- Apply a general problem-solving strategy to solve word problems

### APPROACH WORD PROBLEMS WITH A POSITIVE ATTITUDE

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The world is full of word problems. How much money do I need to fill the car with gas? How much should I tip the server at a restaurant? How many socks should I pack for vacation? How big a turkey do I need to buy for Thanksgiving dinner, and what time do I need to put it in the oven? If my sister and I buy our mother a present, how much will each of us pay?

Now that we can solve equations, we are ready to apply our new skills to word problems. Do you know anyone who has had negative experiences in the past with word problems? Have you ever had thoughts like the student in the cartoon below?



*Negative thoughts about word problems can be barriers to success.*

When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. We need to calm our fears and change our negative feelings.

Start with a fresh slate and begin to think positive thoughts like the student in the cartoon below. Read the positive thoughts and say them out loud.



*When it comes to word problems, a positive attitude is a big step toward success.*

If we take control and believe we can be successful, we will be able to master word problems.

Think of something that you can do now but couldn't do three years ago. Whether it's driving a car, snowboarding, cooking a gourmet meal, or speaking a new language, you have been able to learn and master a new skill. Word problems are no different. Even if you have struggled with word problems in the past, you have acquired many new math skills that will help you succeed now!

### Use a Problem-Solving Strategy for Word Problems

In earlier chapters, you translated word phrases into algebraic expressions, using some basic mathematical vocabulary and symbols. Since then you've increased your math vocabulary as you learned about more algebraic procedures, and you've had more practice translating from words into algebra.

You have also translated word sentences into algebraic equations and solved some word problems. The word problems applied math to everyday situations. You had to restate the situation in one sentence, assign a variable, and then write an equation to solve. This method works as long as the situation is familiar to you and the math is not too complicated.

Now we'll develop a strategy you can use to solve any word problem. This strategy will help you become successful with word problems. We'll demonstrate the strategy as we solve the following problem.

#### Example

Pete bought a shirt on sale for \$18, which is one-half the original price. What was the original price of the shirt?

#### Solution

Step 1. **Read** the problem. Make sure you understand all the words and ideas. You may need to read the problem two or more times. If there are words you don't understand, look them up in a dictionary or on the Internet.

- *In this problem, do you understand what is being discussed? Do you understand every word?*

Step 2. **Identify** what you are looking for. It's hard to find something if you are not sure what it is! Read the problem again and look for words that tell you what you are looking for!

- *In this problem, the words "what was the original price of the shirt" tell you what you are looking for: the original price of the shirt.*

Step 3. **Name** what you are looking for. Choose a variable to represent that quantity. You can use any letter for the variable, but it may help to choose one that helps you remember what it represents.

- *Let  $p$  = the original price of the shirt*

Step 4. **Translate** into an equation. It may help to first restate the problem in one sentence, with all the important information. Then translate the sentence into an equation.

$$\begin{array}{ccccccc} \underbrace{18} & \underbrace{\text{is}} & \underbrace{\text{one-half}} & \underbrace{\text{of}} & \underbrace{\text{the original price}} & . \\ 18 & = & \frac{1}{2} & \cdot & p \end{array}$$

Step 5. **Solve** the equation using good algebra techniques. Even if you know the answer right away, using algebra will better prepare you to solve problems that do not have obvious answers.

Write the equation.

$$18 = \frac{1}{2}p$$

Multiply both sides by 2.

$$2 \cdot 18 = 2 \cdot \frac{1}{2}p$$

Simplify.

$$36 = p$$

Step 6. **Check** the answer in the problem and make sure it makes sense.

- We found that  $p = 36$ , which means the original price was \$36. Does \$36 make sense in the problem? Yes, because 18 is one-half of 36, and the shirt was on sale at half the original price.

Step 7. **Answer** the question with a complete sentence.

- The problem asked "What was the original price of the shirt?" The answer to the question is: "The original price of the shirt was \$36."

If this were a homework exercise, our work might look like this:

Let  $p$  = the original price.

18 is one-half the original price.

$$18 = \frac{1}{2}p$$

$$2 \cdot 18 = 2 \cdot \frac{1}{2}p$$

$$36 = p$$

Check:

Is \$36 a reasonable price for a shirt? *Yes.*

Is 18 one-half of 36? *Yes.*

The original price of the shirt was \$36.

### TRY IT

Translate the following verbal statement into an algebraic equation and then solve:

- Dion bought a bookcase on sale for \$180, which was one-fourth of the original price.
- What was the original price of the bookcase?
- Use  $p$  for your variable.

### Answer

We list the steps we took to solve the previous example.

### PROBLEM-SOLVING STRATEGY

1. **Read** the word problem. Make sure you understand all the words and ideas. You may need to read the problem two or more times. If there are words you don't understand, look them up in a dictionary or on the internet.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to first restate the problem in one sentence before translating.
5. **Solve** the equation using good algebra techniques.

6. **Check** the answer in the problem. Make sure it makes sense.
7. **Answer** the question with a complete sentence.

For a review of how to translate algebraic statements into words, watch the following video.<sup>1</sup>



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1883#oembed-1>

### Examples

Let's use this approach with another example.

Yash brought apples and bananas to a picnic. The number of apples was three more than twice the number of bananas. Yash brought **11** apples to the picnic. How many bananas did he bring?

#### **Solution**

1. *Write Algebraic Expressions from Statements: Form  $ax+b$  and  $a(x+b)$*  by James Sousa (Mathispower4u.com) for Lumen Learning. <https://youtu.be/Hub7ku7UHT4>. CC BY License

Step 1. <b>Read</b> the problem.	
Step 2. <b>Identify</b> what you are looking for.	How many bananas did he bring?
Step 3. <b>Name</b> what you are looking for. Choose a variable to represent the number of bananas.	Let $b = \text{number of bananas}$
	$11 \Rightarrow$ The number of apples $= \Rightarrow$ was
Step 4. <b>Translate.</b> Restate the problem in one sentence with all the important information. Translate into an equation.	$3 \Rightarrow$ three $+$ $\Rightarrow$ more than $2b \Rightarrow$ twice the number of bananas
Step 5. <b>Solve</b> the equation.	$11 = 2b + 3$
Subtract 3 from each side.	$11 - 3 = 2b + 3 - 3$
Simplify.	$8 = 2b$
Divide each side by 2.	$\frac{8}{2} = \frac{2b}{2}$
Simplify.	$4 = b$
Step 6. <b>Check:</b> First, is our answer reasonable? Yes, bringing four bananas to a picnic seems reasonable. The problem says the number of apples was three more than twice the number of bananas. If there are four bananas, does that make eleven apples? Twice 4 bananas is 8. Three more than 8 is 11.	
Step 7. <b>Answer</b> the question.	Yash brought 4 bananas to the picnic.

## TRY IT

Translate the following scenario into an algebraic equation and then solve:

- Use  $j$  for your variable to represent journals.
- Pavati bought a total of 7 textbooks at the bookstore. She bought four more textbooks than journals.
- How many journals did she buy?

## Answer

In the next example, we will apply our Problem-Solving Strategy to applications of percent.

### Example

Nga's car insurance premium increased by \$60, which was 8% of the original cost. What was the original cost of the premium?

### Solution

Step 1. **Read** the problem. Remember, if there are words you don't understand, look them up.

Step 2. **Identify** what you are looking for.

the original cost of the premium

Step 3. **Name.** Choose a variable to represent the original cost of premium.

Let  $c$  = the original cost

Step 4. **Translate.** Restate as one sentence. Translate into an equation.

$$\underbrace{\$60}_{60} \text{ was } \underbrace{8\%}_{0.08} \text{ of } \underbrace{\text{the original cost}}_c.$$

Step 5. **Solve** the equation.

$$60 = 0.08c$$

Divide both sides by 0.08.

$$\frac{60}{0.08} = \frac{0.08c}{0.08}$$

Simplify.

$$c = 750$$

Step 6. **Check:** Is our answer reasonable? Yes, a \$750 premium on auto insurance is reasonable. Now let's check our algebra. Is 8% of 750 equal to 60?  $750 = c$

$$0.08(750) = 60$$

$$60 = 60 \quad \checkmark$$

Step 7. **Answer** the question.

The original cost of Nga's premium was \$750.

### TRY IT

Translate the following verbal statement into an algebraic equation and then solve:

- Laila's rent increased by 4%. The increase was \$54.4.
- What was the original amount of Laila's rent?
- Use  $r$  for your variable.

### Solution

\$1,360

### CHAPTER ATTRIBUTION

Apply a Problem-Solving Strategy to Basic Word Problems in *Mathematics for the Liberal Arts* by Lumen Learning. CC BY License.

## 1.6 Break-Even Analysis

Should you start up the Internet business. Right now, all you have are some projected costs and a forecasted level of sales. You imagine you are going to sell 400 units. Is that possible? Is it reasonable to forecast this many sales?

Now you may say to yourself, “400 units a month . . . that’s about 13 per day. What’s the big deal?” But let’s gather some more information. What if you looked up your industry in Statistics Canada data and learned that the product in question sells just 1,000 units per month in total? Statistics Canada also indicates that there are eight existing companies selling these products. How does that volume of 400 units per month sound now? Unless you are revolutionizing your industry, it is unlikely you will receive a 40% market share in your first month of operations. With so few unit sales in the industry and too many competitors, you might be lucky to sell 100 units. If this is the case, are you still profitable?

Simply looking at the fixed costs, variable costs, potential revenues, contribution margins, and typical net income is not enough. Ultimately, all costs in a business need to be recovered through sales. Do you know how many units have to be sold to pay your bills? The answer to this question helps assess the feasibility of your business idea.

### WHAT IS **BREAK-EVEN ANALYSIS**?

---

If you are starting your own business and head to the bank to initiate a start-up loan, one of the first questions the banker will ask you is your break-even point. You calculate this number through break-even analysis, which is the analysis of the relationship between costs, revenues, and net income with the sole purpose of determining the point at which total revenue equals total cost. This break-even point is the level of output (in units or dollars) at which all costs are paid but no profits are earned, resulting in a net income equal to zero. To determine the break-even point, you can calculate a break-even analysis in two different ways, involving either the number of units sold or the total revenue in dollars. Each of these two methods is discussed in this section.

### METHOD 1: **BREAK-EVEN ANALYSIS IN UNITS**

---

In this method, your goal is to determine the level of output that produces a net income equal to zero. This method requires unit information, including the unit selling price and unit variable cost.

It is helpful to see the relationship of total cost and total revenue on a graph. Assume that a company has the following information:

$$TFC = \$400 \quad S = \$100 \quad VC = \$60$$

The graph shows dollar information on the  $\diamond$ -axis and the level of output on the  $\diamond$ -axis.

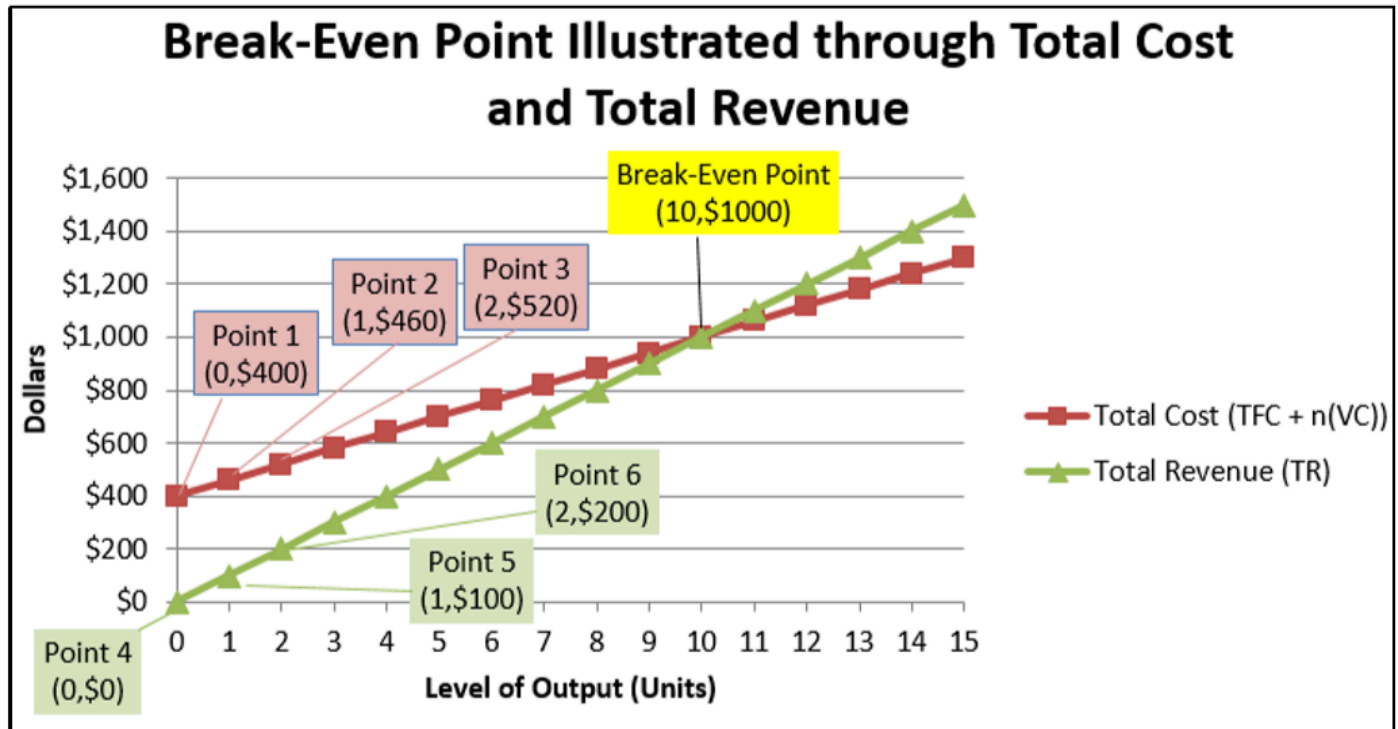


Table 1.1 Break-Even Point Illustrated Through Total Cost and Total Revenue

Here is how you construct such a graph:

Plot the total costs:

- At zero output you incur the total fixed costs of \$400. Denote this as Point 1 (0, \$400).
- As you add one level of output, the total cost rises in the amount of the unit variable cost. Therefore, total cost is  $TFC + n(VC) = \$400 + 1(\$60) = \$460$ . Denote this as Point 2 (1, \$460).
- As you add another level of output (2 units total), the total cost rises once again in the amount of the unit variable cost, producing  $\$400 + 2(\$60) = \$520$ . Denote this as Point 3 (2, \$520).
- Repeat this process for each subsequent level of output and plot it onto the figure. The red line plots these total costs at all levels of output.

Plot the total revenue:

- At zero output, there is no revenue. Denote this as Point 4 (0, \$0).

- b. As you add one level of output, total revenue rises by the selling price of the product. Therefore, total revenue is  $n(S) = 1(\$100) = \$100$ . Denote this as Point 5 (1, \$100).
- c. As you add another level of output (2 units total), the total revenue rises once again in the amount of the selling price, producing  $2(\$100) = \$200$ . Denote this as Point 6 (2, \$200).
- d. Repeat this process for each subsequent level of output and plot it onto the figure. The green line plots the total revenue at all levels of output.

The purpose of break-even analysis is to determine the point at which total cost equals total revenue. The graph illustrates that the break-even point occurs at an output of 10 units. At this point, the total cost is  $\$400 + 10(\$60) = \$1,000$ , and the total revenue is  $10(\$100) = \$1,000$ . Therefore, the net income is  $\$1,000 - \$1,000 = \$0$ ; no money is lost or gained at this point.

## THE FORMULA

---

The Formula 1.2 Net Income Using a Total Revenue and Cost Approach states that the net income equals total revenue minus total costs.

In break-even analysis, net income is set to zero, resulting in

$$0 = n(S) - (TFC + n(VC))$$

Rearranging and solving this formula for  $n$  gives the following:

$$0 = n(S) - TFC - n(VC)$$

$$TFC = n(S) - n(VC)$$

$$TFC = n(S - VC)$$

$$\frac{TFC}{(S - VC)} = n$$

Formula 1.3 states that

$$CM = S - VC$$

therefore, the denominator becomes just CM.

The calculation of the break-even point using this method is thus summarized in Formula 1.4

**n is Breakeven Level of Output (Units):** This is the level of output in units that produces a net income equal to zero.

**TFC is Total Fixed Costs:** The total of all costs that are not affected by the level of output.

Formula 1.4 Unit Break-Even:  $n = \frac{TFC}{CM}$

**CM is Unit Contribution Margin:** The amount of money left over per unit after you have recovered your variable costs. Calculate this by taking the unit selling price and subtracting the unit variable cost (Formula 1.4). You use this money to pay off your fixed costs.

## HOW IT WORKS

Follow these steps to calculate the break-even point in units:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the unit contribution margin (cm) by applying any needed techniques or formulas.

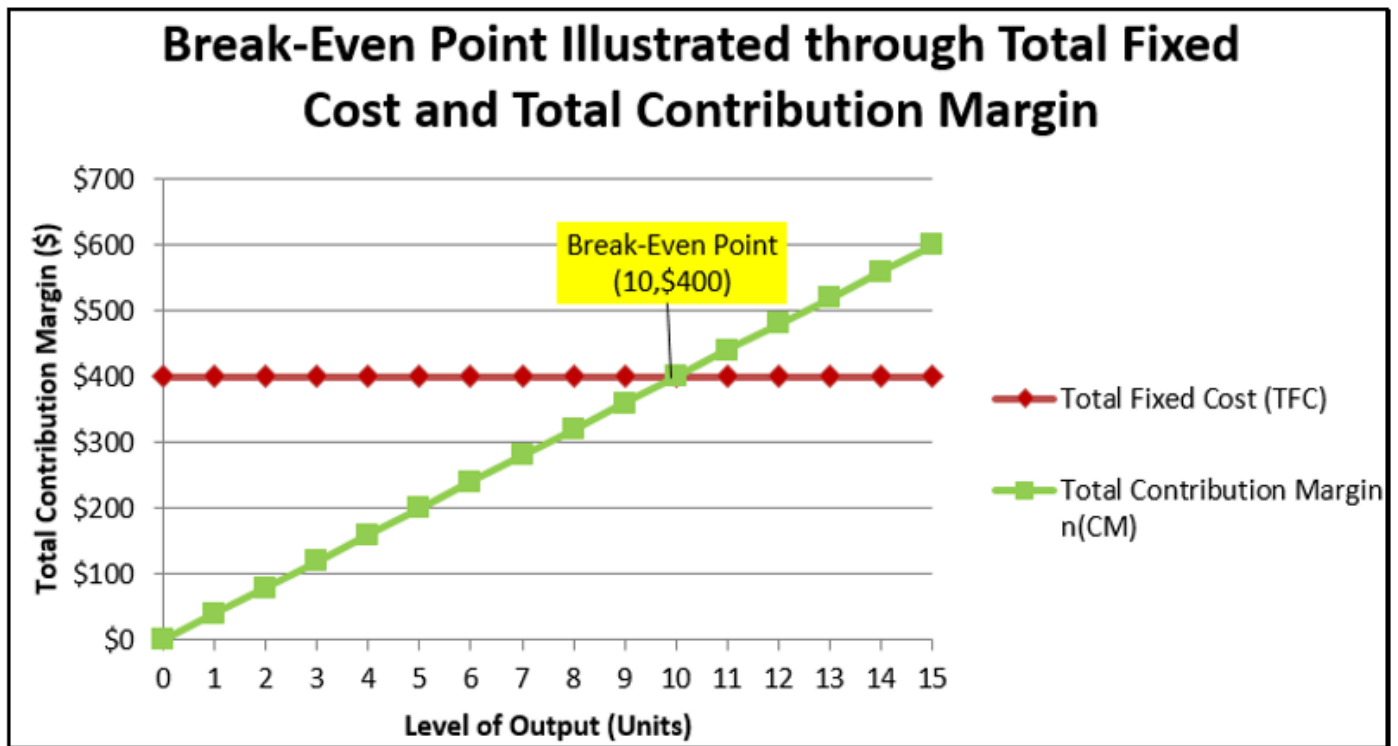
Step 3: Apply Formula 1.4.

Continuing with the example that created the graph on the previous page:

Step 1: Total fixed costs are known,  $TFC = 400$

Step 2: The unit contribution margin is  $\$100 - \$60 = \$40$  For each unit sold this is the amount left over that can be applied against total fixed costs.

Step 3: Applying Formula 1.4 results in  $n = \$400 \div \$40 = 10$  units.



#### IMPORTANT NOTES

When you calculate the break-even units, the formula may produce a number with decimals. For example, a break-even point might be 324.39 units. How should you handle the decimal? A partial unit cannot be sold, so the rule is always to round the level of output up to the next integer, regardless of the decimal. Why? The main point of a break-even analysis is to show the point at which you have recovered all of your costs. If you round the level of output down, you are 0.39 units short of recovering all of your costs. In the long-run, you always operate at a loss, which ultimately puts you out of business. If you round the level of output up to 325, all costs are covered and a tiny dollar amount, as close to zero as possible, is left over as profit. At least at this level of output you can stay in business.

#### EXAMPLE – THE BREAK-EVEN UNITS FOR YOUR PLANNED INTERNET BUSINESS

Let's determine the break-even point in units. The total fixed costs are \$638.03 and the unit contribution margin is \$3.57.

#### SOLUTION

Calculate the break-even point in units sold, or  $\diamond$  at break-even.

#### What You Already Know

Step 1: The total fixed costs are known:  $TFC = \$638.03$ .

Step 2: The contribution rate is known: CM=\$3.57.

### How You Will Get There

Step 3: Apply Formula 1.4.

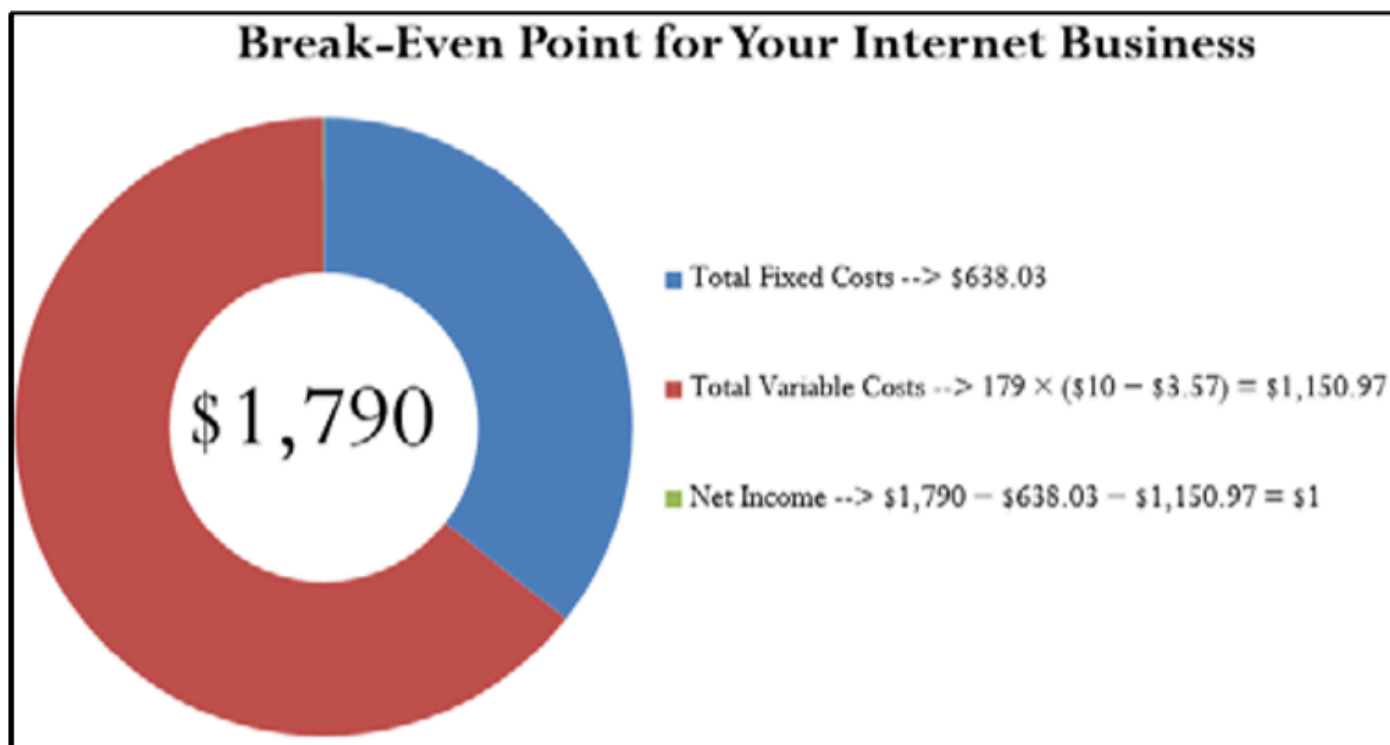
$$n = \frac{\$638.03}{\$3.57} = 178.719888$$

Round this up to 179.

### Calculator Instructions

FC	VC	P	PFT	Q
683.03	6.43	10	0	Answer: 178.719888

In order for your Internet business to break even, you must sell 179 units. At a price of \$10 per unit, that requires a total revenue of \$1,790. At this level of output your business realizes a net income of \$1 because of the rounding.



## METHOD 2: BREAK-EVEN ANALYSIS IN DOLLARS

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The income statement of a company does not display unit information. All information is aggregate, including total revenue, total fixed costs, and total variable costs. Typically, no information is listed about unit selling price, unit variable costs, or the level of output. Without this unit information, it is impossible to apply Formula 1.3.

The second method for calculating the break-even point relies strictly on aggregate information. As a result, you cannot calculate the break-even point in units. Instead, you calculate the break-even point in terms of aggregate dollars expressed as total revenue.

### THE FORMULA

---

To derive the break-even point in dollars, once again start with Formula 1.2, where total revenue at break-even less total fixed costs and total variable costs must equal a net income of zero:

$$NI = TR - (TFC + TVC)$$

$$0 = TR - (TFC + TVC)$$

Rearranging this formula for total revenue gives:

$$0 = TR - TFC - TVC$$

$$TR = TFC + TVC$$

Thus, at the break-even point the total revenue must equal the total cost. Substituting this value into the numerator of gives you:

$$CR = (TR - TVC)/TR \times 100$$

$$CR = ((TFC + TVC) - TVC)/TR \times 100$$

$$CR = TFC/TR \times 100$$

A final rearrangement results in Formula 1.5, which expresses the break-even point in terms of total revenue dollars.

**TR is Total Revenue at Break-even:** The total amount of dollars that the company must earn as revenue to pay for all of the fixed and variable costs. At this level of revenue, the net income equals zero.

**TFC is Total Fixed Costs:** The total of all costs that are not affected by the level of output.

**Formula 1.5 Dollar Break-Even:**

$$TR = \frac{TFC}{CR}$$

**CR is Contribution Rate:** The percentage of the selling price, expressed in decimal format, available to pay for fixed costs.

## HOW IT WORKS

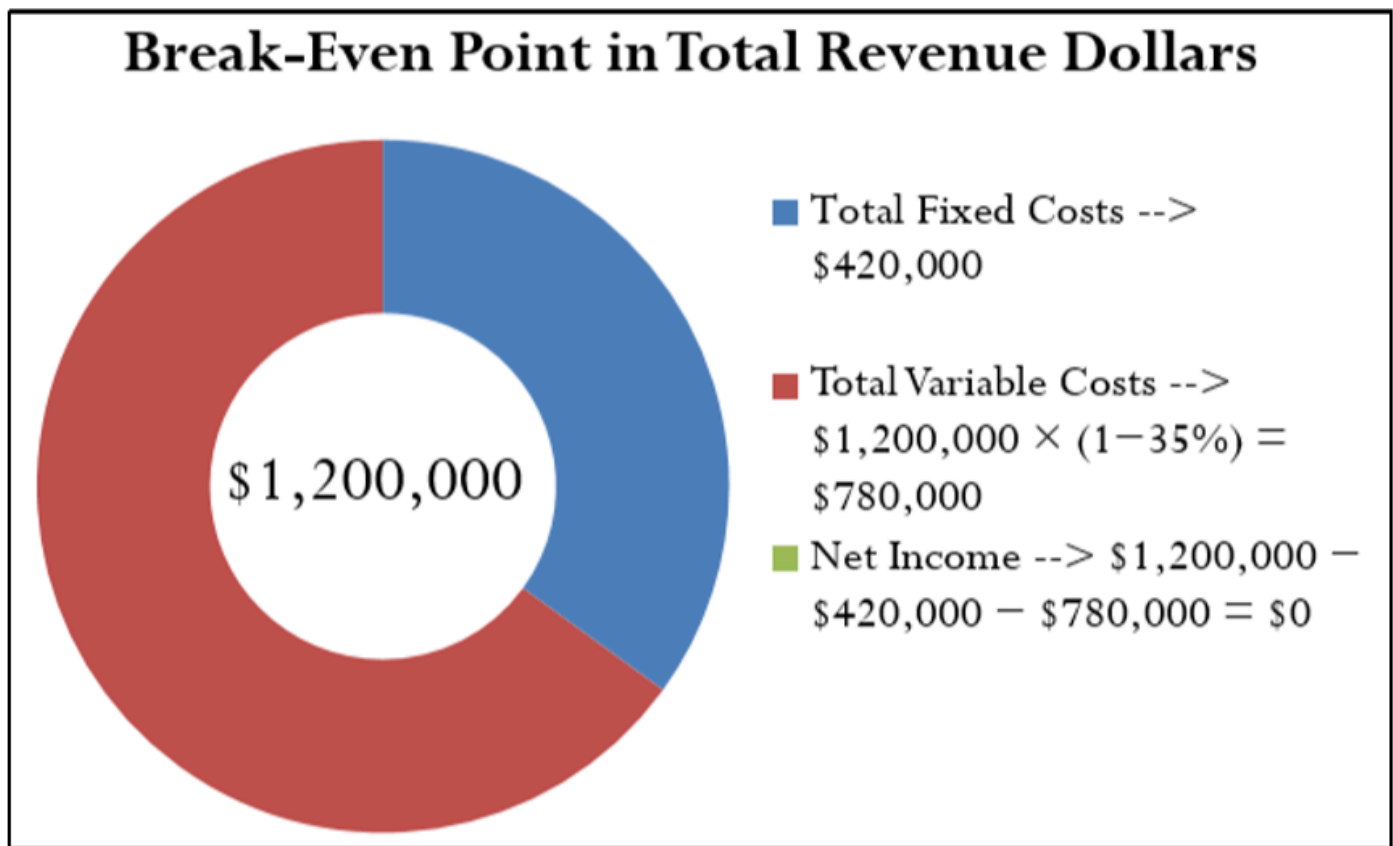
Follow these steps to calculate the break-even point in total revenue dollars:

Step 1: Calculate or identify the total fixed costs (TFC).

Step 2: Calculate the contribution rate (CR), by applying any needed techniques or formulas. If not provided, typically the CR is calculated using Formula 1.6, which requires aggregate information only.

$$CR = \frac{TR - TV}{TR} \times 100$$

Step 3: Apply Formula 1.5 to calculate the break-even point in dollars.



Assume that you are looking at starting your own business. The fixed costs are generally easier to calculate than the variable costs. After running through the numbers, you determine that your total fixed costs are \$420,000, or  $TFC = \$420,000$ . You are not sure of your variable costs but need to gauge your break-even point. Many of your competitors are publicly traded companies, so you go online and pull up their annual financial reports. After analyzing their financial statements, you see that your competitors have a contribution rate of 35%, or  $CR = 0.35$ , on average. What is your estimate of your break-even point in dollars?

Step 1: Total fixed costs are  $TFC = \$420,000$ .

Step 2: The estimated contribution rate is  $CR = 0.35$ .

Step 3: Applying Formula 1.5 results in  $TR = \$420,000 \div 0.35 = \$1,200,000$ . If you average a similar contribution rate, you require total revenue of \$1,200,000 to cover all costs, which is your break-even point in dollars.

#### IMPORTANT NOTES

You need to be very careful with the interpretation and application of a break-even number. In particular, the break-even must have a point of comparison, and it does not provide information about the viability of the business.

Break-Even Points Need to Be Compared. The break-even number by itself, whether in units or dollars, is meaningless. You need to compare it against some other quantity (or quantities) to determine the feasibility of the number you have produced. The other number needs to be some baseline that allows you to grasp the scope of what you are planning. This baseline could include but is not limited to the following:

- Industry sales (in units or dollars)
- Number of competitors fighting for market share in your industry
- Production capacity of your business

For example, in your Internet business the break-even point is 179 units per month. Is that good? In the section opener, you explored a possibility where your industry had total monthly sales of 1,000 units and you faced eight competitors. A basic analysis shows that if you enter the industry and if everyone split the market evenly, you would have sales of 1,000 divided by nine companies, equal to 111 units each. To just pay your bills, you would have to sell almost 61% higher than the even split and achieve a 17.9% market share. This doesn't seem very likely, as these other companies are already established and probably have satisfied customers of their own that would not switch to your business.

Break-Even Points Are Not Green Lights. A break-even point alone cannot tell you to do something, but it can tell you not to do something. In other words, break-even points can put up red lights, but at no point does it give you the green light. In the above scenario, your break-even of 179 units put up a whole lot of red lights since it does not seem feasible to obtain. However, what if your industry sold 10,000 units instead of 1,000 units? Your break-even would now be a 1.79% market share (179 units out of 10,000 units), which certainly seems realistic and attainable. This does not mean "Go for it," however. It just means that from a strictly financial point of view breaking even seems possible. Whether you can actually sell that many units depends on a whole range of factors beyond just a break-even number. For instance, if your Google ad is written poorly you might not be able to generate that many sales. The break-even analysis cannot factor in this non-quantitative variable, and for that reason it cannot offer a "go ahead."

## EXERCISE : GIVE IT SOME THOUGHT

---

What would happen to the break-even point in each of the following situations? Would it increase, decrease, or remain the same?

1. The unit contribution margin increases.
2. The total fixed costs increase.
3. The contribution rate decreases.

### EXAMPLE: DETERMINING THE BREAK-EVEN DOLLARS

---

In the annual report to shareholders, Borland Manufacturing reported total gross sales of \$7,200,000, total variable costs of \$4,320,000, and total fixed costs of \$2,500,000. Determine Borland's break-even point in dollars.

#### Solution

---

Calculate the dollar break-even point, which is the total revenue (TR) at break-even.

#### What You Already Know

---

Step 1:

The total fixed costs are known:  $TFC = \$2,500,000$ .

Other known information includes the following:

$$TR = \$7,200,000, TVC = \$4,320,000$$

#### How You Will Get There

---

Step 2: Calculate the contribution rate by applying Formula 1.6.

Step 3: Apply Formula 1.5.

Perform

$$\text{Step 2: } CR = \frac{\$7,200,000 - \$4,320,000}{\$7,200,000} \times 100 = \frac{\$2,880,000}{\$7,200,000} \times 100 = 40$$

$$\text{Step 3: } TR = \frac{\$2,500,000}{40} = \frac{\$2,500,000}{0.4} = \$6,250,000$$

Borland Manufacturing achieves its break-even point at \$6,250,000 in total revenue. At this point, total fixed costs are \$2,500,000 and total variable costs are \$3,750,000, producing a net income of zero.

### CHAPTER ATTRIBUTION

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## 1.7 Symbols and Formulas

### SYMBOLS USED

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BEDMAS: An acronym for Brackets, Exponents, Division, Multiplication, Addition, and Subtraction.

CM = Contribution Margin

CR = Contribution Rate

d = Discount Rate

D\$: = Markdown Amount

n = a frequency or count or total

N = in merchandising it is the net price; in single payment compound interest it is the number of compound periods; in annuity compound interest it is the number of annuity payments

NI = net income

TFC = Total Fixed Costs

TR = Total Revenue

TV = Total Variable

TVC = Total Variable Cost

VC: = Unit Variable Cost

x = any individual piece of data

---

FORMULAS USED

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**Formula 1.1 Unit Variable Cost**

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$$VC = \frac{TVC}{n}$$

**Formula 1.2 Net Income Using a Total Revenue and Cost Approach**

---

$$NI = n(S) - (TFC + n(VC))$$

**Formula 1.3 Break-Even Analysis**

---

$$CM = S - VC$$

**Formula 1.4 Unit Break-Even**

---

$$n = \frac{TVC}{CM}$$

**Formula 1.5 Dollar Break-Even**

---

$$TR = \frac{TVC}{CR}$$

**Formula 1.6 Contribution Rate**

---

$$CR = \frac{TR - TV}{TR} \times 100$$

## 2 MATHEMATICS OF MERCHANDISING

### WHAT IS IT GOING TO COST ME?

---

When you buy an iPod, it is very important that the right price is set. The price should

- be seen as fair by you the buyer,
- pay for the costs (plastics, battery, buttons, circuit boards, headset) and expenses (employees, factory, electricity, distribution) of making the iPod, and
- allow the seller's business to make some extra money as profit so that it can grow its business further.

If your business is selling a product, you will need to pay close attention to price adjustments because they affect profitability. Various discounts, like putting items on sale, may increase sales while lowering the amount of profit per transaction. How do you know where to set the balance to maximize profit overall?

As a student in a business program, consider this chapter essential to the success of any business. Whether your pricing strategy is high or low, the company must ensure that it can still pay its bills as a minimum requirement. And that requires careful juggling of many factors. If it fails to manage its pricing properly, the company will go bankrupt!

This chapter will make you a smarter business professional and a wiser consumer. You shop retail almost every day and regularly purchase goods and services. If you understand how product pricing works, you can make sense of "deals." You can easily explain why the same product sells for two different prices at two different stores.

In this chapter, you must learn the language of marketers to perform merchandising mathematics involving product costs, expenses, prices, markups, markdowns, and ultimately profitability. Once the study of the various pricing components is complete, we will see how the various pieces of the pricing puzzle fit together into a cohesive merchandising environment.

*Adaptation note: Chapter 6 in Business Math: A Step-by-Step handbook*

## 2.1 Figuring Out the Cost: Discounts

### HOW MUCH?

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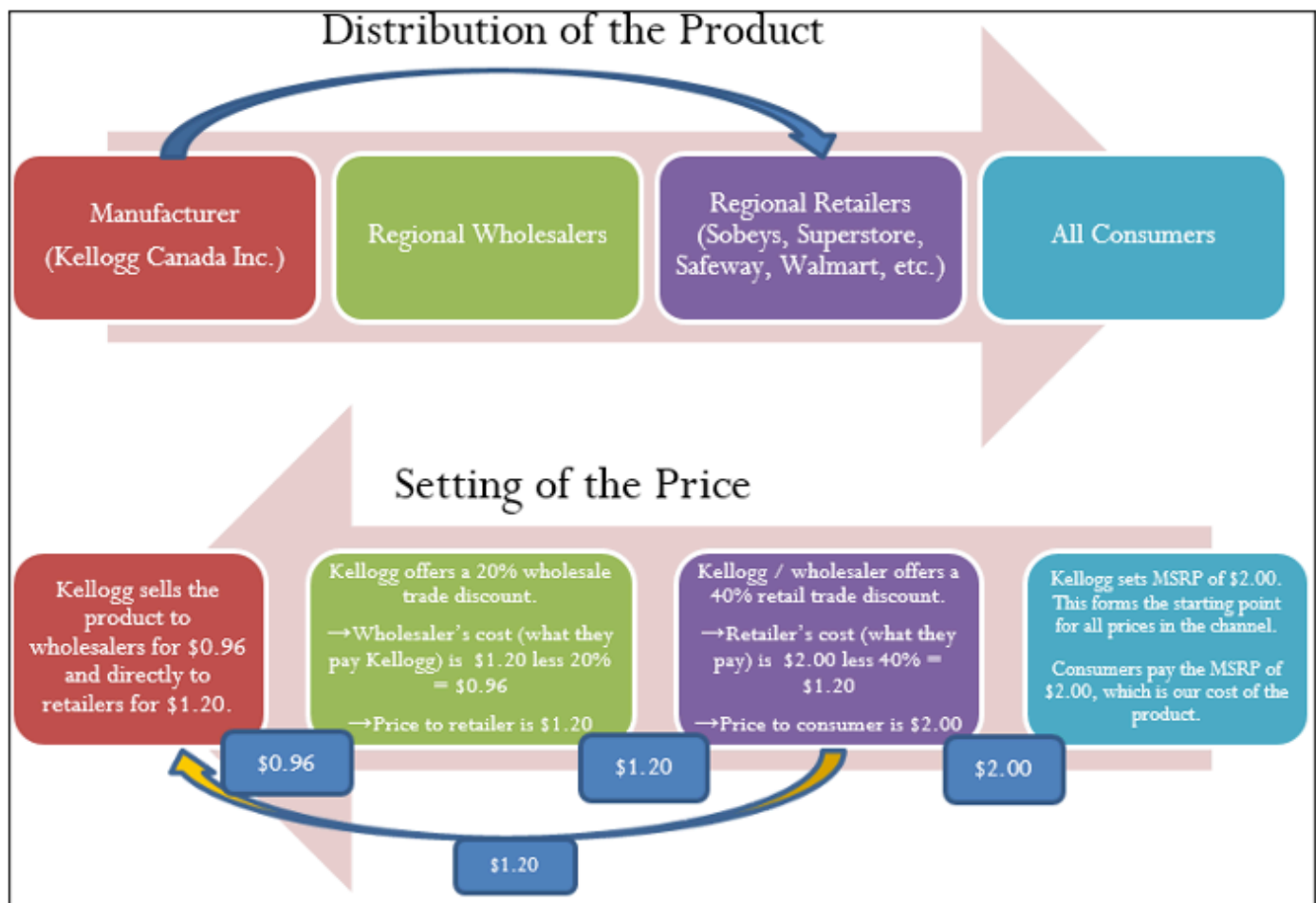
You mutter in exasperation, “Why can’t they just set one price and stick with it?” Your mind boggles at all the competing discounts you encounter at the mall in your search for that perfect Batman toy for your nephew. Walmart is running their Rollback promotion and is offering a Batmobile for 25% off, regularly priced at \$49.99. Toys R’ Us has an outlet in the parking lot where the regular price for the same toy is \$59.99, but all Batman products are being cleared out at 40% off. You head over to The Bay for a warehouse clearance event that has the same toy priced at \$64.99 but at 35% off. It is also Bay Days, which means you can scratch and win a further 10% to 20% off the sale price. You go to Dairy Queen for a Blizzard to soothe your headache while you figure things out.

The **cost** of a product is the amount of money required to obtain the merchandise. If you are a consumer, the ticketed price tag on the product is your cost. If you are a reseller (also known as a *middleman* or *intermediary*), what you pay to your supplier for the product is your cost. If you are a manufacturer, then your cost equals all of the labour, materials, and production expenditures that went into creating the product.

A **discount** is a reduction in the price of a product. As a consumer, you are bombarded with discounts all the time. Retailers use various terms for discounts, including *sales* or *clearance*. If your business purchases a product from a supplier, any discount it receives lowers how much the business pays to acquire the product. When a business buys products, the price paid is the cost to the business. Therefore, a lower price means a lower cost.

If your business is the one selling the product, any discount offered lowers the selling price and reduces revenue per sale. Since the revenue must cover all costs and expenses associated with the product, the lower price means that the business reduces profits per sale. In business, it is common practice to express a discount as a percentage off the regular price.

## HOW DISTRIBUTION AND PRICING WORK



Start with distribution in the top half of the figure and work left to right. As an example, let's look at a manufacturer such as Kellogg Canada Inc. (which makes such products as Pop-Tarts, Eggo Waffles, and Rice Krispies). Kellogg's Canadian production plant is located in London, Ontario. To distribute its products to the rest of Canada, Kellogg Canada uses various regional wholesalers. Each wholesaler then resells the product to retailers in its local trade area; however, some retailers (such as the Real Canadian Superstore) are very large, and Kellogg Canada distributes directly to these organizations, bypassing the wholesaler as represented by the blue arrow. Finally, consumers shop at these retailers and acquire Kellogg products.

The relationship of distribution to pricing is illustrated in the bottom half of the figure, working right to left. For now, focus on understanding how pricing works; the mathematics used in the figure will be explained later in this chapter. Kellogg Canada sets a **manufacturer's suggested retail price**, known as the MSRP. This is a recommended retail price based on consumer market research. Since grocery retailers commonly carry thousands or tens of thousands of products, the MSRP helps the retailer to determine the retail price at which the product should be listed. In this case, assume a \$2.00 MSRP, which is the price consumers will pay for the product.

The retailer must pay something less than \$2.00 to make money when selling the product. Kellogg Canada understands its distributors and calculates that to be profitable most retailers must pay approximately 40% less than the MSRP. Therefore, it offers a 40% discount. If the retailer purchases directly from Kellogg, as illustrated by the yellow arrow, the price paid by the retailer to acquire the product is \$2.00 less 40%, or \$1.20. Smaller retailers acquire the product from a wholesaler for the same price. Thus, the retailer's cost equals the wholesaler's price (or Kellogg Canada's price if the retailer purchases it directly from Kellogg).

The wholesaler's price is \$1.20. Again, Kellogg Canada, knowing that the wholesaler must pay something less than \$1.20 to be profitable, offers an additional 20% discount exclusively to the wholesaler. So the price paid by the wholesaler to acquire the product from Kellogg Canada is \$1.20 less 20%, or \$0.96. This \$0.96 forms Kellogg Canada's price to the wholesaler, which equals the wholesaler's cost.

In summary, this discussion illustrates two key pricing concepts:

1. Companies higher up in the distribution channel pay lower prices than those farther down the channel. Companies receive discounts off the MSRP based on their level in the distribution system. This may result in multiple discounts, such as a wholesaler receiving both the retailer's discount and an additional discount for being a wholesaler.
2. One organization's price becomes the next organization's cost (assuming the typical distribution channel structure):
  - Manufacturer's Price = Wholesaler's Cost
  - Wholesaler's Price = Retailer's Cost
  - Retailer's Price = Consumer's Cost

## TYPES OF DISCOUNTS

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You will perform discount calculations more effectively if you understand how and why single pricing discounts and multiple pricing discounts occur. Businesses or consumers are offered numerous types of discounts, of which five of the most common are trade, quantity, loyalty, sale, and seasonal.

### *Trade Discounts*

---

A **trade discount** is a discount offered to businesses only based on the type of business and its position in the distribution system (e.g., as a retailer, wholesaler, or any other member of the distribution system that resells the product). Consumers are ineligible for trade discounts. In the discussion of the figure, two trade discounts are offered. The first is a 40% retail trade discount, and the second is a 20% wholesale trade discount. Typically, a business that is higher up in the distribution system receives a combination of these trade discounts. For example, the wholesaler receives both the 40% retail trade discount and the 20% wholesale trade discount from the MSRP. The wholesaler's cost is calculated as an MSRP of \$2.00 less 40% less 20% = \$0.96.

### *Quantity Discounts*

---

A **quantity discount** (also called a *volume discount*) is a discount for purchasing larger quantities of a certain product. If you have ever walked down an aisle in a Real Canadian Superstore, you probably noticed many shelf tags that indicate quantity discounts, such as “Buy one product for \$2” or “Take two products for \$3.” Many Shell gas stations offer a Thirst Buster program in which customers who purchase four Thirst Busters within a three-month period get the fifth one free. If the Thirst Busters are \$2.00 each, this is equivalent to buying five drinks for \$10.00 less a \$2.00 quantity discount.

### *Loyalty Discounts.*

---

A **loyalty discount** is a discount that a seller gives to a purchaser for repeat business. Usually no time frame is specified; that is, the offer is continually available. As a consumer, you see this regularly in marketing programs such as Air Miles or with credit cards that offer cash back programs. For example, Co-op gas stations in Manitoba track consumer gasoline purchases through a loyalty program and mails an annual loyalty discount cheque to its customers, recently amounting to 12.5¢ per litre purchased. In business-to-business circles, sellers typically reward loyal customers by deducting a loyalty discount percentage, commonly ranging from 1% to 5%, from the selling price.

### *Sale Discounts*

---

A **sale discount** is a temporary lowering of the price from a product's regular selling price. Businesses put items on sale for a variety of reasons, such as selling excess stock or attracting shoppers. You see such promotional events all the time: LED monitors are on sale at Best Buy; Blu-Ray discs are half off at Walmart; The Brick is having a door crasher event Saturday morning.

### *Seasonal Discounts*

---

A **seasonal discount** is a discount offered to consumers and businesses for purchasing products out of season. At the business level, manufacturers tend to offer seasonal discounts encouraging retailers, wholesalers, or distributors to purchase products before they are in season. Bombardier Inc. manufactures Ski-Doos, which are sold in Canada from approximately November through March—a time of year when most of the country has snow and consumers would want to buy one. To keep production running smoothly from April through October, Bombardier could offer seasonal discounts to its wholesalers and retailers for the coming winter season. At a retail level, the examples are plentiful. On November 1 most retailers place their Halloween merchandise on seasonal discount to clear out excess inventory, and many retailers use Boxing Day (or Boxing Week) to clear their out-of-season merchandise.

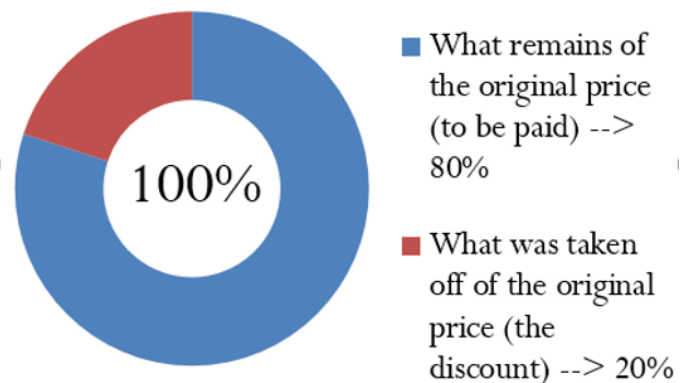
## Single Discounts

Let's start by calculating the cost when only one discount is offered. Later in this section you will learn how to calculate a cost involving multiple discounts.

### THE FORMULA

Figuring out the price after applying a single discount is called a net price calculation. When a business calculates the net price of a product, it is interested in what you still have to pay, not in what has been removed. Note in Formula 2.1 below that you take 1 and subtract the discount rate to determine the rate owing. If you are eligible for a 20% discount, then you must pay 80% of the list price, as illustrated in the figure to the right.

#### 20% Discount Interpretation



### FORMULA 2.1

**N is Net Price:** The **net price** is the price of the product after the discount is removed from the list price. It is a dollar amount representing what price remains after you have applied the discount.

Formula 2.1 – Single Discount: 
$$N = L \times (1 - d)$$

**L is List Price:** The **list price** is the normal or regular dollar price of the product before any discounts. It is the **Manufacturer's Suggested Retail Price (MSRP)** - a price for a product that has been published or advertised in some way), or any dollar amount before you remove the discount.

**d is Discount Rate:** The discount rate represents the percentage (in decimal format) of the list price that is deducted.

Formula 2.1 once again applies Formula 2.2 on rate, portion, and base, where the list price is the base, the  $(1 - d)$  is the rate, and the net price represents the portion of the price to be paid.

Notice that Formula 2.1 requires the discount to be in a percentage (decimal) format; sometimes a discount is expressed as a dollar amount, though, such as "Save \$5 today." Formulas 2.2a and 2.2b relate the discount dollar amount to the list price, discount percent, and net price. Choose one formula or the other depending on which variables are known.

## FORMULA 2.2

**D\$ is Discount Amount:** Determine the discount amount in one of two ways, depending on what information is known. If the list price and discount rate are known, apply Formula 6.2a. If the list price and net price are known, apply Formula 6.2b. Either of these formulas can be algebraically manipulated to solve for any other unknown variable as well.

**Formula 2.2a – Discount Amount:**  $D\$ = L \times d$

**Formula 2.2b – Discount Amount:**  $D\$ = L - N$

**L is List Price:** The dollar amount of the price before any discounts.

**N is Net Price:** The dollar amount of the price after you have deducted all discounts.

**d is Discount Rate:** The percentage (in decimal format) of the list price to be deducted. This time you are interested in figuring out the amount of the discount, therefore you do not take it away from 1.

## HOW IT WORKS

Follow these steps to calculate the net price involving a single discount. These steps are adaptable if the net price is a known variable and one of the other variables is unknown.

**Step 1:** Identify any known variables, including list price, discount rate, or discount amount.

**Step 2:** If the list price is known, skip this step. Otherwise, solve for list price using an appropriate formula.

**Step 3:** Calculate the net price.

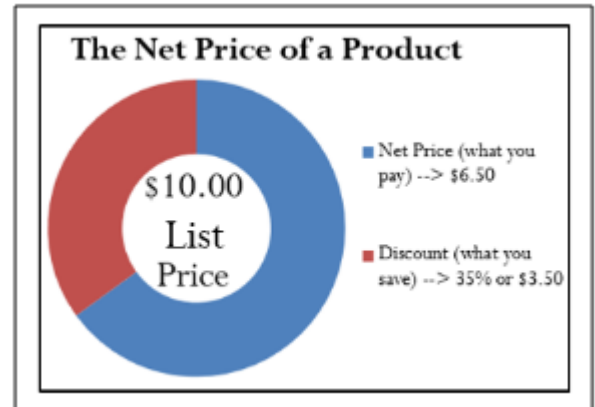
- If the list price and discount are known, apply Formula 2.1.
- If the list price and discount amount are known, apply Formula 2.2b and rearrange for N.

Assume a product sells for \$10 and is on sale at 35% off the regular price. Calculate the net price for the product.

**Step 1:** The list price of the product is  $L = \$10$ . It is on sale with a discount rate of  $d = 0.35$ .

**Step 2:** List price is known, so this step is not needed.

**Step 3:** Applying Formula 2.1 results in a new price of  $N = \$10 \times (1 - 0.35) = \$6.50$ . Note that if you are interested in learning the discount amount, you apply Formula 2.2b to calculate  $D\$ = \$10 - \$6.50 = \$3.50$ .



## IMPORTANT NOTES

You can combine Formula 2.1 and either version of Formula 2.2 in a variety of ways to solve any single discount situation for any of the three variables. As you deal with increasingly complicated pricing formulas, your algebraic skills in solving linear equations and substitution become very important.

Many of the pricing problems take multiple steps that combine various formulas, so you need to apply the PUPP model systematically. In any pricing problem, you must understand which variables are provided and match them up to the known formulas. To get to your end goal, you must look for formulas in which you know all but one variable. In these cases, solving for variables will move you forward toward solving the overall pricing problem.

If you find you cannot produce a formula with only one unknown variable, can you find two formulas with the same two unknowns? If so, you can use your algebraic skills to find the roots of the two equations simultaneously. Alternatively, you can solve one formula for a variable then substitute it into the other formula, allowing you to isolate the remaining variable. Throughout the examples in this chapter you will see many applications of these algebraic skills.

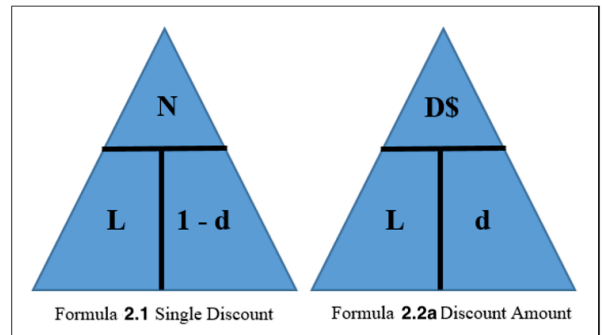
## THINGS TO WATCH OUT FOR

Remember to apply the rounding rules:

1. Until you arrive at the final solution, avoid rounding any interim numbers unless you have some special reason to do so.
2. Round all dollar amounts to the nearest cent. If the dollar amount has no cents, you may write it either without the cents or with the ".00" at the end.
3. Round all percentages to four decimals when in percent format.

## PATHS TO SUCCESS

When working with single discounts, you are not always solving for the net price. Sometimes you must calculate the discount percent or the list price. At other times you know information about the discount amount but need to solve for list price, net price, or the discount rate. The triangle technique can remind you how to rearrange the formulas for each variable, as illustrated in the figure to the right.



### Examples

1. Will you pay more than, less than, or exactly \$10.00 for a product if you are told that you are paying:
  - a. a net price of \$10.00 when there is a discount of 25%?
  - b. a list price of \$10.00 when there is a discount of 25%?
2. If an item is subject to a 40% discount, will the net price be more than or less than half of the list price of the product?

### Solution

- a. Exactly \$10. The net price is the price after the discount.
  - b. Less than \$10. The discount needs to be removed from the list price.
2. More than half. A 40% discount means that you will pay 60% of the list price

### Example 2.1A: Determining the Retailer's Net Price for a Pair of Jeans

A manufacturer that sells jeans directly to its retailers uses market research to find out it needs to offer a 25% trade discount. In doing so, the retailers will then be able to price the product at the MSRP of \$59.99. What price should retailers pay for the jeans?

### Solution

#### Plan

Calculate how much a retailer should pay for the jeans after the regular price has been discounted to accommodate the trade discount. This is called the net price for the product, or  $N$ .

**What You Already Know**

---

Step 1: The list price and the discount rate are known:

$$L = \$59.99 \quad d = 0.25$$

**How You Will Get There**

---

Step 2: List price is known, so skip this step.

Step 3: Apply Formula 2.1.

**Perform**

---

$$\text{Step 3: } N = \$59.99 \times (1 - 0.25) = \$59.99 \times 0.75 = \$44.99$$

**Present**

---

The manufacturer should sell the jeans to the retailers for \$44.99

**Example 2.1B: Determining the List Price of a Jacket**

Winners pays a net price of \$27.50 for a winter jacket after receiving a retail trade discount of 45%. What was the MSRP of the jacket?

**Solution****Plan**

---

Calculate the MSRP for the jacket before Winners received the retail trade discount to arrive at the net price. This is called the list price for the product, or L.

**What You Already Know**

---

Step 1: The net price and the discount rate are known:

$$N = \$27.50 \quad d = 0.45$$

**How You Will Get There**

---

Step 2: List price is the unknown variable; skip this step.

Step 3: Apply Formula 2.1, rearranging for L.

### Perform

Step 3:  $\$27.50 = L(1 - 0.45)$   $L = \$27.50 \div (1 - 0.45)$

$L = \$27.50 \div 0.55 = \$50.00$

### Present

The MSRP, or list price, of the winter jacket is \$50.00.

You are shopping at Mountain Equipment Co-op for a new environmentally friendly water bottle. The price tag reads

\$14.75, which is \$10.24 off the regular price. Determine the discount rate applied.

### Example 2.1C: Determining the Discount Percent and Discount Amount

You are shopping at Mountain Equipment Co-op for a new environmentally friendly water bottle. The price tag reads \$14.75, which is \$10.24 off the regular price. Determine the discount rate applied.

### Solution

#### Plan

You need to find out how the sale price translates into the discount rate, or d.

#### What You Already Know

Step 1: The discount amount and net price are known:

$D\$ = \$10.24$   $N = \$14.75$

#### How You Will Get There

Step 2: Use Formula 2.2b to calculate the list price, rearranging for L.

Step 3: Convert the discount amount into a percentage by applying Formula 2.2a, rearranging for d.

### Perform

Step 2: List price:  $\$10.24 = L - \$14.75$ , which results in  $L = \$24.99$  Step 3: Discount rate:  $d = D\$ \div L$

$$d = \$10.24 = 0.409764 \text{ or } 40.9764\%$$

\$24.99

### Present

---

The water bottle today has been reduced in price by the amount of \$10.24. This represents a sale discount of 40.9674%.

## MULTIPLE DISCOUNTS

---

You are driving down the street when you see a large sign at Old Navy that says, “Big sale, take an additional 25% off already reduced prices!” In other words, products on sale (the first discount) are being reduced by an additional 25% (the second discount). Because Formula 2.1 handles only a single discount, you must use an extended formula in this case.

## THE FORMULA

---

Businesses commonly receive more than one discount when they make a purchase. Consider a transaction in which a business receives a 30% trade discount as well as a 10% volume discount. First, you have to understand that this is not a  $30\% + 10\% = 40\%$  discount. The second discount is always applied to the net price after the first discount is applied. Therefore, the second discount has a smaller base upon which it is calculated. If there are more than two discounts, you deduct each subsequent discount from continually smaller bases. Formula 2.3 expresses how to calculate the net price when multiple discounts apply.

### FORMULA 2.3 MULTIPLE DISCOUNTS

**N is Net Price:** The dollar amount of the price after all discounts have been deducted.

**L is List Price:** The dollar amount of the price before any discounts.

**Formula 2.3 – Multiple Discounts:**

$$N = L \times (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

**$d_1$  is First Discount,  $d_2$  is Second Discount,  $d_n$  is  $n$ th Discount:** When there is more than one discount, you must extend beyond Formula 6.1 by multiplying another discount expression. These discounts are represented by the same  $d$  symbol; however, each discount receives a subscript to make its symbol unique. Therefore, the first discount receives the symbol of  $d_1$ , the second discount receives the symbol  $d_2$ , and so on. Recall that the symbol  $n$  represents the number of pieces of data (a count), so you can expand or contract this formula to the exact number of discounts being offered.

It is often difficult to understand exactly how much of a discount is being received when multiple discounts are involved. Often it is convenient to summarize the multiple discount percentages into a single percentage. This makes it easier to calculate the net price and aids in understanding the discount benefit. Simplifying multiple percent discounts into a single percent discount is called finding the **single equivalent discount**. Whether you apply the multiple discounts or just the single equivalent discount, you arrive at the same net price. The conversion of multiple discount percentages into a single equivalent discount percent is illustrated in Formula 2.4.

### FORMULA 2.4

**$d_{\text{equiv}}$  (or just  $d$ ) is the single equivalent discount rate** that is equal to the series of multiple discounts. Recall that taking  $(1 - d)$  calculates what you pay. Therefore, if you take 1, which represents the entire amount, and reduce it by what you pay, the rate left over must be what you did not pay. In other words, it is the discount rate.

**Formula 2.4 – Single Equivalent Discount:**

$$d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

**$d_1$  is First Discount,  $d_2$  is Second Discount,  $d_n$  is  $n$ th Discount:** This is the same notation as in Formula 6.3. Since there are multiple discounts, each discount receives a numerical subscript to give it a unique identifier. You can expand or contract the formula to the exact number of discounts being offered.

## HOW IT WORKS

Refer back to the steps in calculating net price. The procedure for calculating a net price involving a single discount extends to a more generic procedure involving multiple discounts. As with the single discount procedures, you can adapt the model if the net price is known and one of the other variables is unknown. Follow these steps to calculate the net price involving any number of discounts:

**Step 1:** Identify any known variables, including list price, discount rate(s), or discount amount.

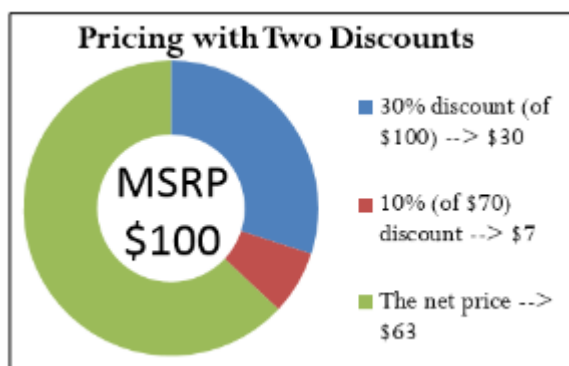
**Step 2:** If the list price is known, skip this step. Otherwise, solve for list price.

- If only one discount is involved, apply Formula 2.2a.
- If more than one discount is involved, the discount amount represents the total discount amount received from all of the discounts combined. This requires you first to convert the multiple discount rates into an equivalent single discount rate using Formula 2.4 and then to apply Formula 2.2a.

**Step 3:** Calculate the net price.

- If the list price and only a single known discount rate are involved, apply Formula 2.1.
- If the list price and multiple discount rates are known and involved, apply Formula 2.3.
- If the list price and the total discount amount are known, apply Formula 2.2b and rearrange for N.

Assume a product with an MSRP of \$100 receives a trade discount of 30% and a volume discount of 10%.



### Calculate the net price.

- Step 1: The list price and discounts are  $L = \$100$ ,  $d_1 = 0.30$  and  $d_2 = 0.10$ .
- Step 2: List price is known, so skip this step.
- Step 3: Apply Formula 2.3 to calculate the net price:

$$N = \$100 \times (1 - 0.30) \times (1 - 0.10) = \$63$$

The net price is \$63, which is illustrated above.

If you are solely interested in converting multiple discounts into a single equivalent discount, you need only substitute into Formula 2.4. In the above example, the product received a trade discount of 30% and a volume discount of 10%. To calculate the single equivalent discount, apply Formula 2.4:

$$d_{equiv} = 1 - (1 - 0.30) \times (1 - 0.10)$$

$$d_{equiv} = 1 - (0.70)(0.90)$$

$$d_{equiv} = 1 - 0.63 = 0.37$$

Therefore, whenever discounts of 30% and 10% are offered together, the single equivalent discount is 37%. Whether it is the multiple discounts or just the single equivalent discount that you apply to the list price, the net price calculated is always the same.

### IMPORTANT NOTES

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**Order of Discounts:** The order of the discounts *does not* matter in determining the net price. Remember from the rules of BEDMAS that you can complete multiplication in any order. Therefore, in the above example you could have arrived at the \$63 net price through the following calculation:

$$\$100 \times (1 - 0.10) \times (1 - 0.30) = \$63$$

The order of the discounts *does* matter if trying to interpret the value of any single discount. If the trade discount is applied before the quantity discount and you are wanting to know the quantity discount amount, then the quantity discount needs to be second. Thus,

$$\$100 \times (1 - 0.30) = \$70$$

$$\$70 \times 0.10 = \$7$$

which is the amount of the quantity discount.

### Price Does Not Affect Single Equivalent Discount

---

Notice in Formula 2.4 that the list price and the net price are not involved in the calculation of the single equivalent discount. When working with percentages, whether you have a net price of \$6.30 and a list price of \$10, or a net price of \$63 and a list price of \$100, the equivalent percentage always remains constant at 37%.

## Things To Watch Out For

A common mistake when working with multiple discounts is to add the discounts together to calculate the single equivalent discount. This mistaken single discount is then substituted into Formula 2.1 to arrive at the wrong net price. Remember that if two discounts of 30% and 10% apply, you cannot sum these discounts. The second discount of 10% is applied on a smaller price tag, not the original price tag. To calculate the net price you must apply Formula 2.3.

## PATHS TO SUCCESS

If you happen to know any two of the net price (N), list price (L), or the total discount amount (D\$), then you could also use Formula 2.2 to solve for the single equivalent discount, dequiv. For example, if you know the net price is \$63 and the total discount amount for all discounts is \$37, you could use Formula 2.2b to figure out that the list price is \$100, then convert the discount amount into a percentage using Formula 2.2a. This method will also produce a single equivalent discount of 37%.

Another method of calculating the single equivalent discount is to recognize Formula 2.2a as an application involving percent change. The variable  $d$  is a discount rate, which you interpret as a negative percent change. The discount amount,  $D\$$ , is the difference between the list price (representing the Old price) and the net price (representing the New price after the discount). Therefore, Formula 2.2a can be rewritten as follows:

$$D\$ = L \times d$$

becomes

$$\text{New} - \text{Old} = \text{Old} \times \%$$

$$\frac{\text{New} - \text{Old}}{\text{Old}} = \triangle \%$$

Therefore, any question about a single equivalent discount where net price and list price are known can be solved as a percent change. Using our ongoing net price example, you have:

$$\frac{\$63 - \$100}{\$100} = -0.37 \text{ or } -37\%$$

This is a discount of 37%.

## GIVE IT SOME THOUGHT

If you are offered discounts in the amount of 25%, 15%, 10%, and 5%, will your total discount percent be 55%, less than 55%, or more than 55%?

Solution

Less than 55%. Each percent discount is calculated from a smaller base.

#### Example 2.1D: Retailer Purchasing Ski-Doods with Multiple Discounts

A retail dealership purchases some Expedition TUV Yeti II Ski-Doods to stock in its stores. Examining the merchandising terms of the manufacturer, Bombardier, the dealership notices that it would be eligible to receive a 35% trade discount, 15% volume discount, and 3% loyalty discount. Because it is June and Ski-Doods are out of season, Bombardier offers a seasonal discount of 12% for purchases made before June 30. If the MSRP for the Ski-Doo is \$12,399.00 and the dealership purchases this item on June 15, what price would it pay?

#### Solution

##### Plan

You are looking for the net price that the retail dealership will pay for the Ski-Doo, or N.

##### What You Already Know

Step 1: The retail dealership is eligible for all four discounts (it qualifies for the seasonal discount since it is purchasing before June 30).

Therefore,  $L = \$12,399.00$ ,  $d_1 = 0.35$ ,  $d_2 = 0.15$ ,  $d_3 = 0.03$ , and  $d_4 = 0.12$

##### How You Will Get There

Step 2: You know the list price, so skip this step.

Step 3: Apply Formula 2.3.

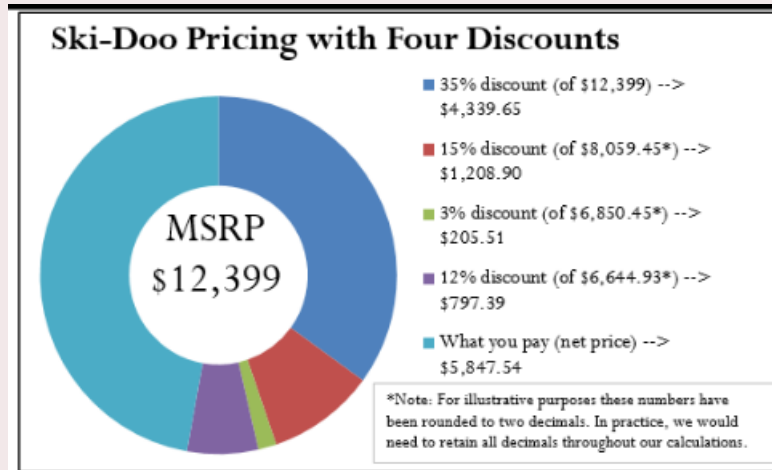
##### Perform

Step 3:  $N = \$12,399 \times (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$

$N = \$12,399 \times 0.65 \times 0.85 \times 0.97 \times 0.88 = \$5,847.54$

##### Present

After all four discounts, the retail dealership could purchase the Ski- Doo for \$5,847.54.



### Example 2.1E: Reducing Multiple Discounts to a Single Equivalent Discount

The retail dealership in Example 2.1D purchases more products subject to the same discounts. It needs to simplify its calculations. Using the information from Example 2.1D, what single equivalent discount is equal to the four specified discounts?

#### Solution

##### Plan

You are looking for a single equivalent discount that is equal to the four discount percentages, or dequiv (or just d).

##### What You Already Know

You know the discount rates:

$$d_1 = 0.35, d_2 = 0.15, d_3 = 0.03, \text{ and } d_4 = 0.12$$

##### How You Will Get There

Apply Formula 2.4.

##### Perform

$$d_{\text{equiv}} = 1 - (1 - 0.35) \times (1 - 0.15) \times (1 - 0.03) \times (1 - 0.12)$$

$$d_{\text{equiv}} = 1 - (0.65)(0.85)(0.97)(0.88) = 1 - 0.471614 = 0.528386 \text{ or } 52.8386\%$$

## Present

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The retail dealership can apply a 52.8386% discount to all the products it purchases.

### Example 2.1F: Making a Smart Consumer Purchase

You are shopping on Boxing Day for an 80" HDTV. You have just one credit card in your wallet, a cashback Visa card, which allows for a 1% cash rebate on all purchases. While scanning flyers for the best deal, you notice that Visions is selling the TV for \$5,599.99 including taxes, while Best Buy is selling it for \$5,571.99 including taxes. However, because of a computer glitch Best Buy is unable to accept Visa today. Where should you buy your television?

## Solution

### Plan

---

You want to know which store you should buy the television at. You must calculate the net price (N) for each of the stores.

### What You Already Know

---

Step 1: You know the list price for each of the stores. You also know the discount available from Visa. Thus,

$L_{\text{Best Buy}} = \$5,571.99$  with no discounts since Visa cannot be used there

$L_{\text{Visions}} = \$5,599.99$ ,  $d = 0.01$ , since you can use your Visa card there

### How You Will Get There

---

Step 2: List price is known, so skip this step.

Step 3: Apply Formula 2.1.

## Perform

---

Step 3: Best Buy: No discounts apply, so the list price equals the net price and  $N = \$5,571.99$ .

Visions:  $N = \$5,599.99 \times (1 - 0.01) = \$5,599.99 \times 0.99 = \$5,543.99$

## Present

---

The net price for Visions is \$5,543.99. You save  $\$5,571.99 - \$5,543.99 = \$28.00$  by purchasing your TV at Visions.

**Example 2.1G: Understanding the Price**

An advertisement claims that at 60% off, you are saving \$18. However, today there is an additional 20% off. What price should you pay for this item? What percent savings does this represent?

**Solution****Plan**

You are looking for how much you should pay after the discounts (N), and the single equivalent percentage that represents the two discounts (dequiv or just d).

**What You Already Know**

Step 1: You know the discount amount for the first discount only, as well as the two discount rates:  $D\$1 = \$18$ ,  $d1 = 0.60$ , and  $d2 = 0.20$ .

**How You Will Get There**

Step 2: Calculate the list price by applying Formula 2.2a and rearranging for L.

Step 3: To calculate the net price, apply Formula 2.3. Step 4: To calculate the single equivalent discount, apply Formula 2.4.

**Perform**

$$\text{Step 2: } \$18.00 = L \times 0.60$$

$$L = \$18.00 \div 0.6 = \$30.00$$

$$\text{Step 3: } N = \$30 \times (1 - 0.60) \times (1 - 0.20) = \$30 \times 0.40 \times 0.80 = \$9.60 \quad \text{Step 4: dequiv} = 1 - (1 - 0.60) \times (1 - 0.20) = 1 - (0.40)(0.80) = 1 - 0.32 = 0.68 \text{ or } 68\%$$

**Present**

You should pay \$9.60 for the item, which represents a 68

**EXERCISES**

Round all money to two decimals and percentages to four decimals in each of the following questions.

## Mechanics

For questions 1–4, solve for the unknown variables (identified with a ?) based on the information provided. “N/A” indicates that the particular variable is not applicable in the question.

Question	List Price of MSRP	First Discount	Second Discount	Third Discount	Net price	Equivalent Single Discount Rate	Total Discount Amount
1.	\$980.00	42%	N/A	N/A	?	N/A	?
2.	?	25%	N/A	N/A	\$600.00	N/A	?
3.	\$1,975.00	25%	15%	10%	?	?	?
4.	?	18%	4%	7%	\$366.05	?	?

## Applications

- A wholesaler of stereos normally qualifies for a 35% trade discount on all electronic products purchased from its manufacturer. If the MSRP of a stereo is \$399.95, what net price will the wholesaler pay?
- Mary is shopping at the mall where she sees a sign that reads, “Everything in the store is 30% off, including sale items!” She wanders in and finds a blouse on the clearance rack. A sign on the clearance rack states, “All clearance items are 50% off.” If the blouse is normally priced at \$69.49, what price should Mary pay for it?
- A distributor sells some shoes directly to a retailer. The retailer pays \$16.31 for a pair of shoes that has a list price of \$23.98. What trade discount percent is the distributor offering to its retailers?
- A retailer purchases supplies for its head office. If the retailer pays \$16.99 for a box of paper and was eligible for a 15% volume discount, what was the original MSRP for the box of paper?
- Mountain Equipment Co-op has purchased a college backpack for \$29 after discounts of 30%, 8%, and 13%. What is the MSRP for the backpack? What single discount is equivalent to the three discounts?
- Walmart purchased the latest CD recorded by Selena Gomez. It received a total discount of \$10.08 off the MSRP for the CD, which represents a discount percent of 42%.
  - What was the MSRP?
  - What was the net price paid for the CD?
- Best Buy just acquired an HP Pavilion computer for its electronics department. The net price on the computer is \$260.40 and Best Buy receives discounts of 40% and 38%.
  - What single discount is equivalent to the two discounts?
  - What is the list price?
  - What is the total discount amount?

12. TELUS retails a Samsung cellphone at the MSRP of \$399.99. TELUS can purchase the phone from its supplier and receive a 20% trade discount along with a 5% volume discount.
  - a. What is the single equivalent percent discount?
  - b. What net price does TELUS pay for the phone?
  - c. How much of a discount in dollars does this represent?
13. A wholesaler offers the following discounts: 10% seasonal discount for all purchases made between March 1 and May 1, 15% cumulative quantity discount whenever more than 5,000 units are purchased in any month, 5% loyalty discount for customers who have made regular purchases every month for at least one year, and a 33% trade discount to any retailer. Ed's Retail Superstore makes a purchase of 200 watches, MSRP \$10, from the wholesaler on April 29. This month alone, Ed's has ordered more than 5,000 watches. However, Ed's has purchased from the wholesaler for only the past six months. Determine the total price that Ed's should pay for the watches.
14. If a distributor is eligible for a 60% trade discount, 5% volume discount, and 3% seasonal discount, what single equivalent discount rate would it be eligible to receive? If the trade discount is applied first and equals a trade discount of \$48, calculate the net price for the item.

### Challenge & Critical Thinking

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As mentioned in one of the "Paths to Success" sections, discount percentages share a commonality with negative percent changes (Section 3.1). Use the formulas from this chapter to solve questions 15–17 involving percent change.

15. A human resource manager needs to trim labour costs in the following year by 3%. If current year labour costs are \$1,231,498, what are the labour costs next year?
16. At an accounting firm, the number of accountants employed is based on the ratio of 1:400 daily manual journal entries. Because of ongoing increases in automation, the number of manual journal entries declines at a constant rate of 4% per year. If current entries are 4,000 per year, how many years and days will it take until the firm needs to lay off one accountant? (Hint: An accountant is laid off when the number of journal entries drops below 3,600.)
17. An economist is attempting to understand how Canada reduced its national debt from 1999 to 2008. In 1999, Canada's national debt was \$554.143 billion. In 2008, the national debt stood at \$457.637 billion. What percentage had the national debt been reduced by during this time period?
18. Sk8 is examining an invoice. The list price of a skateboard is \$109.00, and the invoice states it received a trade discount of 15% and quantity discount of 10% as well as a loyalty discount. However, the amount of the loyalty discount is unspecified.
  - a. If Sk8 paid \$80.88 for the skateboard, what is the loyalty discount percent?

- b. If the loyalty discount is applied after all other discounts, what amount of loyalty dollars does Sk8 save per skateboard?
19. Currently, a student can qualify for up to six different tuition discounts at a local college based on such factors as financial need or corporate sponsorships. Mary Watson just applied to the college and qualifies for all six discounts: 20%, 15%, 23%, 5%, 3%, and 1%.
- a. She is confused and wants the college to tell her what single discount percent she is receiving. What should the college tell her?
  - b. If her total list tuition comes to \$6,435.00, how much should she pay?
20. Sumandeep is very loyal to her local hairstylist. Because she is loyal, her hairstylist gives her three different discounts: 10%, 5%, and 5%. These discounts amount to \$14.08 in savings.
- a. What was the list price her hairstylist charged her?
  - b. What amount did she pay her hairstylist?
  - c. If her hairstylist increases prices by 5%, what are the list price, net price, and total discount amount?

## Answers



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=373#h5p-67>

## CHAPTER ATTRIBUTION

*Adapted from unit 6.1 in Business Math: A Step-by-Step Handbook*

## 2.2 Markup: Setting the Regular Price

### NEED TO STAY IN BUSINESS

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As you wait in line to purchase your Iced Caramel Macchiato at Starbucks, you look at the pricing menu and think that \$4.99 seems like an awful lot of money for a frozen coffee beverage. Clearly, the coffee itself doesn't cost anywhere near that much. But then gazing around the café, you notice the carefully applied colour scheme, the comfortable seating, the high- end machinery behind the counter, and a seemingly well-trained barista who answers customer questions knowledgeably.

Where did the money to pay for all of this come from? You smile as you realize your \$4.99 pays not just for the macchiato, but for everything else that comes with it.

The process of taking a product's cost and increasing it by some amount to arrive at a selling price is called **markup**. This process is critical to business success because every business must ensure that it does not lose money when it makes a sale. From the consumer perspective, the concept of markup helps you make sense of the prices that businesses charge for their products or services. This in turn helps you to judge how reasonable some prices are (and hopefully to find better deals).

### THE COMPONENTS IN A SELLING PRICE

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Before you learn to calculate markup, you first have to understand the various components of a selling price. Then, in the next section, markup and its various methods of calculation will become much clearer.

When your business acquires merchandise for resale, this is a monetary outlay representing a cost. When you then resell the product, the price you charge must recover more than just the product cost. You must also recover all the selling and operating expenses associated with the product. Ultimately, you also need to make some money, or profit, as a result of the whole process.

### THE FORMULA

---

Most people think that marking up a product must be a fairly complex process. It is not. Formula 2.5 illustrates the relationship between the three components of cost, expenses, and profits in calculating the selling price.

## FORMULA 2.5

**S is Selling Price:** Once you calculate what the business paid for the product (cost), the bills it needs to cover (expenses), and how much money it needs to earn (profit), you arrive at a selling price by summing the three components.

**C is Cost:** The cost is the amount of money that the business must pay to purchase or manufacture the product. If manufactured, the cost represents all costs incurred to make the product. If purchased, this number results from applying an appropriate discount formula from Section 6.1. There is a list price from which the business will deduct discounts to arrive at the net price. The net price paid for the product equals the cost of the product. If a business purchases or manufactures a product for \$10 then it must sell the product for at least \$10. Otherwise, it fails to recover what was paid to acquire or make the product in the first place—a path to sheer disaster!

**Formula 2.5 The Selling Price of a Product:**  $S = C + E + P$

**E is Expenses:** Expenses are the financial outlays involved in selling the product. Beyond just purchasing the product, the business has many more bills to pay, including wages, taxes, leases, equipment, electronics, insurance, utilities, fixtures, décor, and many more. These expenses must be recovered and may be calculated as

- a. A fixed dollar amount per unit
- b. A percentage of the product cost. For example, if a business forecasts total merchandise costs of \$100,000 for the coming year and total business expenses of \$50,000, then it may set a general guideline of adding 50% ( $\$50,000 \div \$100,000$ ) to the cost of a product to cover expenses.
- c. A percentage of the product selling price based on a forecast of future sales. For example, if a business forecasts total sales of \$250,000 and total business expenses of \$50,000, then it may set a general guideline of adding 20% ( $\$50,000 \div \$250,000$ ) of the selling price to the cost of a product to cover expenses.

**P is Profit:** Profit is the amount of money that remains after a business pays all of its costs and expenses. A business needs to add an amount above its costs and expenses to allow it to grow. If it adds too much profit, though, the product's price will be too high, in which case the customer may refuse to purchase it. If it adds too little profit, the product's price may be too low, in which case the customer may perceive the product as shoddy and once again refuse to purchase it. Many businesses set general guidelines on how much profit to add to various products. As with expenses, this profit may be expressed as

- a. A fixed dollar amount per unit
- b. A percentage of the product cost
- c. A percentage of the selling price

## HOW IT WORKS

Follow these steps to solve pricing scenarios involving the three components:

**Step 1:** Four variables are involved in Formula 2.5. Identify the known variables. Note that you may have to calculate the product's cost by applying the single or multiple discount formulas (Formulas 2.1 and 2.3, respectively). Pay careful attention to expenses and profits to capture how you calculate these amounts.

**Step 2:** Apply Formula 2.5 and solve for the unknown variable.

Assume a business pays a net price of \$75 to acquire a product. Through analyzing its finances, the business estimates expenses at \$25 per unit, and it figures it can add \$50 in profit. Calculate the selling price.

Step 1: The net price paid for the product is the product cost. The known variables are  $C = \$75$ ,  $E = \$25$ , and  $P = \$50$ .

Step 2: According to Formula 2.5, the unit selling price is  $S = C + E + P = \$75 + \$25 + \$50 = \$150$ .

### IMPORTANT NOTES

---

In applying Formula 2.5 you must adhere to the basic rule of linear equations requiring all terms to be in the same unit. That is, you could use Formula 2.5 to solve for the selling price of an individual product, where the three components are the unit cost, unit expenses, and unit profit. When you add these, you calculate the unit selling price. Alternatively, you could use Formula 2.5 in an aggregate form where the three components are total cost, total expenses, and total profit. In this case, the selling price is a total selling price, which is more commonly known as total revenue. But you cannot mix individual components with aggregate components.

### THINGS TO WATCH OUT FOR

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The most common mistake in working with pricing components occurs in the “Understand” portion of the PUPP model. It is critical to identify and label information correctly. You have to pay attention to details such as whether you are expressing the expenses in dollar format or as a percentage of either cost or selling price. Systematically work your way through the information provided piece by piece to ensure that you do not miss an important detail.

### GIVE IT SOME THOUGHT

---

- What three components make up a selling price? In what units are these components commonly expressed?
- In what three ways are expenses and profits expressed?
- What is the relationship between net price and cost?

#### Solutions

1. Cost, expenses (also called overhead), and profit. They are expressed either per unit or as a total.
2. A specific dollar amount, a percentage of cost, or a percentage of the selling price.
3. The net price paid for a product is the same as the cost of the product.

**Example 2.2A: Setting a Price on Fashion in Dollars**

Mary's Boutique purchases a dress for resale at a cost of \$23.67. The owner determines that each dress must contribute \$5.42 to the expenses of the store. The owner also wants this dress to earn \$6.90 toward profit. What is the regular selling price for the dress?

**Solution****Plan**

You are looking for the regular selling price for the dress, or  $S$ .

**What You Already Know**

Step 1: The unit cost of the dress and the unit expense and the unit profit are all known:  $C = \$23.67$ ,  $E = \$5.42$ ,  $P = \$6.90$

**How You Will Get There**

Step 2: Apply Formula 2.5.

**Perform**

Step 2:  $S = \$23.67 + \$5.42 + \$6.90 = \$35.99$

**Present**

Mary's Boutique will set the regular price of the dress at \$35.99.

**Example 2.2B: Setting the Price Using Percentage of Cost**

John's Discount Store just completed a financial analysis. The company determined that expenses average 20% of the product cost and profit averages 15% of the product cost. John's Discount Store purchases Chia Pets from its supplier for an MSRP of \$19.99 less a trade discount of 45%. What will be the regular selling price for the Chia Pets?

**Solution****Plan**

You are looking for the regular selling price for the Chia pets, or  $S$ .

### What You Already Know

Step 1: The list price, discount rate, expenses, and profit are known:

- $L = \$19.99$
- $d = 0.45$
- $E = 20\%$  of cost, or  $0.20C$
- $P = 15\%$  of cost, or  $0.15C$

### How You Will Get There

Step 1 (continued): Although the cost of the Chia Pets is not directly known, you do know the MSRP (list price) and the trade discount. The cost is equal to the net price. Apply Formula 2.1.

Step 2: To calculate the selling price, apply Formula 2.5.

### Perform

$$\text{Step 1: } N = \$19.99 \times (1 - 0.45) = \$19.99 \times 0.55 = \$10.99 = C$$

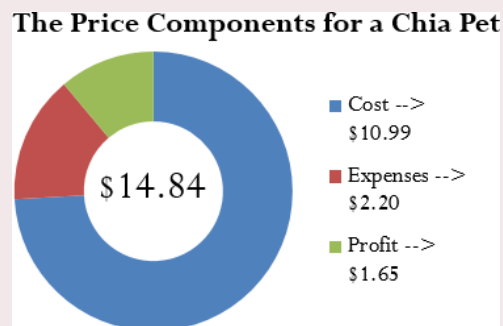
$$\text{Step 2: } S = \$10.99 + 0.20C + 0.15C$$

$$S = \$10.99 + 0.20(\$10.99) + 0.15(\$10.99)$$

$$S = \$10.99 + \$2.20 + \$1.65 = \$14.84$$

### Present

John's Discount Store will sell the Chia Pet for \$14.84.



Based on last year's results, Benthall Appliance learned that its expenses average 30% of the regular selling price. It wants a 25% profit based on the selling price. If Benthall Appliance purchases a fridge for \$1,200, what is the regular unit selling price?

**Solution****Plan**

---

You are looking for the regular unit selling price for the fridge, or  $S$ .

**What You Already Know**

---

Step 1: The cost, expenses, and profit for the fridge are known:

- $E = 30\% \text{ of } S, \text{ or } 0.3$
- $SP = 25\% \text{ of } S, \text{ or } 0.25$
- $SC = \$1,200.00$

**How You Will Get There**

---

Step 2: Apply Formula 2.5

**Perform**

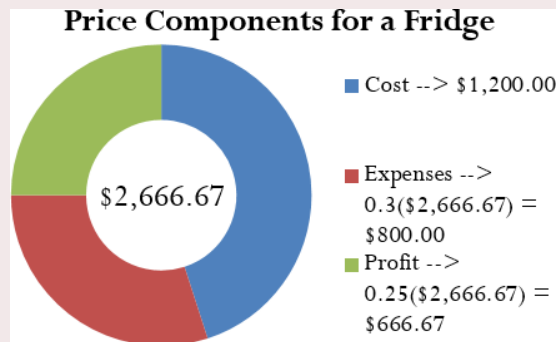
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Step 2:  $S = \$1,200.00 + 0.3S + 0.25S$

$$S = \$1,200.00 + 0.55S$$
$$S - 0.55S = \$1,200.00$$
$$0.45S = \$1,200.00$$
$$S = \$2,666.6$$
**Present**

---

Benthall Appliance should set the regular selling price of the fridge at \$2,666.67.



### Example 2.2D: Using Selling Price to Figure Out the Cost

If a company knows that its profits are 15% of the selling price and expenses are 30% of cost, what is the cost of an MP3 player that has a regular selling price of \$39.99?

#### Solution

#### Plan

You are looking for the cost of the MP3 player, or C.

#### What You Already Know

Step 1: The expenses, profits, and the regular unit selling price are as follows:

- $S = \$39.99$
- $P = 15\% \text{ of } S, \text{ or } 0.15S$
- $E = 30\% \text{ of cost, or } 0.3C$

#### How You Will Get There

Step 2: Apply Formula 2.5, rearranging for C.

#### Perform

$$\text{Step 2: } \$39.99 = C + 0.3C + 0.15(\$39.99)$$

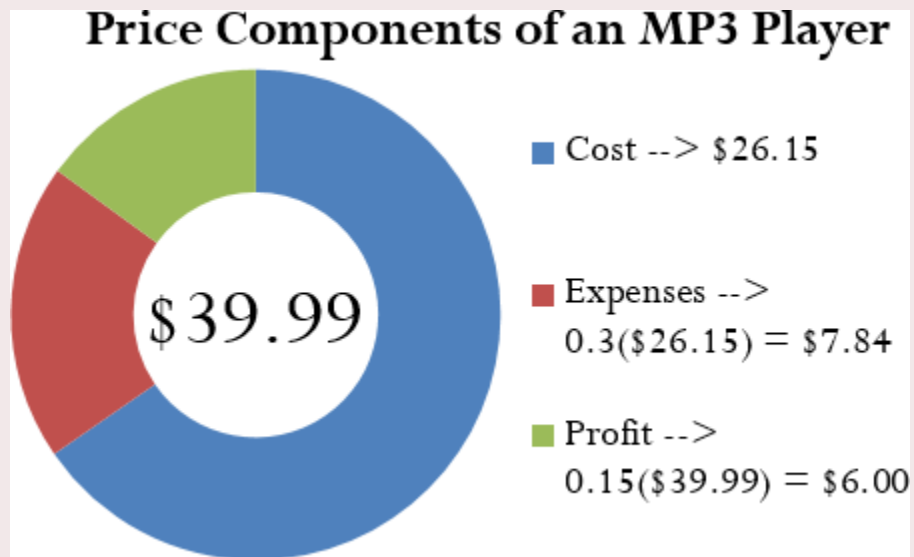
$$\$39.99 = 1.3C + \$6.00$$

$$\$33.99 = 1.3C$$

$$\$26.15 = C$$

### Present

The cost of the MP3 Player is \$26.15.



### Example 2.2E: Determining Profitability

Peak of the Market considers setting the regular unit selling price of its strawberries at \$3.99 per kilogram. If it purchases these strawberries from the farmer for \$2.99 per kilogram and expenses average 40% of product cost, does Peak of the Market make any money?

#### Solution

#### Plan

In asking whether Peak of the Market makes money, you are looking for the profit, or P.

#### What You Already Know

Step 1: The cost, expenses, and proposed regular unit selling price for the strawberries are as follows:

- S = \$3.99
- C = \$2.99
- E = 40% of cost, or 0.4C

### How You Will Get There

Step 2: Apply Formula 2.5, rearranging for P.

### Perform

Step 2:  $\$3.99 = \$2.99 + 0.4(\$2.99) + P$

$\$3.99 = \$4.19 + P$

$-\$0.20 = P$

### Present

The negative sign on the profit means that Peak of the Market would take a loss of \$0.20 per kilogram if it sells the strawberries at \$3.99. Unless Peak of the Market has a marketing reason or sound business strategy for doing this, the company should reconsider its pricing.

## CALCULATING THE MARKUP DOLLARS

Most companies sell more than one product, each of which has different price components with varying costs, expenses, and profits. Can you imagine trying to compare 50 different products, each with three different components? You would have to juggle 150 numbers! To make merchandising decisions more manageable and comparable, many companies combine expenses and profit together into a single quantity, either as a dollar amount or a percentage. This section focuses on the markup as a dollar amount.

### THE FORMULA - 2.6

One of the most basic ways a business simplifies its merchandising is by combining the dollar amounts of its expenses and profits together as expressed in Formula 2.6.

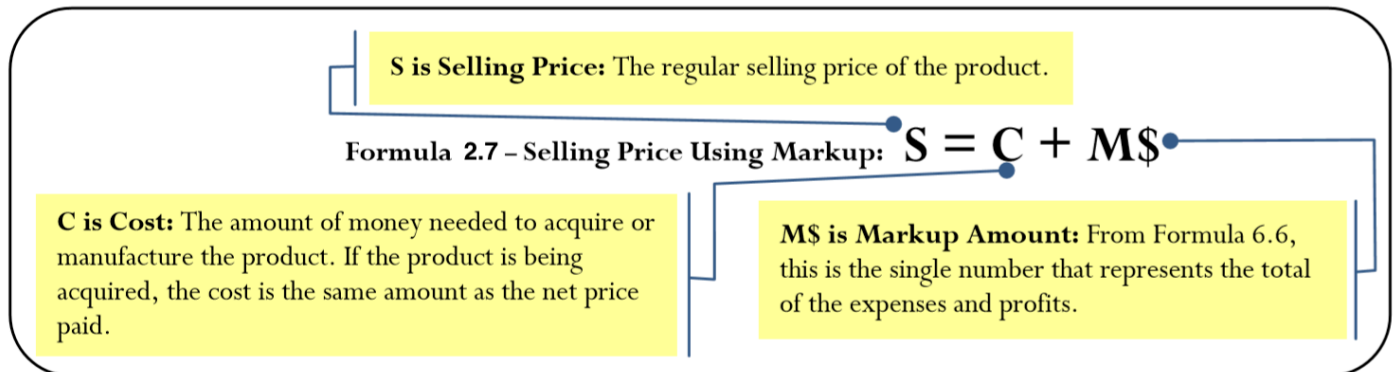
**M\$ is Markup Amount:** Markup is taking the cost of a product and converting it into a selling price. The **markup amount** represents the dollar amount difference between the cost and the selling price.

Formula 2.6 Markup Amount:  **$M\$ = E + P$**

**E is Expenses:** The expenses associated with the product.

**P is Profit:** The profit earned when the product sells.

Note that since the markup amount (M\$) represents the expenses (E) and profit (P) combined, you can substitute the variable for markup amount into Formula 2.5 to create Formula 2.7, which calculates the regular selling price.



## HOW IT WORKS

Follow these steps when you work with calculations involving the markup amount:

**Step 1:** You require three variables in either Formula 2.6 or Formula 2.7. At least two of the variables must be known. If the amounts are not directly provided, you may need to calculate these amounts by applying other discount or markup formulas.

**Step 2:** Solve either Formula 2.6 or Formula 2.7 for the unknown variable. Recall from Example 2.2D that the MP3 player's expenses are \$7.84, the profit is \$6.00, and the cost is \$26.15.

Calculate the markup amount and the selling price.

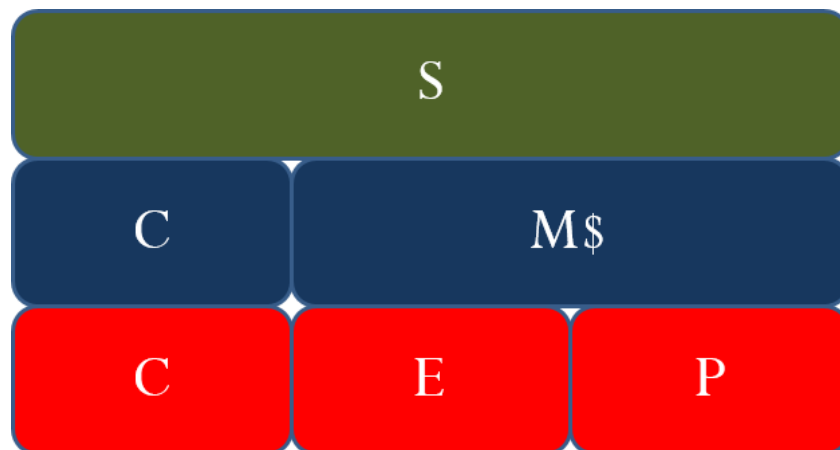
Step 1: The known variables are  $E = \$7.84$ ,  $P = \$6.00$ , and  $C = \$26.15$ .

Step 2: According to Formula 2.6, the markup amount is the sum of the expenses and profit, or  $M\$ = \$7.84 + \$6.00 = \$13.84$ .

Step 2 (continued): Applying Formula 2.7, add the markup amount to the cost to arrive at the regular selling price, resulting in  $S = \$26.15 + \$13.84 = \$39.99$

## PATHS TO SUCCESS

You might have already noticed that many of the formulas in this chapter are interrelated. The same variables appear numerous times but in different ways. To help visualize the relationship between the various formulas and variables, many students have found it helpful to create a markup chart [see below].



This chart demonstrates the relationships between Formulas 2.5, 2.6, and 2.7. It is evident that the selling price (the green line) consists of cost, expenses, and profit (the red line representing Formula 2.5); or it can consist of cost and the markup amount (the blue line representing Formula 2.7). The markup amount on the blue line consists of the expenses and profit on the red line (Formula 2.6).

#### Example 2.2F: Markup as a Dollar Amount

A cellular retail store purchases an iPhone with an MSRP of \$779 less a trade discount of 35% and volume discount of 8%. The store sells the phone at the MSRP. What is the markup amount? If the store knows that its expenses are 20% of the cost, what is the store's profit?

#### Solution

##### Plan

First of all, you need to calculate the markup amount ( $M\$$ ). You also want to find the store's profit ( $P$ ).

##### What You Already Know

Step 1: The smartphone MSRP and the two discounts are known, along with the expenses and selling price:

- $L = \$270$   $d_1 = 0.35$   $d_2 = 0.08$
- $E = 20\%$  of cost, or  $0.2CS = \$779$

##### How You Will Get There

Step 1 (continued): Calculate the cost of the iPhone by applying Formula 2.3.

Step 2: Calculate the markup amount using Formula 2.7.

Step 2 (continued): Calculate the profit by applying Formula 2.6, rearranging for  $P$ .

## Perform

Step 1 (continued):

- $N = \$779.00 \times (1 - 0.35) \times (1 - 0.08)$
- $N = \$779.00 \times 0.65 \times 0.92 = \$465.84 = C$

Step 2:

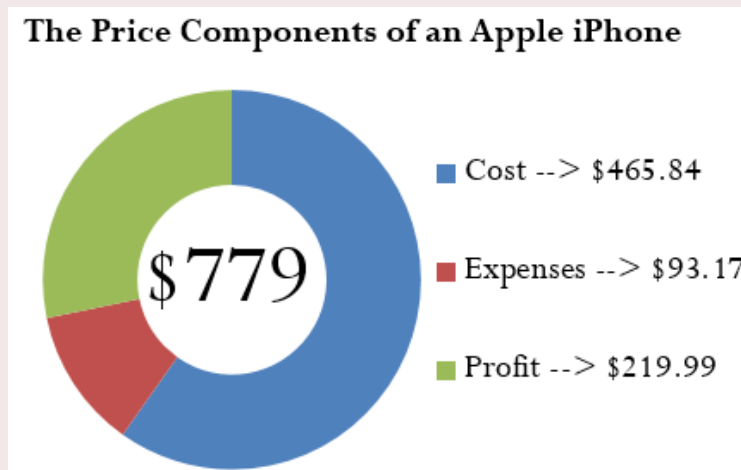
- $\$779.00 = \$465.84 + M\$$
- $\$313.16 = M\$$

Step 2 (continued):

- $\$313.16 = 0.2(\$465.84) + P$
- $\$313.16 = \$93.17 + P$
- $\$219.99 = P$

## Present

The markup amount for the iPhone is \$313.16. When the store sells the phone for \$779.00, its profit is \$219.99.



## CALCULATING THE MARKUP PERCENT

It is important to understand markup in terms of the actual dollar amount; however, it is more common in business practice to calculate the markup as a percentage. There are three benefits to converting the markup dollar amount into a percentage:

1. Easy comparison of different products having vastly different price levels and costs, to help

you see how each product contributes toward the financial success of the company. For example, if a chocolate bar has a 50¢ markup included in a selling price of \$1, while a car has a \$1,000 markup included in a selling price of \$20,000, it is difficult to compare the profitability of these items. If these numbers were expressed as a percentage of the selling price such that the chocolate bar has a 50% markup and the car has a 5% markup, it is clear that more of every dollar sold for chocolate bars goes toward list profitability.

2. Simplified translation of costs into a regular selling price—a task that must be done for each product, making it helpful to have an easy formula, especially when a company carries hundreds, thousands, or even tens of thousands of products. For example, if all products are to be marked up by 50% of cost, an item with a \$100 cost can be quickly converted into a selling price of \$150.
3. An increased understanding of the relationship between costs, selling prices, and the list profitability for any given product. For example, if an item selling for \$25 includes a markup on selling price of 40% (which is \$10), then you can determine that the cost is 60% of the selling price (\$15) and that \$10 of every \$25 item sold goes toward list profits.

You can translate the markup dollars into a percentage using two methods, which express the amount either as a percentage of cost or as a percentage of selling price:

#### *Method 1: Markup as a Percentage of Cost.*

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This method expresses the markup rate using cost as the base. Many companies use this technique internally because most accounting is based on cost information. The result, known as the **markup on cost percentage**, allows a reseller to convert easily from a product's cost to its regular unit selling price.

#### *Method 2: Markup as a Percentage of Selling Price.*

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This method expresses the markup rate using the regular selling price as the base. Many other companies use this method, known as the **markup on selling price percentage**, since it allows for quick understanding of the portion of the selling price that remains after the cost of the product has been recovered. This percentage represents the list profits before the deduction of expenses and therefore is also referred to as the *list profit margin*.

## THE FORMULA

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Both formulas are versions of Formula 2.2 on rate, portion, and base. The markup on cost percentage is expressed in Formula 2.8, while the markup on selling price percentage is expressed in Formula 2.9.

### Formula 2.8 Markup on Cost Percentage

Formula does not parse

In other words, you must add 52.9254% of the cost on top of the unit cost to arrive at the regular unit selling price of \$39.99. Step 2 (continued): To calculate the markup on selling price percentage, apply Formula 2.9:

### *MoC*

In other words, 34.6087% of the selling price represents list profits after the business recovers the \$26.15 cost of the MP3 player.

## IMPORTANT NOTES

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Businesses are very focused on profitability. Your Texas Instruments BAII Plus calculator is programmed with the markup on selling price percentage. The function is located on the second shelf above the number three. To use this function, open the window by pressing 2nd 3. You can scroll between lines using your  $\uparrow$  and  $\downarrow$  arrows.

There are three variables:

- CST is the cost. Use the symbol C.
- SEL is the selling price. Use the symbol S.
- MAR is the markup on selling price percentage. Use the symbol MoS%.

As long as you know any two of the variables, you can solve for the third. Enter any two of the three variables (you need to press ENTER after each), making sure the window shows the output you are seeking, and press CPT.

## THINGS TO WATCH OUT FOR

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Merchandising involves many variables. Nine formulas have been established so far, and a few more are yet to be introduced. Though you may feel bogged down by all of these formulas, just remember that you have encountered most of these merchandising concepts since you were very young and that you interact with retailers and pricing every day. This chapter merely formalizes calculations you already perform on a daily basis, whether at work or at home. The calculation of discounts is no different than going to Walmart and finding your favourite CD on sale. You know that when a business sells a product, it has to recoup the cost of the product, pay its bills, and make some money. And you have worked with percentages since elementary school.

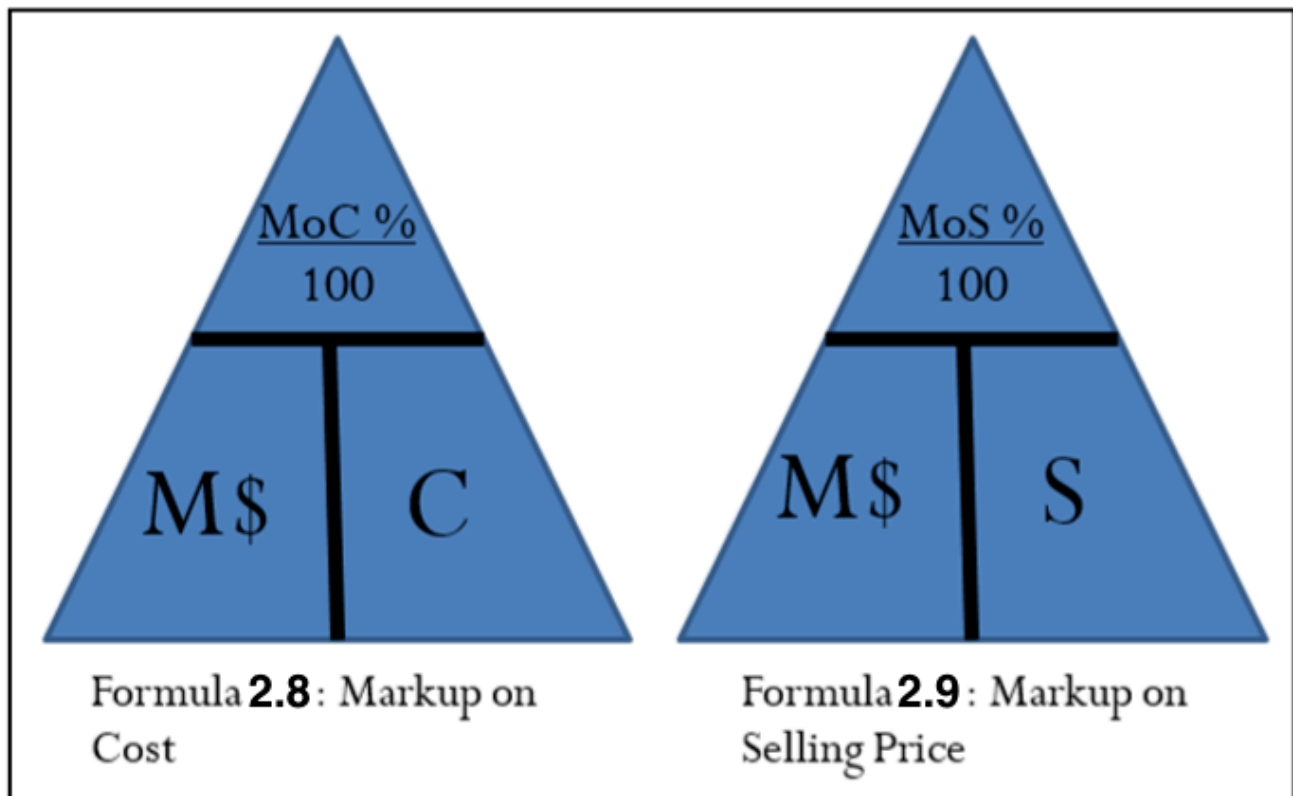
Do not get stuck in the formulas. Think about the concept presented in the question. Change the scenario of the question and put it in the context of something more familiar. Ultimately, if you really have difficulties then look at the variables provided and cross-reference them to the merchandising formulas. Your goal is to find formulas in which only one variable is unknown. These formulas are solvable. Then ask yourself, "How does knowing that new variable help solve any other formula?"

You do not need to get frustrated. Just be systematic and relate the question to what you already know.

## PATHS TO SUCCESS

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The triangle method simplifies rearranging both Formulas 2.8 and 2.9 to solve for other unknown variables as illustrated in the figure below.



Sometimes you need to convert the markup on cost percentage to a markup on selling price percentage, or vice versa. Two shortcuts allow you to convert easily from one to the other:

■ *MoC*

■ *MoS*

Notice that these formulas are very similar. How do you remember whether to add or subtract in the denominator? In normal business situations, the MoC% is always larger than the MoS%. Therefore, if you are converting one to the other you need to identify whether you want the percentage to become larger or smaller.

#### Key Takeaways

- To calculate MoC%, you want a larger percentage. Therefore, make the denominator smaller by subtracting MoS% from 1.
- To calculate MoS%, you want a smaller percentage. Therefore, make the denominator larger by adding MoC% to 1.

#### GIVE IT SOME THOUGHT

Answer the following true/false questions.

1. The markup on selling price percentage can be higher than 100%.
2. The markup dollar amount can be more than the selling price.
3. The markup on cost percentage can be higher than 100%.
4. The markup on cost percentage in most business situations is higher than the markup on selling price percentage.
5. If you know the markup on cost percentage and the cost, you can calculate a selling price.
6. If you know the markup on selling price percentage and the cost, you can calculate a selling price.

#### Solutions

1. False. The markup amount is a portion of the selling price and therefore is less than 100%.
2. False. The markup amount plus the cost equals the selling price. It must be less than the selling price.
3. True. A cost can be doubled or tripled (or increased even more) to reach the price.
4. True. The base for markup on cost percentage is smaller, which produces a larger percentage.
5. True. You could combine Formulas 2.7 and 2.8 to arrive at the selling price.
6. True. You could convert the MoS% to a MoC% and solve as in the previous question.

#### Example 2.2G: Markup as a Percentage

A large national retailer wants to price a Texas Instruments BAII Plus calculator at the MSRP of \$39.99. The retailer can acquire the calculator for \$17.23. What is the markup on cost percentage? What is the markup on selling price percentage?

#### Solution

#### Plan

You have been asked to solve for the markup on cost percentage (MoC%) and the markup on selling price percentage (MoS%).

#### What You Already Know

Step 1: The regular unit selling price and the cost are provided:

- $S = \$39.99$
- $C = \$17.23$

#### How You Will Get There

Step 1 (continued): You need the markup dollars. Apply Formula 2.7, rearranging for M\$.

Step 2: To calculate markup on cost percentage, apply Formula 2.8.

Step 2 (continued): To calculate markup on selling price percentage, apply Formula 2.9.

### Perform

Step 1 (continued):  $\$39.99 = \$17.23 + M\$$

$\$22.76 = M\$$

Step 2:  $\text{MoC}\% = \$^{22.76}/\$_{17.23} \times 100 = 132.0952\%$

Step 2 (continued):  $\text{MoS}\% = \$^{22.76}/\$_{39.99} \times 100 = 56.9142\%$

### Present

The markup on cost percentage is 132.0952%. The markup on selling price percentage is 56.9142%.

## BREAK-EVEN PRICING

In running a business, you must never forget the “bottom line.” In other words, if you fully understand how your products are priced, you will know when you are making or losing money. Remember, if you keep losing money you will not stay in business for long! 15% of new businesses will not make it past their first year, and 49% fail in their first five years. This number becomes even more staggering with an 80% failure rate within the first decade.<sup>1</sup> Do not be one of these statistics! With your understanding of markup, you now know what it takes to break even in your business.

*Break-even* means that you are earning no profit, but you are not losing money either. Your profit is zero.

## THE FORMULA

If the regular unit selling price must cover three elements—cost, expenses, and profit—then the regular unit selling price must exactly cover your costs and expenses when the profit is zero. In other words, if Formula 2.5 is modified to calculate the selling price at the break-even point (SBE) with  $P=0$ , then

$$S_{BE} = C + E$$

This is not a new formula. It just summarizes that at break-even there is no profit or loss, so the profit ( $P$ ) is eliminated from the formula.

## HOW IT WORKS

The steps you need to calculate the break-even point are no different from those you used to calculate the regular selling price. The only difference is that the profit is always set to zero.

Recall Example 2.2D that the cost of the MP3 player is \$26.15 and expenses are \$7.84. The break-even price (SBE) is  $\$26.15 + \$7.84 = \$33.99$ . This means that if the MP3 player is sold for anything more than \$33.99, it is profitable; if it is sold for less, then the business does not cover its costs and expenses and takes a loss on the sale.

1. Statistics Canada, “Failure Rates for New Firms,” The Daily, February 16, 2000, [www.statcan.gc.ca/daily-quotidien/000216/dq000216b-eng.htm](http://www.statcan.gc.ca/daily-quotidien/000216/dq000216b-eng.htm).

### Example 2.2H: Knowing Your Break-Even Price

John is trying to run an eBay business. His strategy has been to shop at local garage sales and find items of interest at a great price. He then resells these items on eBay. On John's last garage sale shopping spree, he only found one item—a Nintendo Wii that was sold to him for \$100. John's vehicle expenses (for gas, oil, wear/tear, and time) amounted to \$40. eBay charges a \$2.00 insertion fee, a flat fee of \$2.19, and a commission of 3.5% based on the selling price less \$25. What is John's minimum list price for his Nintendo Wii to ensure that he at least covers his expenses?

#### Solution

#### Plan

You are trying to find John's break-even selling price (SBE).

#### What You Already Know

Step 1: John's cost for the Nintendo Wii and all of his associated expenses are as follows:

- $C = \$100.00$
- $E \text{ (vehicle)} = \$40.00$
- $E \text{ (insertion)} = \$2.00$
- $E \text{ (flat)} = \$2.19$
- $E \text{ (commission)} = 3.5\%(SBE - \$25.00)$

#### How You Will Get There

Step 1 (continued): You have four expenses to add together that make up the E in the formula.

Step 2: Formula 2.5 states  $S = C + E + P$ .

Since you are looking for the break-even point, then P is set to zero and  $SBE = C + E$ .

#### Perform

Step 1 (continued):  $E = \$40.00 + \$2.00 + \$2.19 + 3.5\%(S_{BE} - \$25.00)$

$$E = \$44.19 + 0.035(S_{BE} - \$25.00) = \$44.19 + 0.035S_{BE} - \$0.875$$

$$E = \$43.315 + 0.035S_{BE}$$

Step 2:

$$S_{BE} = \$100.00 + \$43.315 + 0.035S_{BE}$$

$$S_{BE} = \$143.315 + 0.035S_{BE}$$

$$S_{BE} - 0.035S_{BE} = \$143.315$$

$$0.965S_{BE} = \$143.315$$

$$S_{BE} = \$148.51$$

### Present

At a price of \$148.51 John would cover all of his costs and expenses but realize no profit or loss. Therefore, \$148.51 is his minimum price.

## EXERCISES

Round all money to two decimals and percentages to four decimals for each of the following exercises.

### Mechanics

For questions 1–8, solve for the unknown variables (identified with a ?) based on the information provided

Question	Regular Unit Selling Price	Cost	Expenses	Profit	Markup Amount	Break-Even Price	Markup on Cost	Markup on Selling Price
1.	?	\$188.42	\$48.53	\$85.00	?	?	?	?
2.	\$999.99	?	30% of C	?	?	?	?	?
3.	?	?	?	\$183.28	?	?	155%	?
4.	\$274.99	?	20% of S	?	?	?	?	35%
5.	?	?	45% of C	?	\$540.00	\$1,080.00	?	?
6.	?	?	?	?	?	?	68%	?
7.	?	?	\$100.00	?	\$275.00		?	19%
8.	?	?	15% of C	12% of S	?	\$253.00	?	?

### Applications

- If a pair of sunglasses sells at a regular unit selling price of \$249.99 and the markup is always 55% of the regular unit selling price, what is the cost of the sunglasses?
- A transit company wants to establish an easy way to calculate its transit fares. It has determined that the cost of a transit ride is \$1.00, with expenses of 50% of cost. It requires \$0.75 profit per ride. What is its markup on cost percentage?
- Daisy is trying to figure out how much negotiating room she has in purchasing a new car. The car has an MSRP of \$34,995.99. She has learned from an industry insider that most car dealerships have a 20% markup on selling price. What does she estimate the dealership paid for the car?
- The markup amount on an eMachines desktop computer is \$131.64. If the machine regularly retails for \$497.25 and expenses average 15% of the selling price, what profit will be earned?
- Manitoba Telecom Services (MTS) purchases an iPhone for \$749.99 less discounts of 25% and 15%. MTS's expenses are known to average 30% of the regular unit selling price.
  - What is the regular unit selling price if a profit of \$35 per iPhone is required?
  - What are the expenses?
  - What is the markup on cost percentage?

- d. What is the break-even selling price?
14. A snowboard has a cost of \$79.10, expenses of \$22.85, and profit of \$18.00.
- a. What is the regular unit selling price?
  - b. What is the markup amount?
  - c. What is the markup on cost percentage?
  - d. What is the markup on selling price percentage?
  - e. What is the break-even selling price? What is the markup on cost percentage at this break-even price?

### Challenge, Critical Thinking, & Other Applications

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15. A waterpark wants to understand its pricing better. If the regular price of admission is \$49.95, expenses are 20% of cost, and the profit is 30% of the regular unit selling price, what is the markup amount?
16. Sally works for a skateboard shop. The company just purchased a skateboard for \$89.00 less discounts of 22%, 15%, and 5%. The company has standard expenses of 37% of cost and desires a profit of 25% of the regular unit selling price. What regular unit selling price should Sally set for the skateboard?
17. If an item has a 75% markup on cost, what is its markup on selling price percentage?
18. A product received discounts of 33%, 25%, and 5%. A markup on cost of 50% was then applied to arrive at the regular unit selling price of \$349.50. What was the original list price for the product?
19. Mountain Equipment Co-op (MEC) wants to price a new backpack. The backpack can be purchased for a list price of \$59.95 less a trade discount of 25% and a quantity discount of 10%. MEC estimates expenses to be 18% of cost and it must maintain a markup on selling price of 35%.
- a. What is the cost of backpack?
  - b. What is the markup amount?
  - c. What is the regular unit selling price for the backpack?
  - d. What profit will Mountain Equipment Co-op realize?
  - e. What happens to the profits if it sells the backpack at the MSRP instead?
20. Costco can purchase a bag of Starbucks coffee for \$20.00 less discounts of 20%, 15%, and 7%. It then adds a 40% markup on cost. Expenses are known to be 25% of the regular unit selling price.
- a. What is the cost of the coffee?
  - b. What is the regular unit selling price?
  - c. How much profit will Costco make on a bag of Starbucks coffee?
  - d. What markup on selling price percentage does this represent?
  - e. Repeat questions (a) through (d) if the list price changes to \$24.00.

## 2.3 Markdown: Setting the Sale Price

### EVERYBODY LOVES A SALE

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Flashy signs in a retail store announce, “40% off, today only!” Excitedly you purchase three tax-free products with regular price tags reading \$100, \$250, and \$150. The cashier processing the transaction informs you that your total is \$325. You are about to hand over your credit card when something about the total makes you pause. The regular total of all your items is \$500. If they are 40% off, you should receive a \$200 deduction and pay only \$300. The cashier apologizes for the mistake and corrects your total.

Although most retail stores use automated checkout systems, these systems are ultimately programmed by human beings. A computer system is only as accurate as the person keying in the data. A study by the Competition Bureau revealed that 6.3% of items at various retail stores scanned incorrectly. The average error spread is up to 13% around the actual product’s price!<sup>1</sup> Clearly, it is important for you as a consumer to be able to calculate markdowns.

Businesses must also thoroughly understand markdowns so that customers are charged accurately for their purchases. Businesses must always comply with the Competition Act of Canada, which specifically defines legal pricing practices. If your business violates this law, it faces severe penalties.

### THE IMPORTANCE OF MARKDOWNS

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A **markdown** is a reduction from the regular selling price of a product resulting in a lower price. This lower price is called the **sale price** to distinguish it from the selling price.

Many people perceive markdowns as a sign of bad business management decisions. However, in most situations this is not true. Companies must always attempt to forecast the future. In order to stock products, a reseller must estimate the number of units that might sell in the near future for every product that it carries. This is both an art and a science. While businesses use statistical techniques that predict future sales with a relative degree of accuracy, consumers are fickle and regularly change shopping habits. Markdowns most commonly occur under four circumstances:

1. Competition Bureau, Fair Business Practices Branch, *Price Scanning Report*, Table B, page 5, 1999, [www.competitionbureau.gc.ca/epic/site/cb-bc.nsf/en/01288e.html](http://www.competitionbureau.gc.ca/epic/site/cb-bc.nsf/en/01288e.html).

## Clearing Out Excess or Unwanted Inventory

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In these situations, the business thought it could sell 100 units; however, consumers purchased only 20 units. In the case of seasonal inventory, such as Christmas items on Boxing Day, the retailer wishes to avoid packing up and storing the inventory until the next season.

## Clearing Out Damaged or Discontinued Items

---

Selling a damaged product at a discount is better than not selling it at all. When products are discontinued, this leaves shelf space underused, so it is better to clear the item out altogether to make room for profitable items that can keep the shelves fully stocked.

## Increasing Sales Volumes

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Sales attract customers because almost everyone loves a deal. Though special marketing events such as a 48 hour sale reduce the profitability per unit, by increasing the volume sold these sales can lead to a greater profit overall.

## Promoting Add-On Purchases

---

Having items on sale attracts customers to the store. Many times customers will not only purchase the item on sale but also, as long as they are on the premises, grab a few other items, which are regularly priced and very profitable. Like many others, you may have walked into Target to buy one item but left with five instead.

## THE FORMULA

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Markdowns are no different from offering a discount. Recall that one of the types of discounts is known as a sale discount. The only difference here lies in choice of language. Markdowns are common, so you will find it handy to adapt the discount formulas to the application of markdowns, replacing the symbols with ones that are meaningful in merchandising. Formula 2.1 calculates the net price for a product after it receives a single discount:

$$N = L \times (1 - d)$$

**Formula 2.10** adapts this formula for use in markdown situations.

**S<sub>onsale</sub> is Sale Price:** The sale price is the price of the product after reduction by the markdown percent. Conceptually, the sale price is the same as the net price.

**Formula 2.10 – The Sale Price Of A Product:**  $S_{\text{onsale}} = S \times (1 - d)$

**S is Selling Price:** The regular selling price of the product before any discounts. The higher price is the list price. In merchandising questions, this dollar amount may or may not be a known variable. If the selling price is unknown, you must calculate it using an appropriate formula or combination of formulas from either Section 6.1 or Section 6.2.

**d is Markdown Rate:** A markdown rate is the same as a sale discount rate. Therefore you use the same discount rate symbol from Section 6.1 to represent the percentage (in decimal format) by which you reduce the selling price. As in Formula 6.1, note that you are interested in calculating the sale price and not the amount saved. Thus, you take the markdown rate away from 1 to find out the rate owing.

In markdown situations, the selling price and the sale price are different variables. The sale price is always less than the selling price. In the event that a regular selling price has more than one markdown percent applied to it, you can extend

Formula 2.10 in the same manner that Formula 2.3 calculated multiple discounts. If you are interested in the markdown amount in dollars, recall that Formula 2.2 calculates the discount amount in dollars. Depending on what information is known, the formula has two variations:

Formula 2.2a:  $D\$ = L \times d$

Formula 2.2b:  $D\$ = L - N$

Formulas 2.11a and 2.11b adapt these formulas to markdown situations.

**D\$ is Markdown Amount:** You determine the markdown amount using either formula depending on what information is known. If you know the selling price and markdown percent, apply Formula 6.11a. If you know the selling price and sale price, apply Formula 6.11b.

**Formula 2.11a - Markdown Amount:**  $D\$ = S \times d$

**Formula 2.11b – Markdown Amount:**  $D\$ = S - S_{\text{onsale}}$

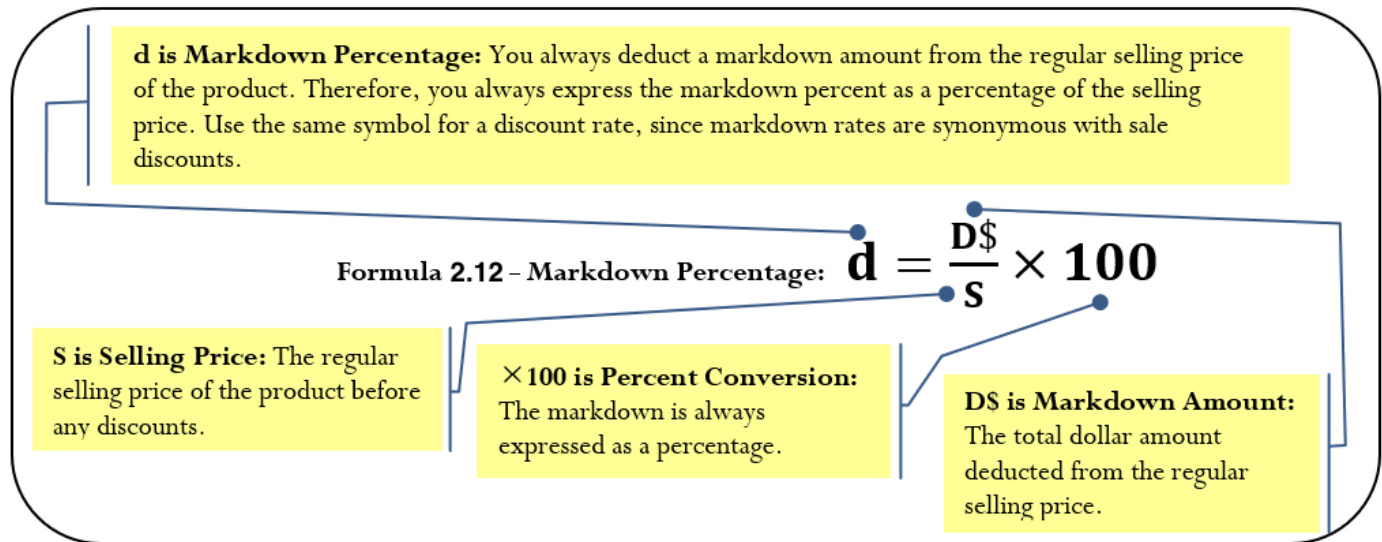
**S is Selling Price:** The regular selling price before you apply any markdown percentages.

**S<sub>onsale</sub> is Sale Price:** The price after you have deducted all markdown percentages from the regular selling price.

**d is Markdown Rate:** the percentage of the selling price to be deducted (in decimal format). In this case, because you are interested in figuring out how much the percentage is worth, you do not take it away from 1 as in Formula 6.10.

The final markdown formula reflects the tendency of businesses to express markdowns as percentages, facilitating easy comprehension and comparison. Recall Formula 2.9 which calculated a markup on selling price percent:

**Formula 2.12** adapts this formula to markdown situations.



## HOW IT WORKS

Follow these steps to calculate a markdown:

Step 1: Across all three markdown formulas, the four variables consist of the selling price (S), sale price (Sonsale), markdown dollars (D\$), and markdown rate (d). Identify which variables are known. Depending on the known information, you may have to calculate the selling price using a combination of discount and markup formulas.

Step 2: Apply one or more of Formulas 2.10, 2.11a, 2.11b, and 2.12 to calculate the unknown variable(s). In the event that multiple markdown rates apply, extend Formula 2.10 to accommodate as many markdown rates as required.

Recall the example of the MP3 player with a regular selling price of \$39.99. Assume the retailer has excess inventory and places the MP3 player on sale for 10% off. What is the sale price and markdown amount?

Step 1: The selling price and markdown percent are  $S = \$39.99$  and  $d = 0.10$ , respectively.

Step 2: Apply Formula 2.10 to calculate the sale price, resulting in  $Sonsale = \$39.99 \times (1 - 0.10) = \$35.99$ .

Step 2 (continued): You could use either of Formulas 2.11a or 2.11b to calculate the markdown amount since the selling price, sale price, and markdown percent are all known. Arbitrarily choosing Formula 2.11a, you calculate a markdown amount of  $D\$ = \$39.99 \times 0.10 = \$4.00$ .

Therefore, if the retailer has a 10% off sale on the MP3 players, it marks down the product by \$4.00 and retails it at a sale price of \$35.99.

## THINGS TO WATCH OUT FOR

---

Avoid getting bogged down in formulas. Recall that the three formulas for markdowns are not new formulas, just adaptations of three previously introduced concepts. As a consumer, you are very experienced with endless examples of sales, bargains, discounts, blowouts, clearances, and the like.

Every day you read ads in the newspaper and watch television commercials advertising percent savings. This section simply crystallizes your existing knowledge. If you are puzzled by questions involving markdowns, make use of your shopping experiences at the mall!

## PATHS TO SUCCESS

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Three of the formulas introduced in this section can be solved for any variable through algebraic manipulation when any two variables are known. Recall that the triangle technique helps you remember how to rearrange these formulas, as illustrated here.

### Example 3.3A: Determining the Sale Price and Markdown Amount

The MSRP for the “Guitar Hero: World Tour” video game is \$189.99. Most retail stores sell this product at a price in linewith the MSRP. You have just learned that a local electronics retailer is selling the game for 45% off. What is the sale price for the video game and what dollar amount is saved?

#### **Solution**

#### **Plan**

---

There are two unknown variables. The first is the video game’s sale price ( $S_{\text{onsale}}$ ). The second is the markdown amount ( $D\$$ ) that is realized at that sale price.

#### **What You Already Know**

---

Step 1: The regular selling price for the video game and the markdown rate are known:  $S = \$189.99$   $d = 0.45$

### How You Will Get There

Step 2: Calculate the sale price by applying Formula 2.10. Step 2 (continued):

Calculate the markdown amount by applying Formula 2.11b.

### Perform

Step 2:  $S_{\text{onsale}} = \$189.99 \times (1 - 0.45) = \$189.99 \times 0.55 = \$104.49$

Step 2 (continued):  $D\$ = \$189.99 - \$104.49 = \$85.50$

### Present

The sale price for the video game is \$104.49. When purchased on sale, “Guitar Hero: World Tour” is \$85.50 off of its regular price.

### Example 3.3B: Markdown Requiring Selling Price Calculation

A reseller acquires an Apple iPad for \$650. Expenses are planned at 20% of the cost, and profits are set at 15% of the cost. During a special promotion, the iPad is advertised at \$100 off. What is the sale price and markdown percent?

### Solution

### Plan

The unknown variables for the iPad are the sale price ( $S_{\text{onsale}}$ ) and the markdown rate ( $d$ ).

### What You Already Know

Step 1: The pricing elements of the iPad along with the markdown dollars are known:  $C = \$650$ ,  $E = 0.2C$ ,  $P = 0.15C$ ,  $D\$ = \$100$

### How You Will Get There

Step 1 (continued): Calculate the selling price of the product by applying Formula 2.5.

Step 2: Calculate the markdown percent by applying Formula 2.12:

Step 2 (continued): Calculate the sale price by applying Formula 2.11b, rearranging for  $S_{\text{onsale}}$ .

**Perform**

Step 1 (continued):  $S = \$650 + 0.2(\$650) + 0.15(\$650) = \$877.50$  Step 2:  $d = \$100 \times 100 = 11.396\%$   $\$877.50$

Step 2 (continued):  $\$100 = \$877.50 - S_{\text{onsale}}$

$S_{\text{onsale}} = \$777.50$

**Present**

When the iPad is advertised at \$100 off, it receives an 11.396% markdown and it will retail at a sale price of \$777.50.

**NEVER-ENDING SALES**

Have you noticed that some companies always seem to have the same item on sale all of the time? This is a common marketing practice. Recall the third and fourth circumstances for markdowns. Everybody loves a sale, so markdowns increase sales volumes for both the marked-down product and other regularly priced items. For example, Michaels has a product line called the Lemax Village Collection, which has seasonal display villages for Christmas, Halloween, and other occasions. When these seasonal product lines come out, Michaels initially prices them at the regular unit selling price for a short period and then reduces their price. For Michaels, this markdown serves a strategic purpose. The company's weekly flyers advertising the Lemax Village Collection sale attract consumers who usually leave the store with other regularly priced items.

**THE FORMULA**

If an item is on sale all the time, then businesses plan the pricing components with the sale price in mind. Companies using this technique determine the unit profitability of the product at the sale price and not the regular selling price. They adapt Formula 2.5 as follows:

$$S = C + E + P$$

$$S_{\text{onsale}} = C + E + P_{\text{onsale}}$$

where  $P_{\text{onsale}}$  represents the planned profit amount when the product is sold at the sale price. This is not a new formula, just a new application of Formula 2.5.

**HOW IT WORKS**

Under normal circumstances, when businesses set their selling and sale prices they follow a three-step procedure:

- Determine the product's cost, expenses, and profit amount.
- Set the regular selling price of the product.
- If a markdown is to be applied, determine an appropriate markdown rate or amount and set the sale price.

However, when a product is planned to always be on sale, businesses follow these steps instead to set the sale price and selling price:

**Step 1:** Set the planned markdown rate or markdown dollars. Determine the pricing components such as cost and expenses. Set the profit so that when the product is marked down, the profit amount is achieved. Alternatively, a planned markup on cost, markup on selling price, or even markup dollars may be set for the sale price.

**Step 2:** Calculate the sale price of the product. If cost, expenses, and profit are known, apply the adapted version of Formula 2.5. Alternatively, adapt and apply any of the other markup formulas (Formulas 2.6 through 2.9) with the understanding that the result is the sale price of the product and not the regular selling price.

**Step 3:** Using the known markdown rate or markdown amount, set the regular selling price by applying any appropriate markdown formula (Formulas 2.10 through 2.12).

#### Example

Assume for the Michael's Lemax Village Collection that most of the time these products are on sale for 40% off. A particular village item costs \$29.99, expenses are \$10.00, and a planned profit of \$8.00 is achieved at the sale price. Calculate the sale price and the selling price.

**Step 1:** The known variables at the sale price are  $C = \$29.99$ ,  $E = \$10.00$ ,  $P = \$8.00$ , and  $d = 0.40$ .

**Step 2:** Adapting Formula 2.5, the sale price is  $S_{\text{onsale}} = C + E + P_{\text{onsale}} = \$29.99 + \$10.00 + \$8.00 = \$47.99$ . This is the price at which Michael's plans to sell the product.

**Step 3:** However, to be on sale there must be a regular selling price. Therefore, if the 40% off results in a price of \$47.99, apply Formula 2.10 and rearrange to get the selling price:  $S = \$47.99 \div (1 - 0.40) = \$79.98$ . Therefore, the product's selling price is \$79.98, which, always advertised at 40% off, results in a sale price of \$47.99. At this sale price, Michael's earns the planned \$8.00 profit.

#### IMPORTANT NOTES

You may ask, "If the product is always on sale, what is the importance of establishing the regular price?" While this textbook does not seek to explain the law in depth, it is worth mentioning that pricing decisions in Canada are regulated by the Competition Act. With respect to the discussion of never-ending sales, the Act does require that the product be sold at a regular selling price for a reasonable period of time or in reasonable quantity before it can be advertised as a sale price. If you revisit the

Michael's example, note in the discussion that the village initially needs to be listed at the regular selling price before being lowered to the sale price.

### GIVE IT SOME **THOUGHT**

- If a product has a markup on cost of 40% and a markdown of 40%, will it sell above or below cost?
- What happens to the profit if a product that is always on sale actually sells at the regular selling price?
- Under normal circumstances, arrange from smallest to largest: regular selling price, cost, and sale price.

#### Solutions

1. Below cost, since the 40% markdown is off of the selling price, which is a larger value.
2. The profit will be increased by the markdown amount.
3. Cost, sale price, regular selling price.

#### **Example 3.3C:** Setting the Price in a Never-Ending Sale

An electronics retailer has 16GB USB sticks on sale at 50% off. It initially priced these USB sticks for a short period of time at regular price, but it planned at the outset to sell them at the sale price. The company plans on earning a profit of 20% of the cost when the product is on sale. The unit cost of the USB stick is \$22.21, and expenses are 15% of the cost.

1. At what price will the retailer sell the USB stick when it is on sale?
2. To place the USB stick on sale, it must have a regular selling price. Calculate this price.
3. If the USB stick is purchased at the regular selling price during the initial time period, how much profit is earned?

#### **Solution**

#### **Plan**

1. This company plans on always having the product on sale, so the pricing needs to be set for the sale price, or  $S_{\text{onsale}}$ .
2. You need the regular selling price, or  $S$ .

3. Solve for the profit at the regular selling price, or P.

### What You Already Know

---

Step 1: You know the unit cost, the retailer's associated expenses, its planned profit at the sale price, and the markdown rate:

- $d = 0.50$
- $C = \$22.21$
- $E = 0.15C$
- $P_{\text{onsale}} = 0.2C$

### How You Will Get There

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Step 2: To solve part (a), apply the adapted version of Formula 2.5:

$$S_{\text{onsale}} = C + E + P_{\text{onsale}}$$

Step 3: After you know the sale price, solve part (b) by applying Formula 2.10, rearranging for S.

Step 4: Solving part (c) requires applying Formula 2.5.

### Perform

---

Step 2:

- $S_{\text{onsale}} = \$22.21 + 0.15(\$22.21) + 0.2(\$22.21)$
- $= \$22.21 + \$3.33 + \$4.44 = \$29.98$

Step 3:

- $\$29.98 = S \times (1 - 0.5)$
- $\$29.98 = S \times 0.5$
- $\$59.96 = S$

Step 4:

- $\$59.96 = \$22.21 + \$3.33 + P$
- $\$59.96 = \$25.54 + P$
- $\$34.42 = P$

### Present

The USB stick is on sale for \$29.98, letting the company achieve its profit of \$4.44 per unit. During the initial pricing period, the USB stick sells for \$59.96 (its regular selling price). If a consumer actually purchases a USB stick during the initial pricing period, the electronics store earns a profit of \$34.42 per unit (which is a total of the \$4.44 planned profit plus the planned markdown amount of \$29.96).

## EXERCISES

Round all money to two decimals and percentages to four decimals for each of the following exercises.

### Mechanics

For questions 1–6, solve for the unknown variables (identified with a ?) based on the information provided.

Question	Regular Selling Price	Markdown Amount	Markdown Percent	Sale Price
1.	\$439.85	?	35%	?
2.	?	\$100.00	?	\$199.95
3.	\$1,050.00	?	?	\$775.00
4.	\$28,775.00	\$3,250.00	?	?
5.	?	?	33%	\$13,199.95
6.	?	\$38.33	12%	?

### Applications

- A pair of Nike athletic shoes is listed at a regular selling price of \$89.99. If the shoes go on sale for 40% off, what is the sale price?
- During its special Bay Days, The Bay advertises a Timex watch for \$39.99 with a regular price of \$84.99. Calculate the markdown percent and markdown amount.
- For spring break you are thinking about heading to Tulum, Mexico. In planning ahead, you notice that a one-week stay at the Gran Bahia Principe Tulum, regularly priced at \$2,349 for air and six nights all inclusive, offers an early-bird booking discount of \$350. What markdown percentage is being offered for booking early?
- A Heritage Infusio deep frying pan is advertised at 70% off with a sale price of \$39.99. What is the frying pan's regular selling price, and what markdown amount does this represent?
- A mass merchandiser uses its Lagostina cookware product line as a marketing tool. The

cookware is always on sale at an advertised price of 45% off. The cost of the cookware is \$199.99, expenses are \$75, and the planned profit at the sale price is \$110. Calculate the sale price and selling price for the cookware.

12. Quicky Mart regularly sells its Red Bull sports drink for \$2.99 per can. Quicky Mart noticed that one of its competitors down the street sells Red Bull for \$1.89. What markdown percentage must Quicky Mart advertise if it wants to match its competitor.
13. A hardware store always advertises a Masterdesigner 75-piece screwdriver set at 80% off for a sale price of \$17.99.
  - a. If the cost of the set is \$10 and expenses are 30% of the sale price, what is the planned profit when the product is on sale?
  - b. What profit is earned if the product actually sells at its regular selling price?
14. A campus food outlet is advertising a “Buy one, get one 25% off” deal. The 25% off comes off the lower-priced item. If you purchase a chicken dinner for \$8.99 and your friend gets the burger combo for \$6.99, what is the markdown percentage on the total price?
15. Blast'em Stereos purchases a stereo system for \$1,900 less two discounts of 40% and 18%. The store uses this product to draw customers to the store and always offers the stereo on sale at 25% off. When the stereo is on sale, it plans on expenses equalling 30% of the cost and a profit of 20% of the sale price.
  - a. What is the sale price for the stereo?
  - b. How much profit does Blast'em make when the stereo sells at the sale price?
  - c. By law, this stereo must sell at the regular selling price for a period of time before going on sale. What is the regular selling price?
  - d. What profit does Blast'em earn if a customer purchases the stereo during this initial period?

### Challenge, Critical Thinking, & Other Applications

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16. Frigid Boards purchases one of its snowboards for \$395 less a retail trade discount of 15% and a loyalty discount of 4%. Its markup on selling price percentage on all snowboards is 21%. At the end of the season, any leftover snowboards are marked down by 10%. What is the sale price for the snowboard?
17. An HP LaserJet printer has an MSRP of \$399.95. It is subject to trade discounts of 30% and 23%. The LaserJet is a featured item for a computer store and is always on sale. The store plans to sell the LaserJet for a sale price that allows it to cover expenses equalling 15% of cost and realize a profit of \$35.00.
  - a. What is the sale price?
  - b. If the MSRP is the regular unit price of the printer, what rate of markdown can the

computer store advertise?

c. What markup on selling price percentage is realized at the sale price?

18. The Brick advertises that when you purchase a queen-size Tempur-Pedic mattress set for \$2,499.97 it will give you a 51" 3-D plasma television with a 3-D starter kit included. The value of this gift is \$1,199.99. What markdown percent does this represent?
19. A Maytag 27 cubic foot refrigerator retails for \$2,400.00 at Landover Appliance Centre. The company, which is celebrating its 30th anniversary this coming weekend, features the fridge for 30% off. The markup on selling price percentage on the fridge at the regular unit selling price is 53%.
  - a. What is the sale price?
  - b. At the sale price, what is the markup on selling price percentage?
  - c. If the expenses are 15% of the regular selling price, what is the profit when the fridge is on sale?
20. Dreger Jewellers is selling a diamond bracelet. It uses this bracelet in its promotions and almost always has it on sale. The cost of the bracelet is \$2,135 less discounts of 20% and 30%. When the bracelet is on sale for 25% off, the expenses are 15% of cost and the profit is 20% of cost.
  - a. What is the sale price?
  - b. What is the bracelet's regular selling price?
  - c. If the bracelet sells at the regular selling price, what are the markup amount and the markup on cost percent?

*Credit: Adapted from unit 6.3 Business math: A Step-by-Step Handbook*

## 2.4 Sales Taxes

### EVERYBODY WANTS A PIECE OF MY PIE

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On your recent cross-Canada road trip, you purchased from many different Tim Hortons' stores. At each store, your products retailed for \$6.99. When you review your credit card receipts after returning home from your trip, you notice that you paid different totals everywhere. In Alberta, they only added GST and your combo cost \$7.34. In British Columbia, they added both PST and GST, resulting in total cost of \$7.83. In Ontario, they added something called HST, resulting in a total cost of \$7.90. You find it interesting that the same combo came to different totals as you travelled across Canada.

### THREE SALES TAXES

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A sales tax is a percent fee levied by a government on the supply of products. In Canada, there are three types of sales taxes: the goods and services tax (GST), provincial sales tax (PST), and the harmonized sales tax (HST). In this section you will learn the characteristics of each of these taxes and then the mathematics for calculating any sales tax.

#### **Goods & Services Tax (GST)**

---

The goods and services tax, better known as GST, is a national federal tax of 5% that applies to the purchase of most goods and services in Canada. Every province and territory has GST. The consumer ultimately bears the burden of this sales tax.

Businesses must collect GST on most of their sales and pay GST on most purchases in the daily course of operations. However, when remitting these taxes, businesses claim a credit with the federal government to recover the GST they paid on eligible purchases. The net result is that businesses do not pay the GST on these eligible purchases. While this may outrage some people, the logic is simple. If a business pays the GST, it becomes a cost of the business, which is then passed on to consumers as it is incorporated into retail prices. When the consumer purchases the product, the consumer would be charged the GST again! In essence, a consumer would be double-taxed on all purchases if businesses paid the GST.

Some goods and services are exempt from GST. While there are many complexities and nuances to the exemptions, generally items that are deemed necessities (such as basic groceries), essential services (such as health, legal aid, and childcare), and charitable activities are nontaxable. You can find a complete listing of exemptions on the Canada Revenue Agency website.

Provincial Sales Tax (PST)

Provincial sales taxes, or PST, are provincially administered sales taxes that are determined by each individual provincial or territorial government in Canada. The table here lists the current PST rates in Canada.

Province / Territory	PST
Alberta	0%
British Columbia	7%
Manitoba	7%
Northwest Territories	0%
Nunavut	0%
Quebec	9.975%
Saskatchewan	5%
Yukon Territory	0%

Similar to GST, PST applies to the purchase of most goods and services in the province, and consumers bear the burden. For the same reasons as with GST, businesses typically pay the PST on purchases for non-resale items (such as equipment and machinery) and do not pay the PST on resale items. Businesses are responsible for collecting PST on sales and remitting the tax to the provincial government. Individual provincial websites list the items and services that are exempt from PST.

Harmonized Sales Tax (HST)

The harmonized sales tax, or HST, is a combination of GST and PST into a single number. Since most goods and services are subjected to both taxes anyway, HST offers a simpler method of collecting and remitting the sales tax—a business has to collect and remit only one tax instead of two. Because there are pros and cons to HST, not all provinces use this method of collection, as summarized in the table below.

Pros	Cons
Items that are previously PST payable to a business are now refunded, lowering input costs and lowering consumer prices	Many items such as utilities, services, and children's clothing that are ineligible for PST become taxed at the full HST rate
Results in overall lower corporate taxes paid	Consumer cost of living increases
Increases the competitiveness of businesses and results in job creation	Tax-exempt items see prices rise because HST is being applied to services and goods such as transportation and gasoline
Businesses only remit one tax and not two, resulting in financial and auditing savings	

Province / Territory	HST
Ontario	13%
New Brunswick	15%
Nova Scotia	15%
Newfoundland and Labrador	15%
Prince Edward Island	15%

To understand HST you can separate it into its GST and PST components. For example, in the table to the left Ontario has an HST of 13%. If GST is 5% and the HST is 13%, then it is clear that the province has a PST of 8%. HST operates mostly in the same manner as GST, in that consumers generally bear the burden for paying this tax, and businesses both collect and pay the HST but have the HST reimbursed on eligible purchases when remitting taxes to the government.

## THE FORMULA

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With respect to sales taxes, you usually calculate two things:

1. The dollar amount of the sales tax.
2. The price of a product including the sales tax.

## CALCULATING THE SALES TAX AMOUNT

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A sales tax is a percent rate calculated on the base selling price of the product. Therefore, if you are interested solely in the amount of the sales tax (the portion owing), apply Formula 2.2 on Rate, Portion, base:

$$Rate = \frac{Portion}{Base} \text{ becomes } TaxRate = \frac{TaxAmount}{PricebeforeTaxes}$$

Rearranging this formula to solve for the tax amount gives the following:

$$Tax\ Amount = Tax\ Rate \times Price\ before\ Taxes\ (which\ is\ the\ same\ as\ Portion = Rate \times Base)$$

## CALCULATING A PRICE INCLUDING TAX – FORMULA 2.13

---

When calculating a selling price including the tax, you take the regular selling price and increase it by the sales tax percentage.

$$\% \text{ change} = \frac{New - Old}{Old} \times 100$$

The Old price is the price before taxes, the percent change is the sales tax percentage, and you need to calculate the New price including the tax. Rearranging the formula for New, you have

$$New = Old + Old\left(\frac{\Delta}{100}\right)$$

which you factor and rewrite as

$$New = Old \times \left(1 + \frac{\Delta}{100}\right)$$

or

$$New = Old + \left(Old \times \frac{\Delta}{100}\right)$$

This is a widespread application of the percent change formula.

**S<sub>tax</sub> is Selling Price Including Tax:** A subscript is added to the regular symbol of S for selling price to denote that this is the price that sums the selling price and sales tax amount together.

**S is Selling Price Without Tax:** The regular selling price before taxes.

**Formula 2.13 Selling Price Including Tax:** 
$$S_{\text{tax}} = S + (S \times \text{Rate})$$

**Rate is Tax Rate:** This is the sales tax in any form of GST, HST, or PST expressed in its decimal format. Round the result of  $(S \times \text{Rate})$  to two decimals. If two taxes are involved in the tax-inclusive price (such as GST and PST), you cannot combine the rates together into a single rate. For example, Manitoba has 5% GST and 8% PST. This is not necessarily equivalent to 13% tax since each tax is rounded to two decimals separately and then summed. If you use a single rate of 13%, you may miscalculate by a penny. Instead, expand the formula to include two separate  $(S \times \text{Rate})$  calculations:

$$S_{\text{tax}} = S + (S \times \text{GST Rate}) + (S \times \text{PST Rate})$$

Ensure that you round off each calculation in brackets to two decimals before adding.

## HOW IT WORKS

Follow these steps to perform calculations involving sales taxes:

**Step 1:** Identify the pricing information. In particular, pay careful attention to distinguish whether the price is before taxes (S) or inclusive of taxes (S<sub>tax</sub>). Also identify all applicable sales taxes, including GST, PST, and HST.

**Step 2:** Apply **Formula 2.13** to solve for the unknown variable.

**Step 3:** If you need to find sales tax amounts, apply Formula 2.2 and rearrange for portion. Ensure that for the base you use the price before taxes.

## EXAMPLE

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Assume a \$549.99 product is sold in British Columbia. Calculate the amount of the sales taxes and the price including the sales taxes.

**Step 1:** The price before taxes is  $S = \$549.99$ . In British Columbia, GST is 5% and PST is 7% (from the PST Table).

**Step 2:** To calculate the price including the sales taxes, apply Formula 2.13:

$$S < sub > tax < /sub > = \$549.99 + (\$549.99 \times 5$$

**Step 3:** Applying the rearranged Formula 2.2 for the GST, you calculate:

$$GSTTaxAmount = 5$$

Applying the same formula for the PST, you calculate:

$$PSTTaxAmount = 7$$

**Note:** You may notice that you could just pull these amounts from the interim calculations in step 2.

Therefore, on a \$549.99 item in British Columbia, \$27.50 in GST and \$38.50 in PST are owing, resulting in a price including sales taxes of \$615.99.

## IMPORTANT NOTES

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To calculate a price including a single sales tax rate, use the percent change function ( $\Delta\%$ ) on the calculator. You can review the full instructions for this function at the end of Chapter 3. When using this function, OLD is the price before taxes, NEW is the price after taxes, and %CH is the single tax rate in its percentage format. You do not use the #PD variable, which therefore defaults to 1.

Note that if more than one tax rate applies on the same base, sum the tax rates together and enter as the percent change (or use as Rate in the formula). However, note that due to the penny rounding on the individual taxes, an error margin of a single penny may occur when trying to make  $S$  plus the taxes equal to  $S_{tax}$ . This may be an unavoidable error and it would remain impossible to exactly equal the price including the sales tax.

## PATHS TO SUCCESS

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You will often need to manipulate Formula 2.13. Most of the time, prices are advertised without taxes and you need to calculate the price including the taxes. However, sometimes prices are advertised including the taxes and you must calculate the original price of the product before taxes. When only one tax is involved, this poses no problem, but when two taxes are involved (GST and PST), combine the taxes into a single amount before you solve for  $S$ .

## GIVE IT SOME THOUGHT

On any given product selling for the same price, put the following provinces in order from highest price to lowest price including taxes (GST and PST, or HST): Alberta, Saskatchewan, British Columbia, Ontario, Prince Edward Island.

### Solutions

PEI (15% HST), Ontario (13% HST), British Columbia (5% GST + 7% PST), Saskatchewan (5% GST + 6% PST), Alberta (5% GST, no PST)

### Example 2.4 A: Calculating Sales tax Across Canada

Dell Canada lists a complete computer system on its Canadian website for \$1,999.99. Calculate the price including taxes if the Canadian buyer is located in: a. Alberta c. Quebec b. Ontario d. British Columbia (BC ).

#### Solution

#### Plan

Four answers are required. For each of the provinces listed, calculate the appropriate GST/PST or HST to add onto the price and arrive at the selling price including taxes ( $S_{\text{tax}}$ ).

#### What You Already Know

Step 1: The price of the computer and tax rates are known:

- $S = \$1,999.99$
- Alberta sales tax = 5% GST
- Ontario sales tax = 13% HST
- Quebec sales tax = 5% GST & 9.975% PST
- BC sales tax = 5% GST & 7% PST

#### How Will You Get There

Step 2: For all provinces, apply Formula 2.13.

- a. For Alberta, the Rate is the 5% GST. Substitute the  $S$  and Rate to solve for  $S_{\text{tax}}$ .
- b. For Ontario, the Rate is the 13% HST. Substitute the  $S$  and Rate to solve for  $S_{\text{tax}}$ .

c. For Quebec, both the 5% GST and 9.975% PST are based on the S. Expand the formula for each tax. Substitute these Rates with the S to arrive at  $S_{\text{tax}}$ .

d. For British Columbia, both the 5% GST and 7% PST are based on the S. Expand the formula for each tax. Substitute these Rates with the S to arrive at  $S_{\text{tax}}$ .

## Perform

Step 2:

a. Alberta:  $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 5\%) = \$2,099.99$

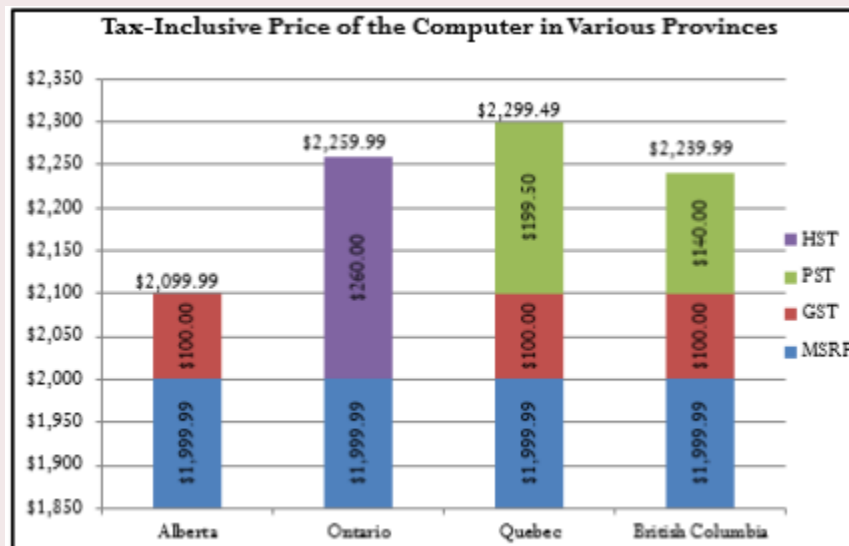
b. Ontario:  $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 13\%) = \$2,259.99$

c. Quebec:  $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 9.975\%) = \$2,299.49$

d. British Columbia:  $S_{\text{tax}} = \$1,999.99 + (\$1,999.99 \times 5\%) + (\$1,999.99 \times 7\%) = \$2,239.99$

## Present

Depending on which province you reside in, the price for the computer from lowest to highest is Alberta at \$2,099.99, British Columbia at \$2,239.99, Ontario at \$2,259.99, and Quebec at \$2,299.49.



“The Brick is having its Midnight Madness sale! Pay no taxes on products purchased during this event!” While this is good marketing, it probably goes without saying that governments do not give up the sales taxes. Essentially The Brick is advertising a tax-inclusive price. Calculate GST and PST amounts for a product advertised at \$729.95, including GST and PST, in Saskatchewan.

### Solution

#### Plan

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To calculate the PST and GST for Saskatchewan, you must calculate the price before taxes (S). Then calculate each tax amount based on the S.

#### What You Already Know

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Step 1: The price after taxes and the tax rates are as follows:

- $S_{\text{tax}} = \$729.95$
- Tax rates = 5% GST and 6% PST

#### How Will You Get There

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Step 2: Apply **Formula 2.13** using the combined PST and GST as the Rate to calculate the S.

Step 3: Apply **Formula 2.2** rearranged for Portion to calculate the tax amounts.

#### Perform

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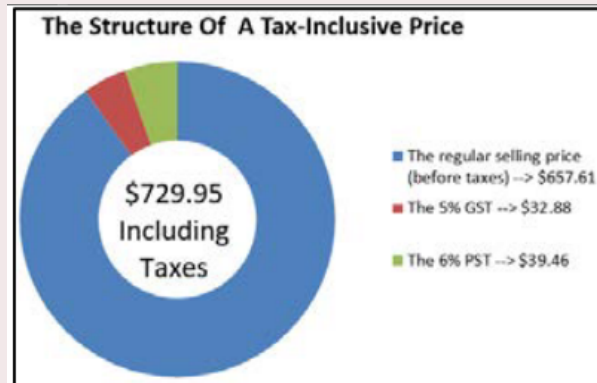
Step 2: Remove the taxes:

- $\$729.95 = S + (S \times 11\%)$
- $\$729.95 = 1.11S$
- $S = \$657.61$

Step 3:

- GST:  $5\% = \text{GST Portion} \div \$657.61$ ;
- GST Portion = \$32.88 PST:  $6\% = \text{PST Portion} \div \$657.61$ ;
- PST Portion = \$39.46

## Present



When paying a tax-inclusive price of \$729.95 in Saskatchewan, the regular selling price of the item is \$657.61 with \$ \$32.88 GST and \$39.46 PST included.

## 2.4.2 THE GST/HST REMITTANCE

When a business collects sales taxes, it is a go-between in the transaction. These sales tax monies do not belong to the business. On a regular basis, the business must forward this money to the government. This payment is known as a tax remittance.

## THE FORMULA

Generally speaking, a business does not pay sales taxes. As a result, the government permits a business to take all eligible sales taxes that it paid through its acquisitions and net them against all sales taxes collected from sales. The end result is that the business is reimbursed for any eligible out-of-pocket sales tax that it paid. Formula 2.14 expresses this relationship.

### Formula 2.14

**Remit is GST/HST Remittance:** This is the dollar amount of the remittance.

- If this amount is positive, it means that the business collected more tax than it paid out; the company must remit this balance to the government.
- If this amount is negative, it means that the business paid out more taxes than it collected; the government must refund this balance to the company.

**Formula 2.14 · GST/HST Remittance:**  $\text{Remit} = \text{Tax Collected} - \text{Tax Paid}$

**Tax Collected and Tax Paid:** Both of these parts of the formula apply Formula 2.2, representing the total amount of sales tax (portion) from all taxable amounts (base) at the appropriate sales tax rate (rate). The taxes collected are based on total tax-eligible revenues. The taxes paid are based on the total tax-eligible acquisitions.

## HOW IT WORKS

Follow these steps to complete a GST/HST remittance:

Step 1: Identify the total amounts of tax-eligible revenues and acquisitions upon which the sales tax is collected or paid, respectively. Identify the applicable sales tax rate of the GST or HST.

Step 2: Calculate the total taxes collected by applying Formula 2.2, where the sales tax is the rate and the total revenue is the base. Solve for portion.

Step 3: Calculate the total taxes paid by applying Formula 2.2, where the sales tax is the rate and the total acquisitions are the base. Solve for portion.

Step 4: Apply **Formula 2.14** to calculate the tax remittance.

## Example

Assume a business has paid GST on purchases of \$153,000. It has also collected GST on sales of \$358,440. Calculate the GST remittance.

**Step 1:** Identifying the variables, you have:

Total Revenue = \$358,440

Total Acquisitions = \$153,000

GST Tax Rate = 5%

**Step 2:** Calculate taxes collected by applying Formula 2.2, where:

GST collected =  $5\% \times \$358,440 = \$17,922$ .

**Step 3:** Calculate taxes paid by applying Formula 2.2, where:

$$\text{GST paid} = 5\% \times \$153,000 = \$7,650.$$

**Step 4:** To calculate the remittance, apply Formula 2.14 and calculate  $\text{Remit} = \$17,922 - \$7,650 = \$10,272$ . The business should remit a cheque for \$10,272 to the government.

## PATHS TO SUCCESS

A shortcut can help you calculate the GST/HST. Remittance using Formula 2.14. If you do not need to know the actual amounts of the tax paid and collected, you can net GST/HST-eligible revenues minus acquisitions and multiply the difference by the tax rate:

$$\text{Remit} = (\text{Revenues} - \text{Acquisitions}) \times \text{Rate}$$

In the example above,  $\text{Remit} = (\$358,440 - \$153,000) \times 5\% = \$10,272$ . If this calculation produces a negative number, then the business receives a refund instead of making a remittance.

### Example 2.4 C Calculating a GST/HST Remittance

An Albertan lumber company reported the following quarterly purchases and sales in its 2013 operating year:

Assuming all purchases and sales are eligible and subject to GST, calculate the GST remittance or refund for each quarter.

#### Solution

#### Plan

Calculate the GST tax remittance, or Tax Remit, for each of the quarters.

#### What You Already Know

Step 1: From the information provided, the total purchases and sales for each quarter along with the GST = 5%, which is the Tax Rate, are known.

#### How Will You Get There

Step 2: For each quarter, calculate the GST collected by rearranging and applying Formula 2.2.

Step 3: Calculate the tax paid.

Step 4: Apply Formula 2.14 for each quarter

**Perform**

	Step 2		Step 3		Step 4
Quarter	Sales	GST Collected (Sales x 5%)	Purchases	GST Paid (Purchases x 5%)	Remittance (Tax Collected – Tax Paid)
Winter	\$911,500	\$45,575	\$1,316,000	\$65,800	\$45,575 – \$65,800 = – \$20,225
Spring	\$1,475,300	\$73,650	\$1,266,200	\$63,310	\$73,650 – \$63,310 = – \$10,340
Summer	\$1,381,700	\$69,085	\$1,053,700	\$52,685	\$69,085 – \$52,685 = – \$16,400
Fall	\$1,124,600	\$56,230	\$878,100	\$43,905	\$56,230 – \$43,905 = – \$12,325

**Present**

In the Winter quarter, the company gets a GST refund of \$20,225. In the other three quarters, the company remits to the government payments of \$10,340, \$16,400, and \$12,325, respectively.

**2.4.3 EXERCISES****Mechanics**

- You are purchasing a new BlackBerry at the MSRP of \$649.99. Calculate the price including taxes in the following provinces or territories:
  - Northwest Territories
  - New Brunswick
  - Nova Scotia
  - British Columbia
- The Brick is advertising a new Serta mattress nationally for a price of \$899.99 including taxes. What is the price before taxes and the sales tax amounts in each of the following provinces?
  - Ontario
  - Saskatchewan
- Audiophonic Electronics is calculating its HST remittance in Prince Edward Island. For each of the following months, calculate the HST remittance or refund on these HST-eligible amounts.

Month	Purchases	Sales
January	\$48,693	\$94,288
February	\$71,997	\$53,639

- Airwaves Mobility is calculating its GST remittance in Alberta. For each of the following quarters, calculate the GST remittance or refund on these GST-eligible amounts.

Quarter	Purchases	Sales
Winter	\$123,698	\$267,122
Spring	\$179,410	\$158,905
Summer	\$216,045	\$412,111
Fall	\$198,836	\$175,003

## Applications

- Elena lives in Nova Scotia and has relatives in Alberta, Saskatchewan, and Quebec. She gets together with them often. She wants to purchase a new aerobic trainer and would like to pay the lowest price. If a family member buys the item, Elena can pick it up at one of their regular family gatherings. The price of the trainer for each province is listed below:
  - Where should Elena have the aerobic trainer purchased and how much would she pay?
  - How much money would she save from her most expensive option?
- Mary Lou just purchased a new digital camera in Nunavut for \$556.49 including taxes. What was the price of the camera before taxes? What amount of sales tax is paid?
- Marley is at Peoples Jewellers in New Brunswick wanting to purchase an engagement ring for his girlfriend. The price of the ring is \$2,699.95. If the credit limit on his credit card is \$3,000, will he be able to purchase the ring on his credit card? If not, what is the minimum amount of cash that he must put down to use his credit card?
- In the IKEA store in Vancouver, British Columbia, you are considering the purchase of a set of kitchen cabinets priced at \$3,997.59. Calculate the amount of GST and PST you must pay for the cabinets, along with the total price including taxes.
- A company in Saskatchewan recorded the following GST-eligible purchases and sales throughout the year. Determine the GST remittance or refund per quarter.

Quarter	Purchases	Sales
1st	\$2,164,700	\$2,522,000
2nd	\$1,571,300	\$2,278,700
3rd	\$1,816,100	\$1,654,000
4th	\$2,395,900	\$1,911,700

- A manufacturer in Nova Scotia recorded the following HST-eligible purchases and sales in its first three months of its fiscal year. Determine the HST remittance or refund per month.

Month	Purchases	Sales
March	\$20,209	\$26,550
April	\$28,861	\$20,480
May	\$22,649	\$42,340

### Challenge, Critical Thinking, & Other Applications

1. If the selling price of an item is 6% higher in Yukon than in Ontario, will the price including taxes be higher in Yukon or Ontario? What percentage more?
2. Colin just travelled across the country on a road trip. He bought some skis in Alberta for \$879.95 plus tax, a boombox in British Columbia for \$145.58 including taxes, a Niagara Falls souvenir in Ontario for \$99.97 plus tax, and some maple syrup in Quebec for \$45.14 including tax. Overall, how much GST, PST, and HST did Colin pay on his trip?
3. Cisco Enterprises in Ontario purchased the following in a single month:
  - 16,000 units of network routers at \$79.25 each, priced at \$97.97 each
  - 12,000 units of wireless LAN adapters at \$129.95 each, priced at \$189.55 each
  - 13,500 units of computer boards at \$229.15 each, priced at \$369.50 each.
 Assuming that all units purchased are sold during the same month and that all purchases and sales are taxable, calculate the tax remittance or refund for the month.
4. In Quebec, the PST used to be calculated on the price including GST. When the PST was calculated in this manner, what PST rate did Quebec set to arrive at the same price including taxes?
5. For each of the following situations, compute the selling price of the product before taxes in the other province/territory that would result in the same selling price including taxes as the item listed.

Price before Tax	Sold In	Find Equivalent Price before Tax in This Province
a. \$363.75	British Columbia	Prince Edward Island
b. \$1,795.00	Alberta	Manitoba
c. \$19,995.95	Saskatchewan	Ontario
d. \$4,819.35	New Brunswick	Quebec

6. A company made the following taxable transactions in a single month. Compute the GST remittance on its operations assuming all sales and purchases are eligible for GST.

Transaction Type	Unit Price	Quantity Involved
Purchase	\$168.70	5,430
Sale	\$130.00	4,000
Sale	\$148.39	3,600
Purchase	\$93.47	2,950
Purchase	\$24.23	3,325
Purchase	\$121.20	2,770
Sale	\$188.88	6,250

Credit: Unit 7.1 in Business Math: A Step-by-Step handbook

## 2.5 Invoicing: Terms of Payment and Cash Discounts

### MAKE SURE YOU BILL THEM

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What a way to start your Monday morning! With dread you face a great stack of envelopes in your in-basket. The first envelope contains an invoice from your major supplier indicating an outstanding balance of \$3,600 with terms of 2/10, 1/20, net 30. In the second envelope, an office supplies bill totals \$500 with terms of 3/15, net 45 EOM. The third envelope holds yet another invoice from your transportation company indicating an outstanding balance of \$21,000 with terms of 2/15, 1/25, net 60 ROG. It also indicates that you have an overdue balance of \$4,000 subject to a 3% monthly penalty. Before opening any more envelopes that probably contain still more bills, you pour yourself another cup of coffee and settle down to figuring out how much you need to pay and when.

In the world of business, most purchases are not paid for in cash. Instead, businesses tend to work through an invoicing system in which they send out bills to their clients on a regular basis. In accounting, this means that the purchase is placed into accounts receivables until such time as a cheque arrives and the purchase is converted to cash. Invoices provide detailed transaction information, listing the amount owed and also indicating the terms on which payment is expected. Companies may offer what are known as cash discounts as incentives for early invoice payment. The rationale for these discounts is simple—a sale is not a sale until you have the cash in hand. The longer an invoice remains in accounts receivable, the less likely that it will be paid. Thus, it might turn into bad debt, which the creditor cannot collect at all.

Invoicing is less common in consumer purchases because of the sheer volume of transactions involved and the higher risk of nonpayment. Imagine purchasing items at Walmart and receiving an invoice to pay your bill next month instead of paying cash. It is hard to fathom the number of invoices Walmart would have to distribute monthly. How much would it cost to collect those debts? How many of those invoices would go unpaid? That is why consumer purchases typically do not involve invoicing.

Invoicing does commonly occur at a consumer level on credit card transactions along with many services where the business may not be able to assess the exact amount of the bill at the time of the transaction or until the service is delivered.

Two examples illustrate this point:

1. Think of your MasterCard bill. You are able to make purchases, say, from March 9 to April 8. Then on April 9 the company sends out a statement saying you have until April 29 to pay your bill. If you do not, interest and late payment penalties are involved.
2. You have a dental visit for a regular cleaning. Before charging you, the dentist's office needs to determine how much your insurance will cover. It may not find out the answer for a day or two, so it sends you an invoice at a later date once it hears from the insurance company. The invoice terms indicate that payment is due upon receipt and you will incur late penalties if payment is not forthcoming.

This section explores the most common aspects of invoicing, including terms of payments and cash discounts. You will learn to calculate the amount required to pay invoices and how to reduce the outstanding balance of an invoice if a partial payment is received. Additionally, the calculation of late payments and penalties is introduced.

## INVOICE TERMS AND INVOICE DATING

---

You must know how to read a business invoice. The figure on the next page<sup>1</sup> provides a sample invoice and highlights the following areas:

1. **Invoicing Company.** The invoice must identify who is sending and issuing the invoice.
2. **Invoice Date.** The date on which the invoice was printed, along with the invoice tracking number (in this case, the order date and shipping date are equivalent to the invoice date). When an invoice is paid, the cheque must reference the invoice number so that the invoicing company can identify which invoice to credit the payment to.
3. **Transaction Details.** The details of the transaction might include number of units, unit prices, and any discounts for which the item is eligible.
4. **Invoice Total.** The total amount owing is indicated, including any taxes or additional charges.
5. **Terms of Payment.** The terms of payment include any cash discounts and due dates. The date of commencement (as discussed below) is determined from this part of the invoice.
6. **Late Penalty.** A penalty, if any, for late payments is indicated on the invoice. Whether or not a company enforces these late penalties is up to the invoice-issuing organization.

1. Invoice provided by and courtesy of Kerry Mitchell Pharmacy Ltd. And Shoppers Drug Mart.

**2**

**1**

**5**

**3**

**6**

**4**

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INVOICE/FACTURE NO. 163737  
ORDER NO. DE COM. ORDER DATE DE LA COMMANDE SHIPPING DATE DE LIVRAISON  
Feb 4 10 Feb 4 10

SHOPPERS DRUG #2428  
K MITCHELL PHCY LTD  
710 ST ANNE'S RD  
WINNIPEG MB  
R2N 0A2

SHOPPERS DRUG #2428  
K MITCHELL PHCY LTD  
710 ST ANNE'S RD  
WINNIPEG MB  
R2N 0A2

ACCOUNT NO. 2358 SALESMAN 41 CUSTOMER ORDER NO. 71777 SHIPPED VIA CANPAR PROVINCIAL TAX NO. TERMS 1% 10 NET 30

PRODUCT NO. NO PRODUIT	DESCRIPTION	ORDERED COMMANDE	QTY À VENDRE	SHIPPED EXPÉDIE	PRICE PRIX	QTY PER CH	AMOUNT MONTANT
07-00192	SPRING 2010 NEW STYLE PK	1	0	1	355.40	PK	355.40
07-0314	H-COMFORT ZONE 314			3		EA	
07-0319	C-COMFORT ZONE 319			3		EA	
07-0304	I-COMFORT ZONE 304			3		EA	
07-0317	G-COMFORT ZONE 317			3		EA	
07-0318	G-COMFORT ZONE 318			3		EA	
07-0315	H-COMFORT ZONE 315			3		EA	
07-0320	G-COMFORT ZONE 320			3		EA	
07-0316	H-COMFORT ZONE 316			3		EA	
07-0321	F-COMFORT ZONE 321			3		EA	
07-0602	P-COMFORT STUD 602			4		EA	
07-0692	T-COMFORT STUD 692			4		EA	
07-0683	S-COMFORT STUD 683			4		EA	
07-0684	S-COMFORT STUD 684			4		EA	
07-0690	T-COMFORT STUD 690			4		EA	
07-0627	P-COMFORT STUD 627			4		EA	
07-0672	P-COMFORT STUD 672			4		EA	
07-0630	P-COMFORT STUD 630			4		EA	

REMARKS/REMARQUES  
\*\* SPRING 2010 NEW STYLE PREPACK \*\*

CONDITIONS OF SALE  
CHECK ALL GOODS UPON RECEIPT. CLAIMS OR RETURNS NOT ACCEPTED  
IF NOT IN WRITING WITHIN 30 DAYS OF RECEIPT. NO CASH REFUND.  
NO RETURNS WITHOUT OUR AUTHORIZATION. RETURNS ARE SUBJECT TO A  
15% RESTOCKING CHARGE. OVERSEA: ADDITIONAL 5% RESTOCK CHARGE OF AT  
LEAST 2% PER MONTH (24% PER ANNUAL). ADDED, BASED ON PREVIOUS  
ORDERING HISTORY.

CONDITIONS DE VENTE  
VÉRIFIEZ TOUTE LA MARCHANDISE À LA RÉCEPTION. TOUTE RÉCLAMATION  
DOIT ÊTRE EN ÉCRIT DANS LES 30 JOURS APRÈS LA RÉCEPTION. PAS DE RÈGLEMENT  
ESPÈCE. REFUSÉ.  
AUCUN RETOUR SANS NOTRE AUTORISATION. LES RETOURS SONT SOUS NOTRE  
ÉVALUATION. COMPTES NON PAYÉS - PRIX DE SERVICE DE 2% PAR MOIS (24%  
PAR ANNUÉ) SONT AJOUTÉS À LA BALANCE.

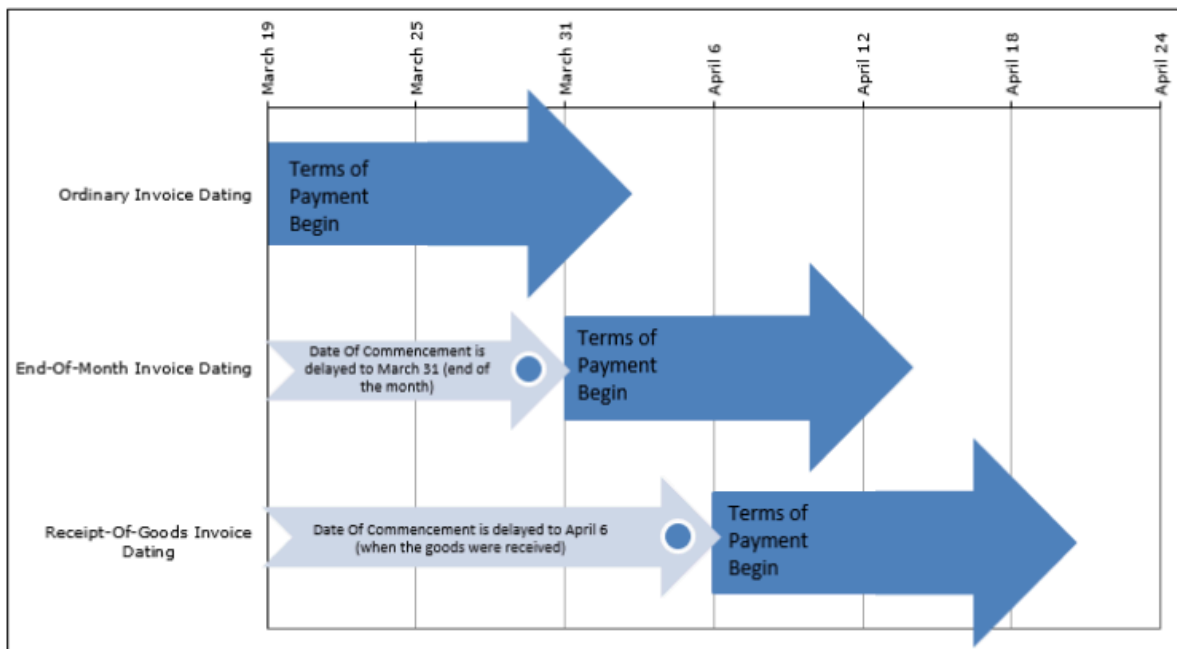
SUB TOTAL/SOUS TOTAL 355.40  
SHIPPING/EXPÉDITION 0.00  
G.S.T./P.S. 17.77  
PROD. TAX PROV. 0.00  
\*\* PROV. TAX 2500 0.00

TOTAL 373.17

\* CODES: EA - EACH DZ - DOZEN PK - PREPACK \* SPECIAL PACK: GE Pack: GE Check: GE  
EA - CHAQUE DZ - DOUZANE PK - PRÉCONDITIONNÉ

## THREE DATES OF COMMENCEMENT

All invoice terms are affected by what is known as the date of commencement, which is the first day from which all due dates stated on the invoice will stem. The date of commencement is determined in one of three ways as illustrated below.



### Ordinary Invoice Dating.

In ordinary invoice dating, or just invoice dating for short, the date of commencement is the same date as the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on March 19. This is the default manner in which most companies issue their invoices, so if an invoice does not specify any other date of commencement you can safely assume it is using ordinary dating.

### End-of -Month Invoice Dating.

End-of -month invoice dating applies when the terms of payment include the wording “end-of -month” or the abbreviation “EOM” appears after the terms of payment. In end-of -month dating, the date of commencement is the last day of the same month as indicated by the invoice date. Therefore, if the invoice is printed on March 19, then all terms of payment commence on the last day of March, or March 31. Many companies use this method of dating to simplify and standardize all of their due dates, in that if the date of commencement is the same for all invoices, then any terms of payment will also share the same dates.

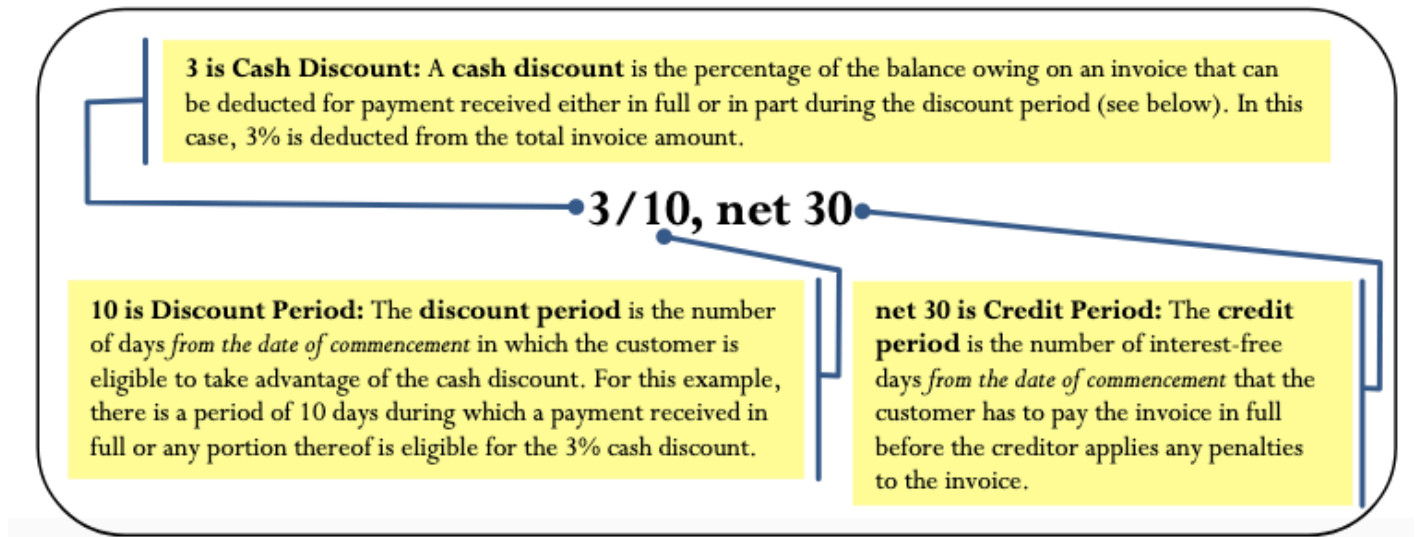
### Receipt-Of-Goods Invoice Dating.

Receipt-of-goods invoice dating applies when the terms of payment include the wording “receipt-of -goods” or the abbreviation “ROG” appears after the terms of payment. In receipt-of -goods dating, the date of commencement is the day on which the customer physically receives the goods. Therefore, if the invoice is printed on March 19 but the goods are not physically received until April 6, then all terms

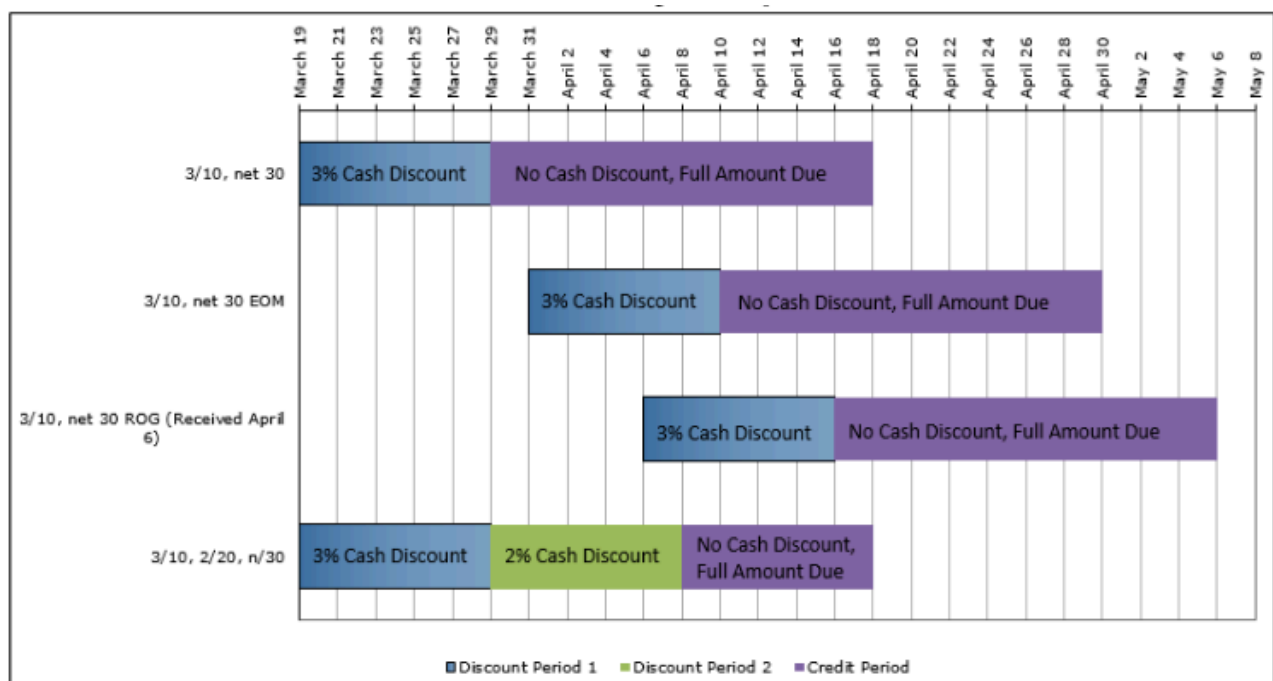
of payment commence on April 6. Companies with long shipping times involved in product distribution or long lead times in production commonly use this method of dating.

## TERMS OF PAYMENT

The most common format for the terms of paying an invoice is shown on the next page and illustrated by an example.



The figure below plots the invoice term of “3/10, net 30” along a time diagram to illustrate the terms of payment. This term is combined with various dates of commencement using the examples from the “Date of Commencement” section.



Note the following observations:

- In all four scenarios, the total length of all bars is the same, since the credit period is 30 days.
- Although all four invoices have the same invoice date of March 19, the date of commencement is modified in the ROG and EOM scenarios, which shift the discount and credit periods into the future. In these cases, any payment before the end of the discount period qualifies for the discount. For example, in the EOM scenario a company could pay its bill early on March 27 and qualify for the 3% cash discount.

You may see invoice terms displayed in several other common formats:

1. 3/10, n/30: In this case, the word “net” has been abbreviated to “n/”. This is illustrated in the fourth scenario in the figure.
2. 3/10, 2/20, n/30. In this case, multiple cash discounts are being offered, meaning 3% within 10 days from the date of commencement and 2% from the 11th to the 20th day from the date of commencement. This is also illustrated in the fourth scenario of the figure.
3. n/30. In this case, no cash discount is offered and only the credit period is identified.

## RULES FOR INVOICE TERMS

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Some common business practices are implemented across most industries. Note that you must always check with the invoicing organization to ensure it is applying these practices.

### No Net Figure

---

If no net figure is stated in the terms of payment, you should assume that the credit period is 20 days after the last discount period. If there are no discount periods, then the credit period is 20 days from the date of commencement. For example, “3/10, 2/15” means that the credit period ends 20 days after the second discount period of 15 days. Hence, the credit period is 35 days after the date of commencement.

### No Cash Discount

---

Not every invoice receives a cash discount. If no terms indicate a cash discount, then the invoicing company seeks full payment only. For example, “n/30” means that no cash discount applies and the credit period ends 30 days from the date of commencement.

### Nonbusiness Days

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Most businesses operate Monday through Friday and are closed on weekends and holidays. As such, any date falling on a nonworking day is moved to the next business day. For example, if an invoice is ordinary dated December 21 and lists a cash discount of 2/10, the end of the discount period falls on January 1. Since this is New Year’s Day, the discount period extends to January 2. In this textbook, you

should apply this practice to any of the five known statutory holidays discussed in Chapter 4 (New Year's Day, Good Friday or Easter Monday in Quebec only, Canada Day, Labour Day, and Christmas Day).

### THREE TYPES OF PAYMENTS

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When businesses pay invoices, three situations can occur:

#### Full Payment

---

A full payment means that the company wants to pay its invoice in full and reduce its balance owing to zero dollars. This is the most common practice.

#### Partial Payment

---

A partial payment means that the company wants to lower its balance owing but will not reduce that balance to zero. A company will generally employ this method when it wants to either lower its accounts payable or demonstrate good faith in paying its invoices. Partial payment may also occur if the company wants to take advantage of a cash discount but lacks the funds to clear the invoice in its entirety.

#### Late Payment

---

A late payment occurs when the company pays its invoice either in full or partially after the credit period has elapsed. Late payments occur for a variety of reasons, but the most common are either insufficient funds to pay the invoice or a simple administrative oversight.

The rest of this section shows you how to handle each of these types of payments mathematically, and it also explores the implications of cash discounts.

### GIVE IT SOME THOUGHT

---

In each of the following cases, determine which term of payment results in the longest credit period extending from the invoice date.

#### Exercises

- 2/10, n/30 or 3/15
- 3/15, n/45 or 2/10, 1/20
- 2/10, n/60 or 3/10, n/30 EOM on an invoice dated January 7
- 3/15, n/45 or 2/10, n/35 ROG, or 2/20 EOM on an invoice dated September 22 and goods received

on September 29

### Solution

1. 30 days or 35 days
2. 45 days or 40 days
3. 60 days or 54 days (24 days left in January plus 30 more)
4. 45 days or 42 days (7 days until received plus 35) or 48 days (8 left in month plus 40 more)

## FULL PAYMENTS

In the opening scenario in this section, the first invoice on your desk was for \$3,600 with terms of 2/10, 1/20, net 30. Suppose that invoice was dated March 19. If you wanted to take advantage of the 2% cash discount, when is the last day that payment could be received, and in what amount would you need to write the cheque? In this section, we will look at how to clear an invoice in its entirety.

## THE FORMULA

The good news is that a cash discount is just another type of discount, and you do the calculations using the exact same formula. Apply Formula 2.1 on single discounts, which is reprinted below.

**N is Net Payment Amount:** The net price is the amount after the discount has been applied. In other words, it is the dollar amount of the actual payment. If there is no cash discount on the payment, then the net price would be the same as the gross price.

Formula 2.1 – Single Discount: 
$$N = L \times (1 - d)$$

**L is Invoice Balance:** The “list” price is the amount of the invoice before any discounts are applied. In other words, it is the total dollar amount of the invoice and represents the balance owing.

**d is Cash Discount Rate:** This discount rate is the cash discount (in decimal format) that is deducted from the total invoice (the gross price) if payment is received within the discount period. This cash discount needs to be clearly stated in the terms of payment.

## HOW IT WORKS

Follow these steps when working with payments and invoices:

Step 1: Look for and identify key information such as the invoice date, date of commencement modifiers such as EOM or ROG, terms of payment, late penalty, and the invoice amount.

Step 2: Draw a timeline similar to the figure on the next page. On this timeline, chronologically arrange from left to right the invoice date and amount, the date of commencement, the end of any discount (if any) or credit periods, cash discounts (if any) that are being offered, and penalties (if any).

Step 3: Determine when payments are being made and in what amount. Locate them on the timeline.

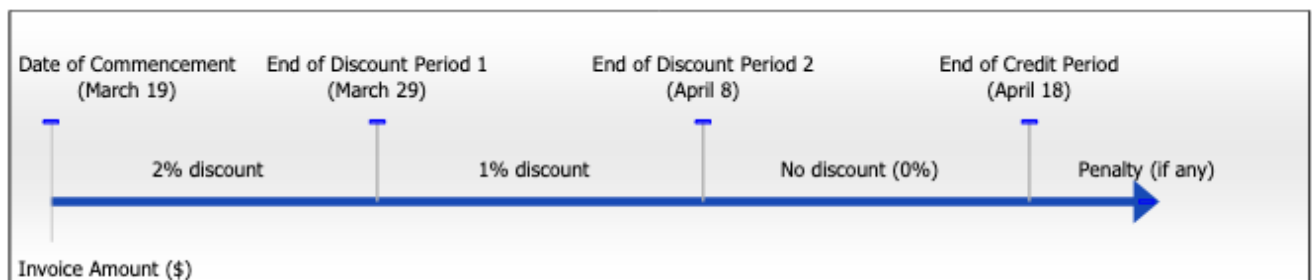
Step 4: Apply the correct calculations for the payment, depending on whether the payment is a full payment, partial payment, or late payment.

## EXAMPLE

Let's continue with the example of the first invoice on your desk. You mail in a cheque to be received by March 29. What amount is the cheque?

**Step 1:** The invoice amount is  $L = \$3,600$ , invoice date is March 19, and terms of payment are 2/10, 1/20, net 30.

**Step 2:** The figure on the next page displays the invoice timeline.



**Step 3:** Note on the timeline that a payment on March 29 is the last day of the 2% discount period.

**Step 4:** According to Formula 6.1, the amount to pay is  $N = \$3,600 \times (1 - 0.02) = \$3,600 \times .98 = \$3,528$ . A cheque for \$3,528 pays your invoice in full. By taking advantage of the cash discount, you reduce your payment by \$72.

## THINGS TO WATCH OUT FOR

If there is one area of invoicing that causes the most confusion, it is assigning information to the net price and invoice amount variables. These are commonly assigned backwards. Remember these rules so that you always get the correct answer:

1. The invoice amount or any reference to the balance owing on an invoice is always a list price.

List Price = Invoice Amount

2. The payment of an invoice, whether in full, partial, or late, is always a net price.

Net Price = Payment Amount

## PATHS TO SUCCESS

Who cares about 1% or 2% discounts when paying bills? While these percentages may not sound like a lot, remember that these discounts occur over a very short time frame. For example, assume you just received an invoice for \$102.04 with terms of 2/10, n/30. When taking advantage of the 2% cash discount, you must pay the bill 20 days early, resulting in a \$100 payment. That is a \$2.04 savings over the course of 20 days. To understand the significance of that discount, imagine that you had \$100 sitting in a savings account at your bank. Your savings account must have a balance of \$102.04 twenty days later. This requires your savings account to earn an annual interest rate of 44.56%! Therefore, a 20-day 2% discount is the same thing as earning interest at 44.56%. Outrageously good!

## GIVE IT SOME THOUGHT

In each of the following cases, determine the cash discount or penalty for which the payment qualifies.

### Exercises

Question	Invoice Date	Terms of Payment	Payment Received On
1.	April 14	2/10, n/30	April 24
2.	July 7	3/10, 2/20, n/30 EOM, 1% per month penalty	August 12
3.	November 12 (goods received November 28)	2/20 ROG, 2% per month penalty	December 29
4.	February 27 (non-leap year)	4/10, 2/15, 1/25 EOM	March 25

### Solution

1. 2% discount
2. 2% discount

3. no discount (0%) or penalty
4. 1% discount

#### Example 2.5A: Paying Your Bills in Full

You receive an invoice for \$35,545.50 dated August 14. The terms of payment are listed as 3/10, 2/20, net 45 EOM. The accounting department is considering paying this debt on one of three days. Determine the full payment required if payment is received by the invoicing company on each of the following dates:

- a. September 3
- b. September 19
- c. September 30

#### Solution

In each case, calculate the full payment (N) after applying any cash discount.

#### What You Already Know

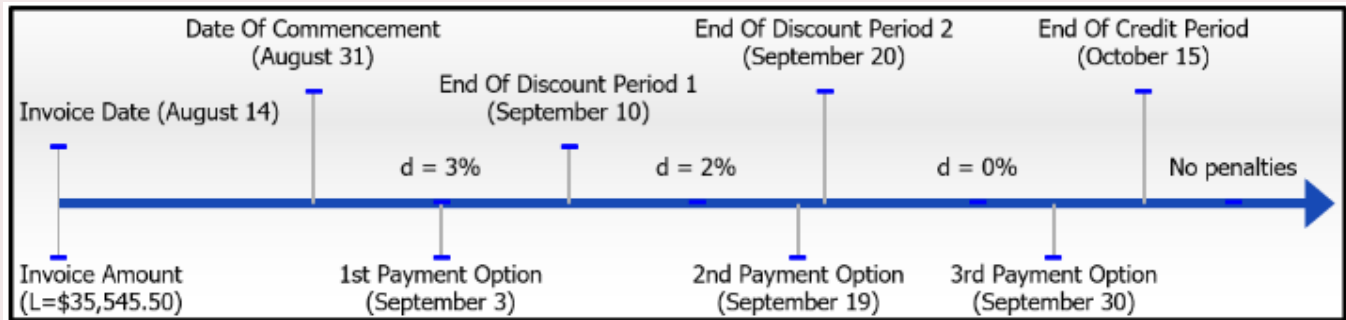
Step 1: The invoice amount, terms of payment, and any payments are known:

- $L = \$35,545.50$
- Invoice date = August 14
- Terms of payment = 3/10, 2/20, net 45 EOM
- Three payment date options are September 3, September 19, and September 30.
- Steps 2 & 3: The figure below illustrates the timeline for the invoice and identification of payments.

#### How You Will Get There

Step 4: Calculate the payment required by applying Formula 6.1. As noted on the timeline:

- a. The first payment option qualifies for  $d = 3\%$ .
- b. The second payment option qualifies for  $d = 2\%$ .
- c. The third payment option qualifies for  $d = 0\%$  (no cash discount).



### Perform

Step 4:

- $N = \$35,545.50 \times (1 - 0.03) = \$35,545.50 \times 0.97 = \$34,479.14$
- $N = \$35,545.50 \times (1 - 0.02) = \$35,545.50 \times 0.98 = \$34,834.59$
- $N = \$35,545.50 \times (1 - 0) = \$35,545.50 \times 1 = \$35,545.50$   
(Note that since there is no discount in part (c), you could have concluded that  $N = L$ )

### Present

If the invoicing company receives payment on September 3, a 3% discount is allowed and \$34,479.14 clears the invoice. If the payment is received on September 19, a 2% discount is allowed and \$34,834.59 pays it off. Finally, if the payment arrives on September 30, there is no discount so the full invoice amount of \$35,545.50 is due.

## PARTIAL PAYMENTS

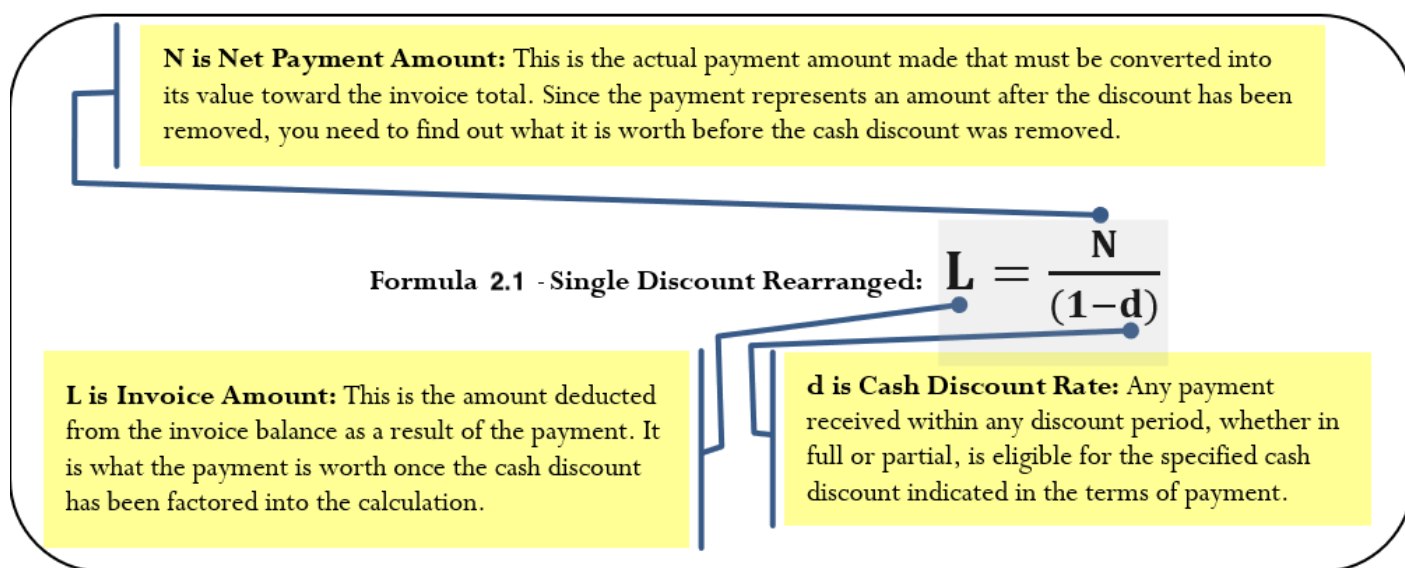
In the section opener, your third invoice is for \$21,000 with terms of 2/15, 1/25, net 60 ROG. After you paid the \$3,528 to clear the first invoice, you realize that your company has insufficient funds to take full advantage of the 2% cash discount being offered by the transportation company. Not wanting to lose out entirely, you decide to submit a partial payment of \$10,000 before the first discount period elapses. What is the balance remaining on the invoice?

Unless you pay attention to invoicing concepts, it is easy to get confused. Perhaps you think that \$10,000 should be removed from the \$21,000 invoice total, thereby leaving a balance owing of \$11,000. Or maybe you think the payment should receive the discount of 2%, which would be \$200. Are you credited with \$9,800 off of your balance? Or maybe \$10,200 off of your balance? In all of these scenarios, you would be committing a serious mistake and miscalculating your balance owing. Let's look at the correct way of handling this payment.

## THE FORMULA

Recall that invoice amounts are gross amounts (amounts before discounts), or  $G$ , and that payment amounts are net amounts (amounts after discounts), or  $N$ . Also recall that an algebraic equation requires all terms to be expressed in the same unit.

In the case of invoice payments, you can think of the balance owing as being in the unit of “pre-discount” and any payment being in the unit of “post-discount.” Therefore, a payment cannot be directly deducted from the invoice balance since it is in the wrong unit. You must convert the payment from a “post-discount” amount into a “pre-discount” amount using a rearranged version of Formula 2.1. You can then deduct it from the invoice total to calculate any balance remaining.

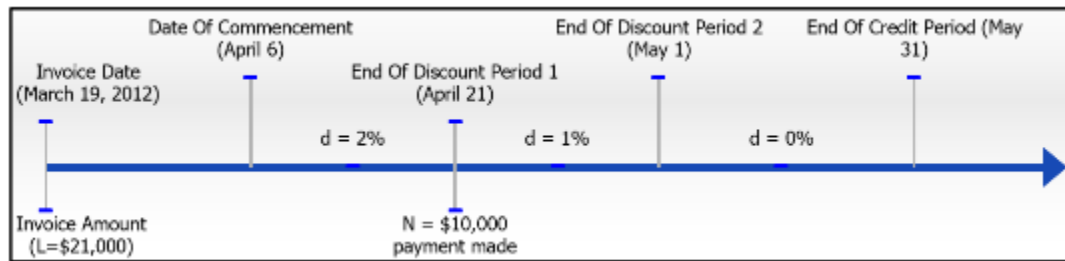


## HOW IT WORKS

Follow the same invoice payment steps even when working with partial payments. Commonly, once you calculate the gross amount of the payment in step 4 you will also have to calculate the new invoice balance by deducting the gross payment amount. Let's continue working with the \$21,000 invoice with terms of 2/15, 1/25, net 60 ROG. Assume the invoice is dated March 19 and the goods are received on April 6. If a \$10,000 payment is made on April 21, what balance remains on the invoice?

**Step 1:** The invoice amount is \$21,000, the invoice date is March 19, goods are received on April 6, and the terms of payment are 2/15, 1/25, net 60. A payment of \$10,000 is made on April 21.

**Step 2:** The figure below shows the invoice timeline.



**Step 3:** The payment of \$10,000 on April 21 falls at the end of the first discount period and qualifies for a 2% discount.

**Step 4:** To credit the invoice, apply the rearranged Formula 6.1:

$$L = \frac{\$10,000}{(100\% - 2\%)} = \frac{\$10,000}{98\%} = \frac{\$10,000}{0.98\%} = \$10,204.08$$

This means that before any cash discounts, the payment is worth \$10,204.08 toward the invoice total. The balance remaining is:  $\$21,000.00 - \$10,204.08 = \$10,795.92$

## IMPORTANT NOTES

In the case where a payment falls within a cash discount period, the amount credited toward an invoice total is always larger than the actual payment amount. If the payment does not fall within any discount period, then the amount credited toward an invoice total is equal to the actual payment amount (since there is no cash discount).

To help you understand why a partial payment works in this manner, assume an invoice is received in the amount of \$103.09 and the customer pays the invoice in full during a 3% cash discount period. What amount is paid? The answer is  $N = \$103.09(100\% - 3\%) = \$100$ . Therefore, any payment of \$100 made during a 3% cash discount period is always equivalent to a pre-discount invoice credit of \$103.09. If the balance owing is more than \$103.09, that does not change the fact that the \$100 payment during the discount period is worth \$103.09 toward the invoice balance.

### Examples – Give It Some Thought

In each of the following situations, determine for the partial payment whether you would credit the invoice for an amount that is larger than, equal to, or less than the partial payment amount.

1. An invoice dated April 7 with terms 4/20, 2/20, n/60 EOM. The goods are received on April 9 and a partial payment is made on May 21.
2. An invoice dated July 26 with terms 3/30, n/45 ROG. The goods are received on August 2 and a partial payment is made on September 3.

3. An invoice dated January 3 with terms  $2\frac{1}{2}/10, 1/20$ . The goods are received on January 10 and a partial payment is made on January 24.

### Solution

1. Since the payment falls within a discount period, the credited amount is larger than the payment.
2. Since the payment does not fall within a discount period, the credited amount is equal to the payment.
3. Since the payment falls within a discount period, the credited amount is larger than the payments

### Example 2.5B: Applying Your Partial Payments toward Your Balance Owning

Heri just received an invoice from his supplier, R&B Foods. The invoice totalling \$68,435.27 is dated June 5 with terms of  $2\frac{1}{2}/10, 1/25, n/45$ . Heri sent two partial payments in the amounts of \$20,000 and \$30,000 that R&B Foods received on June 15 and June 29, respectively. He wants to clear his invoice by making a final payment to be received by R&B Foods on July 18. What is the amount of the final payment?

### Solution

#### Plan

Determine the amount of the final payment (N) after the first two partial payments are credited toward the invoice total. To credit the two partial payments you must find the list amount before any cash discounts (L) such that the payment can be deducted from the invoice total.

#### What You Already Know

Step 1: The invoice amount, terms of payment, and payment amounts are known:

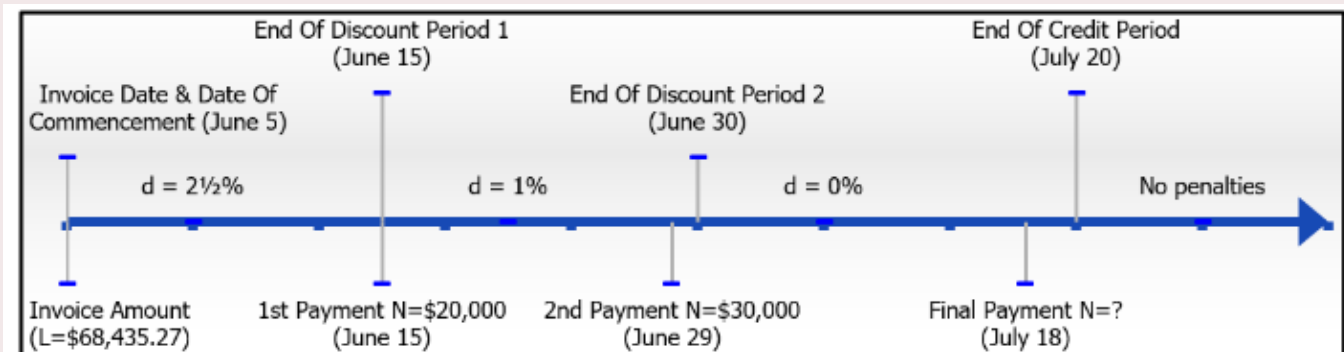
- $L = \$68,435.27$  Invoice date = June 5
- Terms of payment =  $2\frac{1}{2}/10, 1/25, n/45$
- $N_1 = \$20,000$  on June 15
- $N_2 = \$30,000$  on June 29
- $N_3 = ?$  on July 18

Steps 2 & 3: The figure illustrates the timeline for the invoice and identifies the payments.

#### How You Will Get There

Step 4: The first two payments are partial. Apply the rearranged Formula 6.1. As noted on the timeline:

- The first partial payment qualifies for  $d = 2\frac{1}{2}\%$ .
- The second partial payment qualifies for  $d = 1\%$ .
- The last payment is a full payment, where  $d = 0\%$  (no cash discount).



### Perform

#### Step 4

$$a. \quad L = \frac{\$20,000}{(1 - 0.025)} = \frac{\$20,000}{0.975} = \$20,512.82$$

$$\text{Balance Owing} = \$68,435.27 - \$20,512.82 = \$47,922.45$$

$$b. \quad L = \frac{\$30,000}{(1 - 0.01)} = \frac{\$30,000}{0.99} = \$30,303.03$$

$$\text{Balance Owing} = \$47,922.45 - \$30,303.03 = \$17,619.42$$

$$c. \quad \text{Since there is no cash discount, } N_3 = L. \text{ The balance owing is } L = \$17,619.42.$$

Therefore, the payment is the same number and

$$N = \$17,619.42$$

### Present

The first two payments receive a credit of \$20,512.82 and \$30,303.03 toward the invoice total, respectively. This leaves a balance on the invoice of \$17,619.42. Since the final payment is made during the credit period, but not within any cash discount period, the final payment is the exact amount of the balance owing and equals \$17,619.42.

## LATE PAYMENT PENALTIES

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Among all of the invoices that you were paying in the section opener, you realize upon opening that third envelope that you somehow missed an invoice last month and failed to pay the \$4,000 owing, which is now overdue. In looking at the invoice, you notice that it states on the bottom that late payments are subject to a 3% per month penalty. You urgently want to pay this invoice to maintain good relations with your supplier but wonder in what amount you must issue the cheque?

### Why Do Late Penalties Exist?

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Suppliers are doing their business customers a favour when they use invoicing to seek payment. In essence, suppliers provide the products to the customer but are not paid for those products up front. This means customers are granted a period during which they “borrow” the products for free without any associated interest costs that are usually tied to borrowing. If the invoice is not paid in full by the time the credit period elapses, then the supplier starts treating any remaining balance like a loan and charges interest, which is called a late payment penalty.

### THE FORMULA

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When a payment arrives after the credit period has expired, you need to adjust the balance of the invoice upward to account for the penalty. Continue to use Formula 6.1 on Single Discounts for this calculation. In this unique case, the discount rate represents a penalty. Instead of deducting money from the balance, you add a penalty to the balance. As a result, the late penalty is a negative discount. For example, if the penalty is 3% then  $d = -3\%$ .

### HOW IT WORKS

---

Follow the same steps to solve the invoice payment once again when working with late payments. As an example, work with the \$4,000 overdue invoice that is subject to a 3% per month late penalty. If the invoice is paid within the first month of being overdue, what payment is made? Step 1: The invoice balance is  $L = \$4,000$ , and the penalty percent is  $d = -0.03$ .

Steps 2 & 3: Since the focus is strictly on the late payment calculation, you do not need a timeline for the discount or credit periods. The late payment occurs within the first month past the expiry of the credit period. Step 4: To determine the new invoice balance, apply Formula 6.1 to calculate  $N = \$4,000 \times (1 - (-0.03)) = \$4,000 \times 1.03 = \$4,120$ . Therefore, to clear the invoice you must issue a cheque in the amount of \$4,120. This covers the \$4,000 owing along with a late penalty of \$120.

## IMPORTANT NOTES

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Not all invoices have penalties, nor are they always enforced by the supplier. There are many reasons for this:

1. Most businesses pay their invoices in a timely manner, which minimizes the need to institute penalties.
2. Waiving a penalty maintains good customer relations and reinforces a positive, cooperative business partnership.
3. Strict application of late penalties results in difficult situations. For example, if a payment was dropped in the mail on August 13 from your best customer and it arrives one day after the credit period expires on August 15, do you penalize that customer? Is it worth the hassle or the risk of upsetting the customer?
4. The application of late penalties generally involves smaller sums of money that incur many administrative costs. The financial gains from applying penalties may be completely wiped out by the administrative expenses incurred.
5. Businesses have other means at their disposal for dealing with delinquent accounts. For example, if a customer regularly pays its invoices in an untimely manner, the supplier may just decide to withdraw the privilege of invoicing and have the customer always pay up front instead. In this textbook, all penalties are strictly and rigidly applied. If a payment is late, the invoice balance has the appropriate late penalty applied.

## THINGS TO WATCH OUT FOR

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When a late penalty is involved, you must resist the temptation to adjust the payment by the penalty percentage. The discussion and calculations in this section focus on adjusting the outstanding invoice balance to figure out the payment required. For example, assume there is a \$500 outstanding balance subject to a 2% penalty.

- This means that with the penalty, the invoice total is \$510, which determines that a payment of \$510 is required.
- If the customer makes a \$200 partial payment on this late debt, you cannot apply the partial payment procedure, which would give a credit of \$204.08 for the payment, resulting in a balance owing of \$295.92. If you make this mistake, you in fact reward the customer for being tardy!
- Nor can you deduct the 2% penalty from the payment and give credit for  $\$200(1 - 0.02) = \$196$  (resulting in a balance owing of \$304) because the 2% penalty applies to the whole invoice amount and not just the partial payment.
- Instead, the \$200 payment must be deducted directly from the penalty-adjusted balance of \$510. In this case, the customer still owes \$310 to clear the invoice.

## PATHS TO SUCCESS

An alternative method for applying a late penalty is to treat the penalty like a positive percent change. In this case, the penalty is how much you want to increase the balance owing.

Thus, if an invoice for \$500 is subject to a 2% penalty, the  $\Delta\% = 2\%$  and  $\text{Old} = \$500$ .

The goal is to look for the New balance in the following formula:

$$\Delta\% = \frac{\text{New} - \text{Old}}{\text{Old}} \times 100$$

$$2\% = \frac{\text{new} - \$500}{\$500} \times 100$$

$$\text{New} = \$510$$

### Example 2.5C: Whoops! I Missed the Due Date!

You received an invoice dated December 17 in the amount of \$53,455.55 with terms of 4/15, 2/30, n/60 ROG, 2.75% penalty per month for late payments. The merchandise is received on January 24. You sent in a partial payment of \$40,000

on January 31 with intention to pay the remaining balance before the credit period expired. However, you forgot about the invoice and realized your mistake on March 30, when you submit payment in full for the invoice. What amount is the final payment? (Note: Assume this is not a leap year)

#### Solution

#### Plan

The final payment must cover the invoice balance owing on any remaining balance after the partial payment is credited and the outstanding balance is penalized as per the late penalty.

#### What You Already Know

Step 1: The invoice amount, terms of payment, and payment information are known:

- $L = \$53,455.55$
- Invoice date = December 17
- Goods received = January 24
- Terms of payment = 4/15, 2/30, n/60 ROG,
- 2.75% penalty per month for late payments

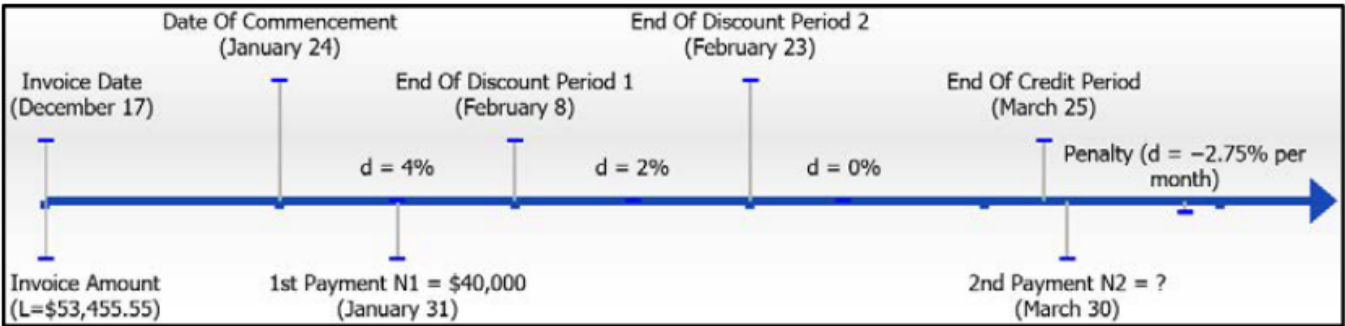
- N1 = \$40,000 on January 31
- N2 = ? on March 30

Steps 2 & 3: The figure below illustrates the timeline for the invoice and identifies the payments.

How You Will Get There

Step 4 (Partial Payment): Credit the partial payment during the discount period ( $d = 4\%$ ) by applying the rearranged Formula 6.1. Step 4 (New Invoice Balance): Deduct the partial payment from the invoice total.

Step 4 (Late Payment): The second payment occurs after the credit period elapses. Therefore the outstanding invoice balance is penalized by  $d = -0.0275$  as per the policy. Apply Formula 6.1. This calculates the outstanding balance including the penalty, which equals the final payment.



Perform

Step 4    (*partialPayment*) :  $L = \frac{\$40,000}{(1 - 0.04)} = \$40,000 \times 1.04 = \$41,666.67$

Step 4    (New Invoice Balance): Balance Owing =  $\$53,455.55 - \$41,666.67 = \$11,788.88$

Step 4    (Late Payment):  $N = \$11,788.88 \times (1 - (-0.0275))$

$N = \$11,788.88 \times 1.0275 = \$12,113.07$

Present

The first payment resulted in a \$41,666.67 deduction from the invoice. The overdue remaining balance of \$11,788.88 has a penalty of \$324.19 added to it, resulting in a final clearing payment of \$12,113.07

2.5.3 EXERCISES

Round all money to two decimals and percentages to four decimals for each of the following exercises. In all questions, assume that the dates indicated are the dates upon which the payment is received.

## Mechanics

For questions 1–3, calculate the full payment required on the payment date that reduces the balance on the invoice to zero. Assume this is not a leap year.

Question	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods	Date of Full Payment
1.	\$136,294.57	January 14	2/10, n/30	January 10	January 22
2.	\$98,482.75	September 14	3/10, 2/20, 1/30, n/50 EOM	October 3	October 19
3.	\$48,190.38	February 21		February 27	April 3

For questions 4–6, calculate the remaining invoice balance after crediting the invoice for the partial payments indicated.

Question	Invoice Amount	Invoice Date	Invoice Terms	Receipt of Goods	Partial Payments and Dates
4.	\$57,775.00	June 16	3/15, $1\frac{1}{2}$ /25	June 12	\$20,000 on July 2; \$15,000 on July 10
5.	\$1,200,310.75	August 25	3/10, 2/20, 1/30, n/50 EOM	September 2	\$500,000 on September 4; \$250,000 on September 14
6.	\$17,481.68	December 3	4/10, 1/30, 1/40, n/60	December 20	\$10,000 on January 2

For questions 7–8, calculate the final payment required on the payment date that reduces the balance of the invoice to zero.

Question	Invoice Amount	Invoice Date	Invoice Terms	Late Payment Penalty	Receipt of Goods Date	Partial Payment and Dates	Final Payment date
7.	\$23,694.50	April 13	3/15, 2/25, 1/40, n/60	1 $\frac{1}{2}$ % per month	April 18	\$10,000 on April 30	June 15
8.	\$332,053.85	October 4	4/10 EOM	2% per month	October 30	\$100,000 on November 5	December 15

## Applications

- Hudson's Bay received an invoice dated July 13 from Nygard International in the amount of \$206,731.75. The terms are 2/10, 1/20, n/30. If Nygard receives full payment on August 1, what amount is paid?
- Time Bomb Traders Inc. in Burnaby just received an invoice dated February 17 (of a leap year) from UrbanEars headphones in the amount of \$36,448.50 with terms of  $1\frac{1}{2}$ /15,  $\frac{1}{2}$ /30, n/45 ROG. The items on the invoice are received on March 3. What amount is the full payment if UrbanEars receives it on March 18?
- Family Foods received an invoice dated July 2 from Kraft Canada in the amount of \$13,002.96 with terms of 2/15, 1/30, n/45 EOM and a late penalty of 2% per month. What amount is paid

in full if Kraft Canada receives a cheque from Family Foods on August 17?

12. An invoice dated November 6 in the amount of \$38,993.65 with terms of  $2\frac{1}{2}/10$ , 1/15 ROG, 2% penalty per month is received by Cargill Limited from Agricare United. The wheat shipment is received on December 2. Cargill Limited made a partial payment of \$15,000 on December 10. What amount should Cargill pay to clear its invoice on December 13?
13. Yamaha Music received two invoices from the same vendor. The first is for \$1,260 dated March 9 with terms 3/10, net 30. The second is for \$2,450 dated March 12 with terms 2/10, 1/20, net 30. If a payment of \$1,000 is made on March 19, what payment amount on March 31 settles both invoices? (Note: Payments are applied to the earlier invoice first.)
14. An invoice is dated July 26 for \$5,345.50 with terms of  $3\frac{1}{4}/10$ , net 45,  $2\frac{1}{2}\%$  per month penalty. If the invoice is paid on September 10, what payment amount is required to pay the entire balance owing?
15. Mohawk College received an invoice dated August 20 from Office Depot for office supplies totalling \$10,235.97 with terms of  $3\frac{3}{4}/15$ , 1½/30, n/45, EOM,  $2\frac{3}{4}\%$  per month penalty. The college made three payments of \$2,000 dated September 4, September 29, and October 10. What payment on October 25 settles the invoice?

### Challenge, Critical Thinking, & Other Applications

16. An invoice for \$100,000 dated February 2 (of a non-leap year) with terms of 4/10, 3/20, 2/30, 1/40, net 60, ROG,  $1\frac{3}{4}\%$  per month penalty. The merchandise is received on February 16. If four equal payments are made on February 20, March 17, April 1, and April 20 resulting in full payment of the invoice, calculate the amount of each payment.
17. An accounting department receives the following invoices from the same vendor and makes the indicated payments. The vendor always applies payments to the earliest invoices first and its late payment policy stands at  $1\frac{1}{2}\%$  per month for any outstanding balance. Calculate the amount of the final payment on August 17 that reduces the total balance owing to zero.
18. John's Home Hardware was invoiced on May 29 for the following non-taxable items with terms of 2/20, 1/40, 2% per month penalty. If a partial payment of \$20,000 is made on July 7, what payment amount on August 30 reduces the balance owing to zero? What is the total dollar amount of the late penalty?

Item	Quantity	Unit Price
2" x 4" x 8" framing studs	13,000	\$0.96
4" x 4" x 8' fence posts	2,400	\$2.87
2" x 8" x 16' wood planks	480	\$8.47
1" x 6" x 5' fence boards	8,000	\$1.22

19. Your company receives a \$138,175.00 invoice dated April 12 with terms of 4/10, 3/20, 2/30, 1/

40, n/60. Due to the large amount, the accounting department proposes two different ways to pay this invoice, depending on company income and financing alternatives. These plans are listed in the table below.

Payment Date	Plan #1	Plan #2
April 21	\$35,000	\$50,000
May 2	\$20,000	\$10,000
May 10	\$25,000	\$50,000
May 13	\$30,000	\$20,000
June 11	Balance Owing	Balance Owing

- a. Which alternative do you recommend?
  - b. If your recommendation is followed, how much money is saved over the other option?
20. You receive an invoice dated July 29 with terms of 3/15, 1 $\frac{3}{4}$ /30, n/50 EOM, 2% penalty per month. If the payments in the following table are made and you pay the invoice in full, what is the total invoice amount?

Payment Date	Payment Amount
August 4	\$67,000.00
August 28	\$83,000.00
September 28	\$15,000.00
September 30	\$83,297.16

*Credit: Adapted from unit 7.4 in Business Math: A Step-by-Step Handbook*

## 2.6 Symbols and Formulas

### SYMBOLS USED

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add

### FORMULAS

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#### Formula 2.1 Single Discount

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$$N = L \times (1 - d)$$

#### Formula 2.2 Discount Amount

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a.  $D\$ = L \times D$

b.  $D\$ = L - N$

#### Formula 2.3 Multiple Discounts

---

$$N = L \times (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

#### Formula 2.4 Single Equivalent Discount

---

$$d_{\text{equiv}} = 1 - (1 - d_1) \times (1 - d_2) \times \dots \times (1 - d_n)$$

#### Formula 2.5 Selling Price of a Product

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$$S = C + E + P$$

#### Formula 2.6 Markup Amount

---

$$M\$ = E + P$$

**Formula 2.7 Selling Price Using Markup**

---

$$S = C + M\$$$

**Formula 2.8 Markup on Cost Percentage**

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$$M_oC\% = \frac{M\$}{C} \times 100$$

**Formula 2.9 Markup on Selling Price Percentage**

---

$$M_oS\% = \frac{M\$}{S} \times 100$$

**Formula 2.10 Sale Price of a Product**

---

$$S_{\text{onsale}} = S \times (1 - d)$$

**Formula 2.11a Markdown Amount**

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$$D\$ = S \times d$$

**Formula 2.11b Markdown Amount**

---

$$D\$ = S - S_{\text{onsale}}$$

**Formula 2.12 Markdown Percentage**

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$$d = \frac{D\$}{S} \times 100$$

## 3 EVALUATING CHOICES: TIME, RISK, AND VALUE

### INTRODUCTION

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Financial decisions can only be made about the future. As much as analysis may tell us about the outcomes of past decisions, the past is “sunk”: it can be known but not decided upon. Decisions are made about the future, which cannot be known with certainty, so evaluating alternatives for financial decisions always involves speculation on both the kind of result and the value of the result that will occur. It also involves understanding and measuring the risks or uncertainties that time presents and the opportunities—and opportunity costs—that time creates.

## 3.1 The Time Value of Money

### Learning Objectives

- Explain the value of liquidity.
- Demonstrate how time creates distance, risk, and opportunity cost.
- Demonstrate how time affects liquidity.
- Analyze how time affects value.

Part of the planning process is evaluating the possible future results of a decision. Since those results will occur sometime in the future, it is critical to understand how the passage of time may affect those benefits and costs—not only the probability of their occurrence, but also their value when they do. Time affects value because time affects liquidity.

Liquidity is valuable, and the liquidity of an asset affects its value: all things being equal, the more liquid an asset is, the more valuable it is. This relationship—how the passage of time affects the liquidity of money and thus its value—is commonly referred to as the **time value of money**, which can actually be calculated concretely as well as understood abstractly.

Suppose you went to Mexico, where the currency is the peso. Coming from Canada, you have dollars to spend. When you get there, you are hungry. You see and smell a taco stand and decide to have a taco. Before you can buy the taco, however, you have to get some pesos so that you can pay for it because the right currency is needed to trade in that market. You have wealth (your fistful of dollars), but you don't have wealth that is liquid. In order to change your dollars into pesos and acquire liquidity, you need to exchange currency. There is a fee to exchange your currency: a **transaction cost**, which is the cost of simply making the trade. It also takes a bit of time, and you could be doing other things, so it creates an opportunity cost. There is also the chance that you won't be able to make the exchange for some reason, or that it will cost more than you thought, so there is a bit of risk involved. Obtaining liquidity for your wealth creates transaction costs, opportunity costs, and risk.

In general, transforming not-so-liquid wealth into liquid wealth creates transaction costs, opportunity costs, and risk, all of which take away from the value of wealth. Liquidity has value because it can be used without any additional costs.

One dimension of difference between not-so-liquid wealth and liquidity is time. Cash flows in the past are sunk, cash flows in the present are liquid, and cash flows in the future are not yet liquid. You can only make choices with liquid wealth, not with cash that you don't have yet or that has already been spent. Separated from your liquidity and your choices by time, there is an opportunity cost: if you had liquidity now, you could use it for consumption or investment and benefit from it now. There is also risk, as there is always some uncertainty about the future: whether or not you will actually get your cash flows and just how much they'll be worth when you do.

The further in the future cash flows are, the farther away you are from your liquidity, the more opportunity cost and risk you have, and the more that takes away from the present value of your wealth, which is not yet liquid. In other words, time puts distance between you and your liquidity, and that creates costs that take away from value. The more time there is, the larger its effect on the value of wealth.

Financial plans are expected to happen in the future, so financial decisions are based on values some distance away in time. You could be trying to project an amount at some point in the future—perhaps an investment payout or college tuition payment. Or perhaps you are thinking about a series of cash flows that happen over time—for example, annual deposits into and then withdrawals from a retirement account. To really understand the time value of those cash flows, or to compare them in any reasonable way, you have to understand the relationships between the nominal or face values in the future and their equivalent present values (i.e., what their values would be if they were liquid today). The equivalent present values today will be less than the nominal or face values in the future because that distance over time, that separation from liquidity, costs us by discounting those values.

#### Key Takeaways

- Liquidity has value because it enables choice.
- Time creates distance or delay from liquidity.
- Distance or delay creates risk and opportunity costs.
- Time affects value by creating distance, risk, and opportunity costs.
- Time discounts value.

#### Exercises

1. How does the expression “a bird in the hand is worth two in the bush” relate to the concept of the time value of money?
2. In what ways can “delay to liquidity” affect the value of your wealth?

**Answers**

An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=738#h5p-64>

**CHAPTER CREDIT**

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Unit 4.1 from Chapter 4 Evaluating Choices: Time, Risk, and Value in Financial Empowerment by Bettina Schneider and Saylor Academy shared under a CC BY-NC-SA 4.0 International License.

## 3.2 Valuing a Series of Cash Flows

### Learning Objectives

- Discuss the importance of the idea of the time value of money in financial decisions.
- Define the present value of a series of cash flows.
- Define an annuity.
- Identify the factors you need to know to calculate the value of an annuity.
- Discuss the relationships of those factors to the annuity's value.
- Define a perpetuity.

It is quite common in finance to value a series of future cash flows (CF), perhaps a series of withdrawals from a retirement account, interest payments from a bond, or deposits for a savings account. The present value of the series of cash flows is equal to the sum of the present value of each cash flow, so valuation is straightforward: find the present value of each cash flow and then add them up.

Often, the series of cash flows is such that each cash flow has the same future value. When there are regular payments at regular intervals and each payment is the same amount, that series of cash flows is an **annuity**. Most consumer loan repayments are annuities, as are, typically, installment purchases, mortgages, retirement investments, savings plans, and retirement plan payouts. Fixed-rate bond interest payments are an annuity, as are stable stock dividends over long periods of time. You could think of your paycheck as an annuity, as are many living expenses, such as groceries and utilities, for which you pay roughly the same amount regularly.

To calculate the present value of an annuity, you need to know:

- the amount of the future cash flows (the same for each),
- the frequency of the cash flows,
- the number of cash flows ( $t$ ), and
- the rate at which time affects value ( $r$ ).

Almost any calculator or the many readily available software applications can do the math for you, but it is important for you to understand the relationships between time, risk, opportunity cost, and value.

If you win the lottery, for example, you are typically offered a choice of payouts for your winnings: a lump sum or an annual payment over twenty years.

The lottery agency would prefer that you took the annual payment because it would not have to give up as much liquidity all at once; it could hold on to its liquidity longer. To make the annual payment more attractive for you—it isn't, because you would want to have more liquidity sooner—the lump-sum option is discounted to reflect the present value of the payment annuity. The discount rate, which determines that present value, is chosen at the discretion of the lottery agency.

Say you win \$10 million. The lottery agency offers you a choice: take \$500,000 per year over twenty years or take a one-time lump-sum payout of \$6.7 million. You would choose the alternative with the greatest value. The present value of the lump-sum payout is \$6.7 million. The value of the annuity is not simply \$10 million, or  $500,000 \times 20$ , because those \$500,000 payments are received over time and time affects liquidity and thus value. So, the question is: What is the annuity worth to you?

Your discount rate or opportunity cost will determine the annuity's value to you, as Table 3.2.1 shows.

**Table 3.2.1 Lottery Present Value with Different Discount Rates**

CF	Rate ( <i>r</i> )	Time ( <i>t</i> ) in Years	PV
500,000	0.0200	20	8,175,717
500,000	0.0400	20	6,795,163
500,000	0.0600	20	5,734,961
500,000	0.0800	20	4,909,074
500,000	0.1000	20	4,256,782
500,000	0.1200	20	3,734,722
500,000	0.1400	20	3,311,565
500,000	0.0416	20	6,700,000

As expected, the present value of the annuity is less if your discount rate (opportunity cost) is more. The annuity would be worth the same to you as the lump-sum payout if your discount rate were 4.16%.

In other words, if your discount rate is about 4% or less—if you don't have more lucrative choices than earning 4% with that liquidity—then the annuity is worth more to you than the immediate payout. You can afford to wait for that liquidity and collect it over twenty years because you have no better choice. On the other hand, if your discount rate is higher than 4%, or if you feel that your use of that liquidity would earn you more than 4%, then you have more lucrative things to do with that money and you want it now: the annuity is worth less to you than the payout.

For an annuity, as when relating one cash flow's present and future value, the greater the rate at which time affects value, the greater the effect on the present value. When opportunity cost or risk is low, waiting for liquidity doesn't matter as much as when opportunity costs or risks are higher. When

opportunity costs are low, you have nothing better to do with your liquidity, but when opportunity costs are higher, you may sacrifice more by having no liquidity. Liquidity is valuable because it allows you to make choices. After all, if there are no more valuable choices to make, you lose little by giving up liquidity. The higher the rate at which time affects value, the more it costs to wait for liquidity, and the more choices pass you by while you wait for liquidity.

When risk is low, it is not really important to have your liquidity firmly in hand any sooner because you'll have it sooner or later anyhow. But when risk is high, getting liquidity sooner becomes more important because it lessens the chance of not getting it at all. The higher the rate at which time affects value, the more risk there is in waiting for liquidity and the more chance that you won't get it at all.

As  $r$  increases, the PV of the annuity decreases.  
As  $r$  decreases, the PV of the annuity increases.

You can also look at the relationship of time and cash flow to annuity value in Table 3.2.2 “Lottery Payout Present Values.” Suppose your payout was more (or less) each year, or suppose your payout happened over more (or fewer) years.

**Table 3.2.2 Lottery Payout Present Values**

CF	Rate	Time	PV
500,000	0.0400	20	6,795,163
500,000	0.0200	20	8,175,717
500,000	0.0600	20	5,734,961
750,000	0.0400	20	10,192,745
250,000	0.0400	20	3,397,582
500,000	0.0400	30	8,646,017
500,000	0.0400	10	4,055,448

As seen in Table 3.2.2, the amount of each payment or cash flow affects the value of the annuity because more cash means more liquidity and greater value.

As CF increases, the PV of the annuity increases.  
As CF decreases, the PV of the annuity decreases.

Although time increases the distance from liquidity, with an annuity, it also increases the number of payments because payments occur periodically. The more periods in the annuity, the more cash flows and the more liquidity there is, thus increasing the value of the annuity.

As  $t$  increases, the PV of the annuity increases.  
As  $t$  decreases, the PV of the annuity decreases.

It is common in financial planning to calculate the FV of a series of cash flows. This calculation is useful when saving for a goal where a specific amount will be required at a specific point in the future (e.g., saving for university, a wedding, or retirement).

It turns out that the relationships between time, risk, opportunity cost, and value are predictable going forward as well. Say you decide to take the \$500,000 annual lottery payout for twenty years. If you deposit that payout in a bank account earning 4%, how much would you have in twenty years? What if the account earned more interest? Less interest? What if you won more (or less) so the payout was more (or less) each year?

What if you won \$15 million and the payout was \$500,000 per year for thirty years; how much would you have then? Or if you won \$5 million and the payout was only for ten years? Table 3.2.3 shows how future values would change.

**Table 3.2.3 Lottery Payout Future Values**

CF	Rate	Time	PV	FV
500,000	0.0400	20	6,795,163	14,889,039
500,000	0.0200	20	8,175,717	12,148,685
500,000	0.0600	20	5,734,961	18,392,796
750,000	0.0400	20	10,192,745	22,333,559
250,000	0.0400	20	3,397,582	7,444,520
500,000	0.0400	30	8,646,017	28,042,469
500,000	0.0400	10	4,055,448	6,003,054

Going forward, the rate at which time affects value ( $r$ ) is the rate at which value grows, or the rate at which your value compounds. It is also called the **rate of compounding**. The bigger the effect of time on value, the more value you will end up with because more time has affected the value of your money while it was growing as it waited for you. So, looking forward at the future value of an annuity:

- As  $r$  increases, the FV of the annuity increases.
- As  $r$  decreases, the FV of the annuity decreases.

The amount of each payment or cash flow affects the value of the annuity because more cash means more liquidity and greater value. If you were getting more cash each year and depositing it into your account, you'd end up with more value.

- As CF increases, the FV of the annuity increases.
- As CF decreases, the FV of the annuity decreases.

The more time there is, the more time can affect value. As payments occur periodically, the more cash flows there are, and the more liquidity there is. The more periods in the annuity, the more cash flows, and the greater the effect of time, thus increasing the future value of the annuity.

- As  $t$  increases, the FV of the annuity increases.
- As  $t$  decreases, the FV of the annuity decreases.

There is also a special kind of annuity called a **perpetuity**, which is an annuity that goes on forever (i.e., a series of cash flows of equal amounts occurring at regular intervals that never ends). It is hard to imagine a stream of cash flows that never ends, but it is actually not as rare as it sounds. The dividends from a share of corporate stock are a perpetuity, because in theory, a corporation has an infinite life (as

a separate legal entity from its shareholders or owners) and because, for many reasons, corporations like to maintain a steady dividend for their shareholders.

The perpetuity represents the maximum value of the annuity, or the value of the annuity with the most cash flows and therefore the most liquidity and therefore the most value.

LIFE IS A SERIES OF CASH FLOWS

Once you understand the idea of the time value of money, and its use for valuing a series of cash flows, and the idea of annuities in particular, you won't believe how you ever got through life without them. These are the fundamental relationships that structure so many financial decisions, most of which involve a series of cash inflows or outflows. Understanding these relationships can be a tool to help you answer some of the most common financial questions about buying and selling liquidity, because loans and investments are so often structured as annuities and certainly take place over time.

Loans are usually designed as annuities, with regular periodic payments that include interest expense and principal repayment. Using these relationships, you can see the effect of a different amount borrowed ( $PV_{\text{annuity}}$ ), interest rate ( $r$ ), or term of the loan ( $t$ ) on the periodic payment (CF).

For example, as seen in Table 3.2.4 "Mortgage Calculations," if you get a \$250,000 (PV), thirty-year ( $t$ ), 6.5% ( $r$ ) mortgage, the monthly payment will be \$1,577 (CF). If the same mortgage had an interest rate of only 5.5% ( $r$ ), your monthly payment would decrease to \$1,423 (CF). If it were a fifteen-year ( $t$ ) mortgage, still at 6.5% ( $r$ ), the monthly payment would be \$2,175 (CF). If you can make a larger down payment and borrow less—say \$200,000 (PV)—then with a thirty-year ( $t$ ), 6.5% ( $r$ ) mortgage your monthly payment would be only \$1,262 (CF).

Table 3.2.4 Mortgage Calculations

CF	Rate ( $r$ )	Time ( $t$ )	PV
1,577	0.0054	360	250,000
1,423	0.0046	360	250,000
2,175	0.0054	180	250,000
1,262	0.0054	360	200,000

Note that in Table 3.2.4, the mortgage rate is the monthly rate—that is, the annual rate divided by twelve (months in the year), or  $r / 12$ , and that  $t$  is stated as the number of months, or the number of years  $\times$  12 (months in the year). That is because the mortgage requires monthly payments, so all the variables must be expressed in units of months. In general, the periodic unit used is defined by the frequency of the cash flows and must agree for all variables. In this example, because you have monthly cash flows, you must calculate using the monthly discount rate ( $r$ ) and the number of months ( $t$ ).

Saving to reach a goal—to provide a down payment on a house, or a child's education, or retirement income—is often accomplished by a plan of regular deposits to an account for that purpose. The savings plan is an annuity, so these relationships can be used to calculate how much would have to be saved each period to reach the goal (CF), or given how much can be saved each period, how long it will take

to reach the goal ( $t$ ), or how a better investment return ( $r$ ) would affect the periodic savings, or the time needed ( $t$ ), or the goal (FV).

For example, if you want to have \$1 million (FV) in the bank when you retire, and your bank pays 3 per cent ( $r$ ) interest per year, and you can save \$10,000 per year (CF) toward retirement, can you afford to retire at age sixty-five? You could if you start saving at age eighteen, because with that annual saving at that rate of return, it will take forty-seven years ( $t$ ) to have \$1,000,000 (FV). If you could save \$20,000 per year (CF), it would only take thirty-one years ( $t$ ) to save \$1,000,000 (FV). If you are already forty years old, you could do it if you save \$27,428 per year (CF) or if you can earn a return of at least 5.34% ( $r$ ). See Table 3.2.5.

**Table 3.2.5 Retirement Savings Calculations**

CF	Rate ( $r$ )	Time ( $t$ )	PV	FV
10,000	0.0300	47	250,000	1,000,000
20,000	0.0300	31	400,000	1,000,000
27,428	0.0300	25	477,606	1,000,000
20,000	0.0534	25	272,621	1,000,000

As you can see, the relationships between time, risk, opportunity cost, and value are some of the most important relationships you will ever encounter in life, and understanding them is critical to making sound financial decisions.

## FINANCIAL CALCULATIONS

Modern tools make it much easier to do the math. Widely available calculators, spreadsheets, and software have been developed to be very user friendly.

Financial calculators have the equations relating the present and future values, cash flows, the discount rate, and time embedded, for single amounts or for a series of cash flows, so that you can calculate any one of those variables if you know all the others.

Personal finance software packages usually come with a planning calculator that performs a similar function. These tools are usually presented as a “mortgage calculator” or a “loan calculator” or a “retirement planner,” and they are set up to answer common planning questions such as, “How much do I have to save every year for retirement?” or, “What will my monthly loan payment be?”

Spreadsheets also have the relevant equations built in, as functions or as macros. There are also stand-alone software applications that may be downloaded. They are useful in answering planning questions, but they lack the ability to store and track your situation in the way that a more complete software package can.

The calculations are discussed here not so that you can perform them—you have many tools to choose from that can do that more efficiently—but so that you can understand them, and most importantly, so that you can understand the relationships that they describe.

## Key Takeaways

1. The idea of the time value of money is fundamental to financial decisions.
2. The present value of a series of cash flows is equal to the sum of the present value of each cash flow.
3. A series of cash flows is an annuity when there are regular payments at regular intervals and each payment is the same amount.
4. To calculate the present value of an annuity, you need to know:
  - the amount of the identical cash flows (CF),
  - the frequency of the cash flows,
  - the number of cash flows ( $t$ ), and
  - the discount rate ( $r$ ) or the rate at which time affects value.
5. The calculation for the present value of an annuity yields valuable insights.
  - The more time ( $t$ ), the more periods and the more periodic payments—that is, the more cash flows, and so the more liquidity and the more value.
  - The greater the cash flows, the more liquidity and the more value.
  - The greater the rate at which time affects value ( $r$ ) or the greater the opportunity cost and risk or the greater the rate of discounting, the more time affects value.
6. The calculation for the future value of an annuity yields valuable insights.
  - The more time ( $t$ ), the more periods and the more periodic payments—that is, the more cash flows, and so the more liquidity and the more value.
  - The greater the cash flows, the more liquidity and the more value.
  - The greater the rate at which time affects value ( $r$ ) or the greater the rate of compounding, the more time affects value.
7. A perpetuity is an infinite annuity.

## Exercise

1. How can you determine if a lump-sum payment or an annuity will have greater value for you?

**Answer**



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=725#h5p-65>

## CHAPTER ATTRIBUTION

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## 3.3 Calculating the Relationship of Time and Value

### Learning Objectives

- Identify the factors you need to know to relate a present value to a future value.
- Write the algebraic expression for the relationship between present and future value.
- Discuss the use of the algebraic expression in evaluating the relationship between present and future values.
- Explain the importance of understanding the relationships among the factors that affect future value.

Financial calculation is not often a necessary skill since it is easier to use calculators, spreadsheets, and software. However, understanding the calculations is important in understanding the relationships between time, risk, opportunity cost, and value.

To do the math, you need to know:

1. what the future cash flows will be,
2. when the future cash flows will be, and
3. the rate at which time affects value (e.g., the costs per time period, or the magnitude [the size or amount] of the effect of time on value).

It is usually not difficult to forecast the timing and amounts of future cash flows. Although there may be some uncertainty about them, gauging the rate at which time affects money can require some judgment. That rate, commonly called the **discount rate** because time discounts value, is the opportunity cost of not having liquidity. Opportunity cost derives from forgone choices or sacrificed alternatives, and sometimes it is not clear what those might have been. It is an important judgment call to make, though, because the rate will directly affect the valuation process.

At times, the alternatives are clear: you could be putting the liquidity in an account earning 3 %, so that's your opportunity cost of not having it. Or you are paying 6.5% on a loan, which you wouldn't be paying if you had enough liquidity to avoid having to borrow; that's your opportunity cost. Sometimes, however, your opportunity cost is not so clear.

Say that today is your twentieth birthday. Your grandparents have promised to give you \$1,000 for your twenty-first birthday, one year from today. If you had the money today, what would it be worth? That is, how much would \$1,000 worth of liquidity one year from now be worth today?

That depends on the cost of its not being liquid today, or on the opportunity costs and risks created by not having liquidity today. If you had \$1,000 today, you could buy things and enjoy them, or you could deposit it in an interest-bearing account. So on your twenty-first birthday, you would have more than \$1,000—you would have the \$1,000 plus whatever interest it had earned. If your bank pays 4% per year (interest rates are always stated as annual rates) on your account, then you would earn \$40 of interest in the next year, or  $1,000 \times .04$ . So, on your twenty-first birthday you would have \$1,040 as expressed in the equation below:

$$\$1,000 + (1,000 \times 0.04) = \$1,000 \times (1 + 0.04) = \$1,040$$

Table 3.3.1 Interest-Bearing Account

Today	Interest Rate	Time (years)	One Year from Now
1,000	0.04	1	$1,040 = 1,000 \times (1+0.04)$

If you left that amount in the bank until your twenty-second birthday, you would have

$$1,040 + (1,040 \times 0.04) = 1,040 \times (1 + 0.04) =$$

$$[1,000 \times (1 + 0.04)] \times (1 + 0.04) =$$

$$1,000 \times (1 + 0.04)^2 = 1,081.60$$

To generalize the computation, if your **present value (PV)**, is your value today,  $r$  is the rate at which time affects value or discount rate (in this case, your interest rate), and if  $t$  is the number of time periods between you and your liquidity, then the **future value (FV)** of your wealth is expressed in the following equation:

$$\mathbf{FV = PV \times (1+r)^t}$$

In this case,  $1,000 \times (1.04)^1 = 1,040$  and  $1,000 \times (1+r)^2 = 1,081.60$ . Table 3.3.2 illustrates how present value, the interest rate, and time equate to future value.

Table 3.3.2 Future Value

Today	Interest Rate	Time (years)	Equation
1,000	0.04	1	$1,040 = 1,000 \times (1+0.04)^1$
1,000	0.04	2	$1,081.60 = 1,000 \times (1+0.04)^2$
PV	$r$	$t$	$FV = PV \times (1+r)^t$

Assuming there is little chance that your grandparents will not be able to give this gift, there is negligible risk. Your only cost of not having liquidity now is the opportunity cost of having to delay consumption or not earning the interest you could have earned.

The cost of delayed consumption is largely derived from a subjective valuation of whatever is consumed, or its **utility** or satisfaction. The more value you place on having something, the more it “costs” you not to have it, and the more time that you are without it affects its value.

Assuming that if you had the money today you would save it, by having to wait until your twenty-first birthday to receive it—and *not* having it today—you miss out on the \$40 it could have earned.

So, what would that nominal \$1,000 (that future value that you get one year from now) actually be worth today? The rate at which time affects your value is 4% because that’s what having a choice (to spend or invest it) could earn for you if only you had received the \$1,000. That’s your opportunity cost; that’s what it costs you to not have liquidity. Since  $PV \times (1+r)^t = FV$ , then  $PV = FV / [(1+r)^t]$ , so  $PV = 1,000 / [(1.04)^1] = 961.5385$ .

Your gift is worth \$961.5385 today (its present value). If your grandparents offered to give you your twenty-first birthday gift on your twentieth birthday, they could give you \$961.5385 today, which would be the equivalent value to you of getting \$1,000 one year from now.

It is important to understand the relationships between time, risk, opportunity cost, and value. This equation describes that relationship:

$$PV \times (1+r)^t = FV$$

The  $r$  is more formally called the “discount rate” because it is the rate at which your liquidity is discounted by time, and it includes not only opportunity costs, but also risk. (On some financial calculators,  $r$  is displayed as  $I$  or  $i$ .)

The  $t$  is how far away you are from your liquidity over time.

Studying this equation yields valuable insights into the relationship it describes. Looking at the equation, you can observe the following relationships:

The more time ( $t$ ) separating you from your liquidity, the more time affects value. The less time separating you from your liquidity, the less time affects value (as  $t$  decreases,  $PV$  increases).

As  $t$  increases, the  $PV$  of your  $FV$  liquidity decreases.

As  $t$  decreases, the  $PV$  of your  $FV$  liquidity increases.

The greater the rate at which time affects value ( $r$ ), or the greater the opportunity cost and risk, the more time affects value. The less your opportunity cost or risk, the less your value is affected.

As  $r$  increases, the  $PV$  of your  $FV$  liquidity decreases.

As  $r$  decreases, the  $PV$  of your  $FV$  liquidity increases.

Table 3.3.3 presents examples of these relationships.

Table 3.3.3 Present Values, Interest Rates, Time, and Future Values				
	Today	Interest Rate	Time (years)	Future Value
<b>Example A</b>	1,000	0.04	1	$1,040 = PV \times 1.04\%$
Greater effect $r$ increases	1,000	0.10	1	$1,100 = PV \times 1.10\%$
More time $t$ increases	1,000	0.04	3	$1,124.86 = PV \times 1.04^3\%$
Less effect $r$ decreases	1,000	0.01	3	$1,010 = PV \times 1.01\%$
Less time $t$ decreases	1,000	0.04	0.5	$1,019.80 = PV \times 1.04^{0.5}\%$
<b>Example B</b>	$961.54 = FV/1.04\%$	0.04	1	1,000
Greater effect $r$ increases	$909.09 = FV/1.10\%$	0.10	1	1,000
More time $t$ increases	$889.00 = FV/1.04^3\%$	0.04	3	1,000
Less effect $r$ decreases	$990.10 = FV/1.01\%$	0.01	1	1,000
Less time $t$ decreases	$980.58 = FV/1.04^{0.5}\%$	0.04	0.5	1,000

The strategy implications of this understanding are simple yet critical. All things being equal, it is more valuable to have liquidity (get paid, or have positive cash flow) *sooner* rather than later and give up liquidity (pay out, or have negative cash flow) *later* rather than sooner.

If possible, accelerate incoming cash flows and decelerate outgoing cash flows: get paid sooner, but pay out later. Or, as Popeye's pal Wimpy used to say, "I'll give you 50 cents tomorrow for a hamburger today."

Please see the following TVM tables developed by Jodi Letkiewicz (York University), in order to practice calculating future value and present value. A sample of the first TVM table—Future Value of a Lump Sum (\$1 at the end of  $t$  periods)—is illustrated below. Remember that  $FV = PV \times (1+r)^t$ .

**TVM Table 1: Future Value of a Lump Sum (\$1 at the end of  $t$  periods)**

$t$	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
1	1.0100	1.0200	1.0300	1.0400	1.0500	1.0600	1.0700	1.0800	1.0900	1.1000
2	1.0201	1.0404	1.0609	1.0816	1.1025	1.1236	1.1449	1.1664	1.1881	1.2100
3	1.0303	1.0612	1.0927	1.1249	1.1576	1.1910	1.2250	1.2597	1.2950	1.3310
4	1.0406	1.0824	1.1255	1.1699	1.2155	1.2625	1.3108	1.3605	1.4116	1.4641
5	1.0510	1.1041	1.1593	1.2167	1.2763	1.3382	1.4026	1.4693	1.5386	1.6105
6	1.0615	1.1262	1.1941	1.2653	1.3401	1.4185	1.5007	1.5869	1.6771	1.7716
7	1.0721	1.1487	1.2299	1.3159	1.4071	1.5036	1.6058	1.7138	1.8280	1.9487
8	1.0829	1.1717	1.2668	1.3686	1.4775	1.5938	1.7182	1.8509	1.9926	2.1436
9	1.0937	1.1951	1.3048	1.4233	1.5513	1.6895	1.8385	1.9990	2.1719	2.3579
10	1.1046	1.2190	1.3439	1.4802	1.6289	1.7908	1.9672	2.1589	2.3674	2.5937
11	1.1157	1.2434	1.3842	1.5395	1.7103	1.8983	2.1049	2.3316	2.5804	2.8531
12	1.1268	1.2682	1.4258	1.6010	1.7959	2.0122	2.2522	2.5182	2.8127	3.1384
13	1.1381	1.2936	1.4685	1.6651	1.8856	2.1329	2.4098	2.7196	3.0658	3.4523
14	1.1495	1.3195	1.5126	1.7317	1.9799	2.2609	2.5785	2.9372	3.3417	3.7975
15	1.1610	1.3459	1.5580	1.8009	2.0789	2.3966	2.7590	3.1722	3.6425	4.1772
16	1.1726	1.3728	1.6047	1.8730	2.1829	2.5404	2.9522	3.4259	3.9703	4.5950
17	1.1843	1.4002	1.6528	1.9479	2.2920	2.6928	3.1588	3.7000	4.3276	5.0545
18	1.1961	1.4282	1.7024	2.0258	2.4066	2.8543	3.3799	3.9960	4.7171	5.5599
19	1.2081	1.4568	1.7535	2.1068	2.5270	3.0256	3.6165	4.3157	5.1417	6.1159
20	1.2202	1.4859	1.8061	2.1911	2.6533	3.2071	3.8697	4.6610	5.6044	6.7275

View the complete TVM tables in PDF format at the following four links:

- TVM Table 1: Future Value of a Lump Sum (\$1 at the end of  $t$  periods)
- TVM Table 2: Present Value of a Lump Sum (\$1)
- TVM Table 3: Present Value of a Series of Payments (Annuity) (\$1)
- TVM Table 4: Future Value of a Series of Payments (Annuity) (\$1)

#### Key Takeaways

1. To relate a present (liquid) value to a future value, you need to know:

- what the present value is or the future value will be,
- when the future value will be, and
- the rate at which time affects value: the costs per time period, or the magnitude of the effect of time on value.

2. The relationship of present value (PV), future value (FV), risk and opportunity cost (the discount rate,  $r$ ), and time ( $t$ ), may be expressed as:  $PV \times (1 + r)^t = FV$ .
3. The above equation yields valuable insights into these relationships:
  - The more time ( $t$ ) creates distance from liquidity, the more time affects value.
  - The greater the rate at which time affects value ( $r$ ), or the greater the opportunity cost and risk, the more time affects value.
  - The closer the liquidity, the less time affects value.
  - The less the opportunity cost or risk, the less value is affected.
4. To maximize value, get paid sooner and pay later.

### Exercises

1. View the TeachMeFinance.com animated audio slide show on “The Time Value of Money” (3:00) by Mark McCracken. This slide show will walk you through an example of how to calculate the present and future values of money. How is each part of the formula used in that lesson equivalent to the formula presented in this text?
2. To have liquidity, when should you increase positive cash flows and decrease negative cash flows, and why?

### Answers



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://pressbooks.nsc.ca/businessmath2500/?p=723#h5p-66>

## CHAPTER CREDIT

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## 4 SIMPLE INTEREST

### Learning Objectives

- Demonstrate the concept of simple interest.
- Determine the number of days between two calendar days using the pre-programmed financial calculator method.
- Calculate the amount of interest, principal, time, interest rate, and maturity value of investments and loans.
- Calculate equivalent payments that replace another payment or a series of payments.
- Use simple interest in solving problems involving business applications such as savings accounts, short term guaranteed investment certificates (GICs), treasury bills, and commercial paper.

## 4.1 Principal, Rate, Time

### SIMPLE INTEREST

---

In a simple interest environment, you calculate interest solely on the amount of money at the beginning of the transaction (amount borrowed or lent).

Assume \$1,000 is placed into an account with 12% simple interest for a period of 12 months. For the entire term of this transaction, the amount of money in the account always equals \$1,000. During this period, interest accrues at a rate of 12%, but the interest is never placed into the account. When the transaction ends after 12 months, the \$120 (12% of \$1,000) of interest and the initial \$1,000 are then combined to total \$1,120.

A loan or investment always involves two parties—one giving and one receiving. No matter which party you are in the transaction, the amount of interest remains unchanged. The only difference lies in whether you are earning or paying the interest.

### The Formula

---

Formula 4.1 Simple Interest:  $I = Prt$  where,

**I is Interest Amount.** The interest amount is the dollar amount of interest that is paid or received.

**P is Present Value or Principal.** The present value is the amount borrowed or invested at the beginning of a period.

**r is Simple Interest Rate.** The interest rate is the rate of interest that is charged or earned during a specified time period. It is expressed as a percent.

**t is Time Period.** The time period or term is the length of the financial transaction for which interest is charged or earned.

### IMPORTANT NOTES

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Recall that algebraic equations require all terms to be expressed with a common unit. This principle remains true for Formula 4.1, particularly with regard to the interest rate and the time period. For example, if you have a 3% annual interest rate for nine months, then either

- The time needs to be expressed annually as  $\frac{9}{12}$  of a year to match the yearly interest rate, or
- The interest rate needs to be expressed monthly as  $\frac{3\%}{12} = 0.25\%$  per month to match the number of months.

It does not matter which you do so long as you express both interest rate and time in the same unit. If one of these two variables is your algebraic unknown, the unit of the known variable determines the unit of the unknown variable. For example, assume that you are solving Formula 4.1 for the time period. If the interest rate used in the formula is annual, then the time period is expressed in number of years.

#### Example 4.1.1: How Much Interest is Owed?

Julio borrowed \$1,100 from Maria five months ago. When he first borrowed the money, they agreed that he would pay Maria 5% simple interest. If Julio pays her back today, how much interest does he owe her?

#### Solution

##### Step 1: Given information:

$P = \$1,100$ ;  $r = 5\%$ ; per year;  $t = 5$  months

**Step 2:** The rate is annual, and the time is in months. Convert the time period from months to years;

$$t = \frac{5}{12}$$

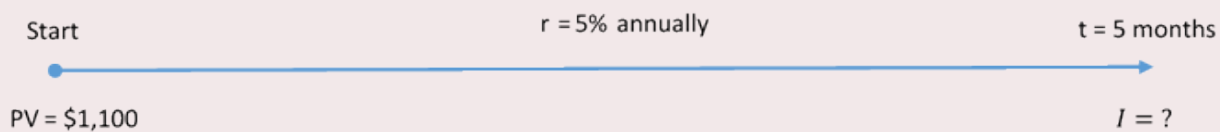


Figure 4.1.1: Timeline [Image Description]

**Step 3:** Solve for the amount of interest,  $I$ .

$$\begin{aligned}
 I &= Prt \\
 &= \$1,100 \times 5\% \times \frac{5}{12} \\
 &= \$1,100 \times 0.05 \times 0.41\bar{6} \\
 &= \$22.92
 \end{aligned}$$

For Julio to pay back Maria, he must reimburse her for the \$1,100 principal borrowed plus an additional \$22.92 of simple interest as per their agreement.

### SOLVING FOR P, R OR T

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Four variables are involved in the simple interest formula, which means that any three can be known, requiring you to solve for the fourth missing variable. To reduce formula clutter, the triangle technique illustrated in the video below will help you remember how to rearrange the simple interest formula as needed.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=26#h5p-1>

### CONCEPT CHECK

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<https://pressbooks.nsc.ca/businessmath2500/?p=26#h5p-2>

#### Example 4.1.2: What did You Start With?

What amount of money invested at 6% annual simple interest for 11 months earns \$2,035 of interest?

#### **Solution**

**Step 1:** Given information:

$r = 6\%$ ;  $t = 11$  months;  $I = \$2,035$

**Step 2:** Convert the time from months to an annual basis to match the interest rate;  $t = \frac{11}{12}$

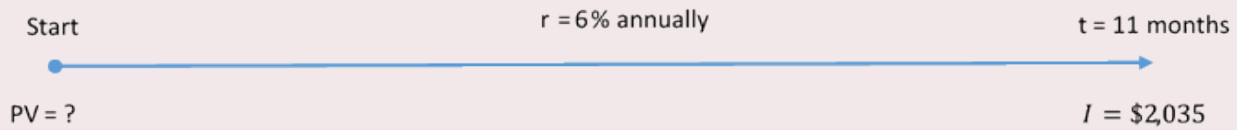


Figure 4.1.2: Timeline [Image Description]

**Step 3:** Solve for the present value, P.

$$\begin{aligned}
 P &= \frac{I}{rt} \\
 &= \frac{\$2,035}{6\% \times \frac{11}{12}} \\
 &= \frac{\$2,035}{0.06 \times 0.91\bar{6}} \\
 &= \$37,000
 \end{aligned}$$

To generate \$2,035 of simple interest at 6% over a time frame of 11 months, \$37,000 must be invested.

#### Example 4.1.3: How long?

For how many months must \$95,000 be invested to earn \$1,187.50 of simple interest at an interest rate of 5%?

#### Solution

**Step 1:** Given information:

$P = \$95,000$ ;  $I = \$1,187.50$ ;  $r = 5\%$ ; per year

**Step 2:** Convert the interest rate to a “per month” format;  $r = \frac{0.05}{12}$

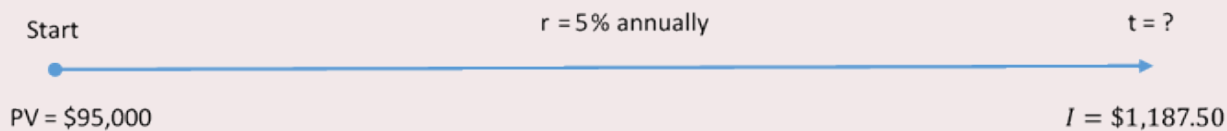


Figure 4.1.3: Timeline [Image Description]

**Step 3:** Solve for the time period,  $t$ .

$$\begin{aligned}
 t &= \frac{I}{Pr} \\
 &= \frac{\$1,187.50}{\$95,000 \times \frac{0.05}{12}} \\
 &= \frac{\$1,187.50}{\$95,000 \times 0.0041\bar{6}} \\
 &= 3 \text{ months}
 \end{aligned}$$


For \$95,000 to earn \$1,187.50 at 5% simple interest, it must be invested for a three-month period.

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. If you want to earn \$1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?
2. If you placed \$2,000 into an investment account earning 3% simple interest, how many months does it take for you to have \$2,025 in your account?
3. A \$3,500 investment earned \$70 of interest over the course of six months. What annual rate of simple interest did the investment earn?



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<https://pressbooks.nsc.ca/businessmath2500/?p=26#h5p-3>

## TIME AND DATES

In the examples of simple interest so far, the time period was given in months. While this is convenient in many situations, financial institutions and organizations calculate interest based on the exact number of days in the transaction, which changes the interest amount.

To illustrate this, assume you had money saved for the entire months of July and August, where

$t = \frac{2}{12}$  or  $t = 0.16666... = 0.1\overline{6}$  of a year. However, if you use the exact number of days, the 31

days in July and 31 days in August total 62 days. In a 365-day year that is  $t = \frac{62}{365}$  or  $t = 0.169863$  of a

year. Notice a difference of 0.003196 ( $0.169863 - 0.16$ ) occurs. Therefore, to be precise in performing simple interest calculations, you must calculate the **exact number of days** involved in the transaction.

## IMPORTANT NOTES

When solving for  $t$ , decimals may appear in your solution. For example, if calculating  $t$  in days, the answer may show up as 45.9978 or 46.0023 days; however, interest is calculated only on complete days. This occurs because the interest amount ( $I$ ) used in the calculation has been rounded off to two decimals. Since the interest amount is imprecise, the calculation of  $t$  is imprecise. When this occurs, round  $t$  off to the nearest integer.

### Example 4.1.4: Time Using Dates

On September 13, 2011, Aladdin decided to pay back the Genie on his loan of \$15,000 at 9% simple interest. If he paid the Genie the principal plus \$1,283.42 of interest, on what day did he borrow the money from the Genie?

#### Solution

**Step 1:** Given variables:

$P = \$15,000$ ;  $I = \$1,283.42$ ;  $r = 9\%$  per year; End Date = September 13, 2011

**Step 2:** The time is in days, but the rate is annual. Convert the rate to a daily rate;  $r = \frac{9\%}{365}$

**Step 3:** Solve for the time,  $t$

$$\begin{aligned} t &= \frac{I}{Pr} \\ &= \frac{\$1,283.42}{\$15,000 \times \frac{0.09}{365}} \\ &= 346.998741 = 347 \text{ days} \end{aligned}$$

**Step 4:** Use the DATE function to calculate the start date (DT1). Use the time in days.

If Aladdin owed the Genie \$1,283.42 of simple interest at 9% on a principal of \$15,000, he must have borrowed the money 347 days earlier, which is October 1, 2010.

## Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

2. Brynn borrowed \$25,000 at 1% per month from a family friend to start her entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?
3. If \$6,000 principal plus \$132.90 of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://pressbooks.nsc.ca/businessmath2500/?p=26#h5p-5>

## CHAPTER ATTRIBUTION

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## 4.2 Maturity Value (or Future Value) Moving Money Involving Simple Interest

### MOVING MONEY INVOLVING SIMPLE INTEREST

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#### Maturity Value (or Future Value)

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The maturity value of a transaction is the amount of money resulting at the end of a transaction, an amount that includes both the interest and the principal together. It is called a maturity value because in the financial world the termination of a financial transaction is known as the “maturing” of the transaction. The amount of principal with interest at some point in the future, but not necessarily the end of the transaction, is known as the future value.

For any financial transaction involving simple interest, the following is true:

$$\text{Amount of money at the end} = \text{Amount of money at the beginning} + \text{Interest}$$

Applying algebra, you can summarize this expression by the following equation, where the future value or maturity value is commonly denoted by the symbol  $S$ .

$$S = P + I$$

Substituting in  $I = Prt$ , yields the equation

$$S = P + Prt \text{ or } S = P(1 + rt)$$

### THE FORMULA

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**Formula 4.2A:** Simple Interest Future Value:  $S = P(1 + rt)$  where,

**I is Interest Amount.** The interest amount is the dollar amount of **P is Present Value or Principal.** The present value is the amount borrowed or invested at the beginning of a period.

**r is Simple Interest Rate.** The interest rate is the rate of interest that is charged or earned during a specified time period. It is expressed as a percent.

**t is Time Period.** The time period or term is the length of the financial transaction for which interest is charged or earned.

From the future value formula  $S=P(1+rt)$  you can derive the present value formula (P):

$$P = \frac{S}{1 + rt}$$

Sometimes you will be required to calculate the simple interest dollar amount (I). The formula is given below.

**Formula 4.2B:** Simple Interest Amount  $I=S-P$

#### Example 4.2.1: Calculating Maturity Value and Interest Amount

Assume that today you have \$10,000 that you are going to invest at 7% simple interest for 11 months. How much money will you have in total at the end of the 11 months? How much interest do you earn?

#### Solution

**Step 1:** Given variables:

$P = \$10,000$ ;  $r = 7\%$ ;  $t = 11$  months

**Step 2:** Express the time in years to match the annual rate;  $t = \frac{11}{12}$

**Step 3:** Solve for the future value, S.

$$\begin{aligned} S &= P \times (1 + rt) \\ &= \$10,000 \times \left( 0.07 \times \frac{11}{12} \right) \\ &= \$10,641.67 \end{aligned}$$

This is the total amount after 11 months.

**Step 4:** Solve for the interest amount, I.

$$I = \$10,641.67 - \$10,000.00 = \$641.67$$

The \$10,000 earns \$641.67 in simple interest over the next 11 months, resulting in \$10,641.67 altogether.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=33#h5p-6>

## Example 4.2.2: Saving for a Down Payment on a Home

You just inherited \$35,000 from your uncle's estate and plan to purchase a house four months from today. If you use your inheritance as your down payment on the house, how much will you be able to put down if your money earns  $4\frac{1}{4}\%$  simple interest? How much interest will you have earned?

**Solution**

Calculate the amount of money four months from now including both the principal and interest earned. This is the maturity value ( $S$ ). Also calculate the interest earned ( $I$ ).

**Step 1:** Given variables:

$$P = \$35,000; t = 4\text{months}; r = 4\frac{1}{4}\% \text{ per year}$$

**Step 2:** Express the time in years to match the annual rate;  $t = \frac{4}{12}$

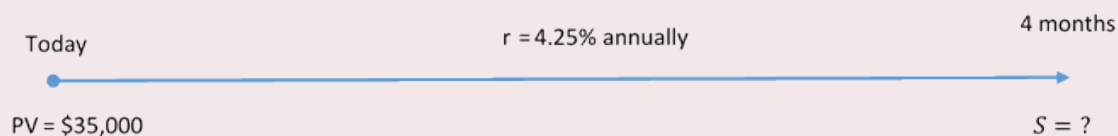


Figure 4.2.2: Timeline [Image Description]

$$\begin{aligned}
 S &= P \times (1 + rt) \\
 &= \$35,000 \times \left(1 + 4\frac{1}{4}\% \times \frac{4}{12}\right) \\
 &= \$35,000 \times (1 + 0.0425 \times 0.\bar{3}) \\
 &= \$35,495.83
 \end{aligned}$$

**Step 3:** Solve for the future value,  $S$ .

**Step 4:** Solve for the amount of interest,  $I$ .

$$I = \$35,495.83 - \$35,000.00 = \$495.83$$

Four months from now you will have \$35,495.83 as a down payment toward your house, which includes \$35,000 in principal and \$495.83 of interest.

#### Example 4.2.3: Saving for Tuition

Recall the section opener, where you needed \$8,000 for tuition in the fall and the best simple interest rate you could find was 4.5%. Assume you have eight months before you need to pay your tuition. How much money do you need to invest today?

#### Solution

Calculate the principal amount of money today  $P$  that you must invest such that it will earn interest and end up at the \$8,000 required for the tuition.

**Step 1:** Given variables:

$$S = \$8,000; r = 4.5\% \text{ per year}; t = 8 \text{ months}$$

**Step 2:** Express the time in years to match the annual rate;  $t = \frac{8}{12}$

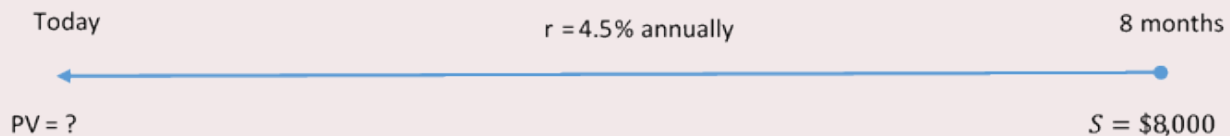


Figure 4.2.3: Timeline [Image Description]

$$\begin{aligned}
 P &= \frac{S}{1 + rt} \\
 &= \frac{\$8,000}{(1 + 4.5\% \times \frac{8}{12})} \\
 &= \frac{\$8,000}{(1 + 0.045 \times 0.\bar{6})} \\
 &= \$7,766.99
 \end{aligned}$$

**Step 3:** Solve for the present value, P.

If you place \$7,766.99 into the investment,

it will grow to \$8,000 in the eight months.

#### Example 4.2.4: What Exactly Are You being Offered?

You are sitting in an office at your local financial institution on August 4. The bank officer says to you, “We will make you a great deal. If we advance that line of credit and you borrow \$20,000 today, when you want to repay that balance on September 1 you will only have to pay us \$20,168.77, which is not much more!” Before answering, you decide to evaluate the statement. Calculate the simple interest rate that the bank officer used in her calculations.

##### Solution

Determine the rate of interest that you would be charged on your line of credit.

**Step 1:** Given variables:

$P = \$20,000$ ;  $S = \$20,168.77$ ;  $t = \text{August 4 to September 1}$

**Step 2:** Calculate the number of days in the transaction.

**Step 3:** Since interest rates are usually expressed annually, convert the time from days to an annual number;  $t = \frac{28}{365}$

**Step 4:** Calculate the amount of interest, I.

$$I = \$20,168.77 - \$20,000 = \$168.77$$

**Step 5:** Solve for r.

$$\begin{aligned} r &= \frac{I}{Pt} \\ &= \frac{\$168.77}{\$20,000.00 \times \frac{28}{365}} \\ &= 0.110002 \text{ or } 11.0002\% \end{aligned}$$

The interest rate on the offered line of credit is 11.0002% (note that it is probably exactly 11%; the extra 0.0002% is most likely due to the rounded amount of interest used in the calculation).

## EQUIVALENT PAYMENTS

- **Late Payments.** If a debt is paid late, then a financial penalty that is fair to both parties involved should be imposed. That penalty should reflect a current rate of interest and be added to the original payment. Assume you owe \$100 to your friend and that a fair current rate of simple interest is 10%. If you pay this debt one year late, then a 10% late interest penalty of \$10 should be added, making your debt payment \$110. This is no different from your friend receiving the \$100 today and investing it himself at 10% interest so that it accumulates to \$110 in one year.
- **Early Payments.** If a debt is paid early, there should be some financial incentive (otherwise, why bother?). Therefore, an interest benefit, one reflecting a current rate of interest on the early payment, should be deducted from the original payment. Assume you owe your friend \$110 one year from now and that a fair current rate of simple interest is 10%. If you pay this debt today, then a 10% early interest benefit of \$10 should be deducted, making your debt payment today \$100. If your friend then invests this money at 10% simple interest, one year from now he will have the \$110, which is what you were supposed to pay.

Notice in these examples that a simple interest rate of 10% means \$100 today is the same thing as having \$110 one year from now. This illustrates the concept that two payments are equivalent payments if, once a fair rate of interest is factored in, they have the same value on the same day. Thus, in general you are finding two amounts at different points in time that have the same value, as illustrated in the figure below.

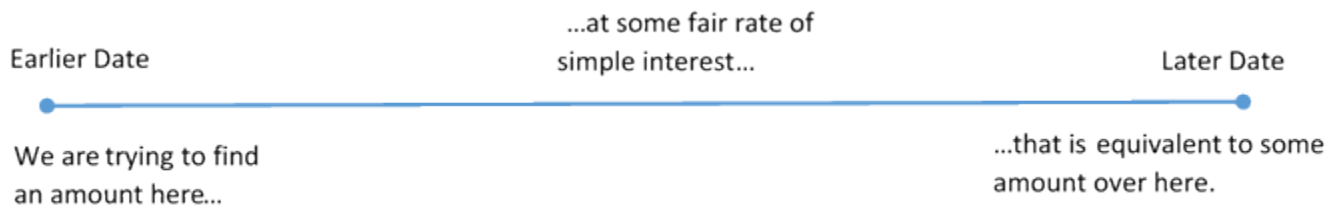


Figure 4.2.E: Timeline for Equivalent Payments [Image Description]

## HOW IT WORKS

The steps required to calculate an equivalent payment are no different from those for single payments. If an early payment is being made, then you know the future value, so you solve for the present value (which removes the interest). If a late payment is being made, then you know the present value, so you solve for the future value (which adds the interest penalty).

### Example 4.2.5: Making a Late Payment

Erin owes Charlotte \$1,500 today. Unfortunately, Erin had some unexpected expenses and is unable to make her debt payment. After discussing the issue, they agree that Erin can make the payment nine months late and that a fair simple interest rate on the late payment is 5%. Use 9 months from now as your focal date and calculate how much Erin needs to pay. What is the amount of her late penalty?

#### Solution

A late payment is a future value amount ( $S$ ). The late penalty is equal to the interest ( $I$ ).

**Step 1:** Given variables:

$P = \$1,500$ ;  $r = 5\%$  annually;  $t = 9$  months

**Step 2:** Express the time in years to match the annual rate;  $t = \frac{9}{12}$

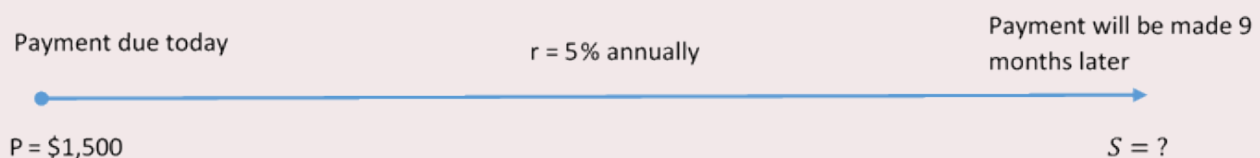


Figure 4.2.5: Timeline [Image Description]

**Step 3:** Calculate the future value,  $S$ .

$$S = P \times (1 + rt)$$

$$\begin{aligned} S &= \$1,500 \times \left(1 + 5\% \times \frac{9}{12}\right) \\ &= \$1,500 \times (1 + 0.05 \times 0.75) \\ &= \$1,556.25 \end{aligned}$$

**Step 4:** Solve for the amount of interest,  $I$ .

$$I = \$1,556.25 - \$1,500.00 = \$56.25$$

Erin's late payment is for \$1,556.25, which includes a \$56.25 interest penalty for making the payment nine months late.

#### Example 4.2.6: Making an Early Payment

Rupert owes Aminata two debt payments: \$600 four months from now and \$475 eleven months from now. Rupert came into some money today and would like to pay off both of the debts immediately. Aminata has agreed that a fair interest rate is 7%. Using today as a focal date, what amount should Rupert pay? What is the total amount of his early payment benefit?

#### Solution

An early payment is a present value amount ( $P$ ). Both payments will be moved to today and summed. The early payment benefit will be the total amount of interest removed ( $I$ ).

**Step 1:** Given variables:

$$r = 7$$

The two payments and payment due dates are known.

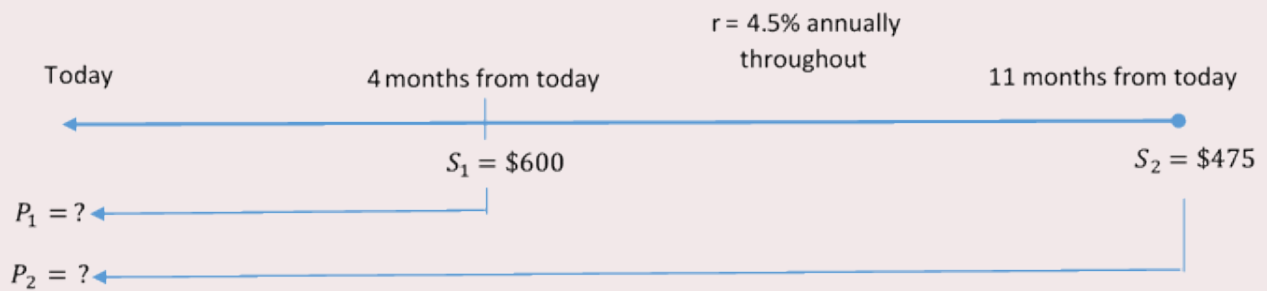
**Payment #1:**  $S_1 = \$600$ ;  $t = 4$  months from now

**Payment #2:**  $S_2 = \$475$ ;  $t = 11$  months from now

Replacement payment is being made today (the focal date).

#### Payment #1:

**Step 2:** Express the time in years to match the annual rate;  $t = \frac{4}{12}$



$$\text{Total Amount Paid } (P) = P_1 + P_2$$

Figure 4.2.6: Timeline [Image Description]

$$P_1 = \frac{S_1}{1 + rt}$$

$$\text{Step 3: Solve for } P_1. \quad = \frac{\$600}{(1 + 0.07 \times 0.\overline{3})}$$

$$= \$586.32$$

#### Payment #2:

**Step 2:** While the rate is annual, the time is in months. Convert the time to an annual number;  $t = \frac{11}{12}$

**Step 3:** Solve for  $P_2$ .

$$P_2 = \frac{S_2}{1 + rt}$$

$$= \frac{\$475}{(1 + 0.07 \times 0.91\overline{6})}$$

$$= \$446.36$$

$$\text{Total Amount Paid Today, } P = P_1 + P_2$$

$$= \$586.32 + \$446.36$$

$$= \$1,032.68$$

**Step 4:** Calculate the total amount of interest,  $I$ .

**Payment #1:**  $I_1 = S_1 - P_1 = \$600 - \$586.32 = \$13.68$

**Payment #2:**  $I_2 = S_2 - P_2 = \$475 - \$446.36 = \$28.64$

$$I = \$13.68 + \$28.64 = \$42.32$$

To clear both debts today, Rupert pays \$1,032.68, which reflects a \$42.32 interest benefit reduction for the early payment.

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was \$10,267.21 for a loan that spanned April 14 to July 31 at 9.1%, how much was the principal and how much was the interest?
2. Suppose Robin borrowed \$3,600 on October 21 and repaid the loan on February 21 of the following year. What simple interest rate was charged if Robin repaid \$3,694.63?
3. Jayne needs to make three payments to Jade requiring \$2,000 each 5 months, 10 months, and 15 months from today. She proposes instead making a single payment eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?
4. Merina is scheduled to make two loan payments to Bradford in the amount of \$1,000 each, two months and nine months from now. Merina doesn't think she can make those payments and offers Bradford an alternative plan where she will pay \$775 seven months from now and another payment seven months later. Bradford determines that 8.5% is a fair interest rate. What is the amount of the second payment?



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://pressbooks.nsc.ca/businessmath2500/?p=33#h5p-7>

Timelines for questions 3 and 4 are included in Solutions to Exercises.

## CHAPTER ATTRIBUTION

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## 4.3 Application of Future Value: Savings Accounts And Short-Term GICs

### SAVINGS ACCOUNTS AND SHORT TERM GICS

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#### Savings Accounts

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A savings account is a deposit account that bears interest and has no stated maturity date. These accounts are found at most financial institutions, such as commercial banks (Royal Bank of Canada, TD Canada Trust, etc.), trusts (Royal Trust, Laurentian Trust, etc.), and credit unions (Credit Union Atlantic, FirstOntario, Steinbach, Assiniboine, Servus, etc.). Owners of such accounts make deposits to and withdrawals from these accounts at any time, usually accessing the account at an automatic teller machine (ATM), at a bank teller, or through online banking.

A wide variety of types of savings accounts are available. This textbook focuses on the most common features of most savings accounts, including how interest is calculated, when interest is deposited, insurance against loss, and the interest rate amounts available.

1. **How Interest Is Calculated.** There are two common methods for calculating simple interest:
    - Accounts earn simple interest that is calculated based on the daily closing balance of the account. The closing balance is the amount of money in the account at the end of the day. Therefore, any balances in the account throughout a single day do not matter. For example, if you start the day with \$500 in the account and deposit \$3,000 at 9:00 a.m., then withdraw the \$3,000 at 4:00 p.m., your closing balance is \$500. That is the principal on which interest is calculated, not the \$3,500 in the account throughout the day.
    - Accounts earn simple interest based on a minimum monthly balance in the account. For example, if in a single month you had a balance in the account of \$900 except for one day, when the balance was \$500, then only the \$500 is used in calculating the entire month's worth of interest.
- When Interest Is Deposited. Interest is accumulated and deposited (paid) to the account once monthly, usually on the first day of the month. Thus, the interest earned on your account for the month of January appears as a deposit on February 1.

2. **Insurance against Loss.** Canadian savings accounts at commercial banks are insured by the national Canada Deposit Insurance Corporation (CDIC), which guarantees up to \$100,000 in savings. At credit unions, this insurance is usually provided provincially by institutions such as the Deposit Insurance Corporation of Ontario (DICO) or the Nova Scotia Credit Union Deposit Insurance Corporation (NSCUDIC), which also guarantees up to \$100,000. This means that if your bank were to fold, you could not lose your money (so long as your deposit was within the maximum limit). Therefore, savings accounts carry almost no risk.
3. **Interest Rate Amounts.** Interest rates are higher for investments that are riskier. Savings accounts carry virtually no risk, which means the interest rates on savings accounts tend to be among the lowest you can earn. At the time of writing, interest rates on savings accounts ranged from a low of 0.05% to a high of 1.95%. Though this is not much, it is better than nothing and certainly better than losing money!

While a wide range of savings accounts are available, these accounts generally follow one of two common structures when it comes to calculating interest. These structures are flat rate savings accounts and tiered savings accounts. Each of these is discussed separately.

## HOW IT WORKS

### Flat-Rate Savings Accounts.

A flat-rate savings account has a single interest rate that applies to the entire balance. The interest rate may fluctuate in sync with short-term interest rates in the financial markets.

Follow these steps to calculate the monthly interest for a flat-rate savings account:

**Step 1:** Identify the interest rate, opening balance, and the monthly transactions in the savings account.

**Step 2:** Set up a flat-rate table as illustrated here. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one.

Table 4.3.1. Example of a Flat-Rate Table for Step 2

Date	Closing Balance in Account	# of Days	Simple Interest Earned
			$I = Prt$
Total Interest earned			

**Step 3:** For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

**Step 4:** Calculate the number of days that the closing balance is maintained for each row.

**Step 5:** Apply simple interest formula  $I = Prt$ , to each row in the table. Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts ( $I$ ).

**Step 6:** Sum the Simple Interest Earned column and round off to two decimals.

When you are calculating interest on any type of savings account, pay careful attention to the details on how interest is calculated and any restrictions or conditions on the balance that is eligible to earn the interest.

#### Example 4.3.1: Savings at the Royal Bank

The RBC High Interest Savings Account pays 0.75% simple interest on the daily closing balance in the account and the interest is paid on the first day of the following month. On March 1, the opening balance in the account was \$2,400. On March 12, a deposit of \$1,600 was made. On March 21, a withdrawal of \$2,000 was made. Calculate the total simple interest earned for the month of March.

#### Solution

Calculate the total interest amount ( $I$ ) for the month.

**Step 1:** Given variables:

$r = 0.75\%$  per year

The following transactions dates are known.

March 1 opening balance = \$2,400

March 12 deposit = \$1,600

March 21 withdrawal = \$2,000

**Step 2:** Set up a flat-rate table.

**Step 3:** Determine the date ranges for each balance throughout the month and calculate the closing balances.

**Step 4:** For each row of the table, calculate the number of days involved.

**Step 5:** Apply simple interest formula  $I = Prt$  to calculate simple interest on each row.

**Step 6:** Sum the Simple Interest Earned.

Table 4.3.2. Flat-Rate Table for Savings at the Royal Bank Example 4.3.1

Dates (Step 2)	Closing Balance in Account (Step 3)	# of Days (Step 4)	Simple Interest Earned ( $I = Prt$ ) (Step 5)
March 1 to March 12	\$2,400	$12 - 1 = 11$	$\begin{aligned} I &= \$2,400(0.0075)\left(\frac{11}{365}\right) \\ &= \$0.542465 \end{aligned}$
March 12 to March 21	$\$2,400 + \$1,600$ $= \$4,000$	$21 - 12 = 9$	$\begin{aligned} I &= \$4,000(0.0075)\left(\frac{9}{365}\right) \\ &= \$0.739726 \end{aligned}$
March 21 to April 1	$\$4,000 - \$2,000$ $= \$2,000$	$31 + 1 - 21$ $= 11$	$\begin{aligned} I &= \$2,000(0.0075)\left(\frac{11}{365}\right) \\ &= \$0.452054 \end{aligned}$
<b>Step 6:</b> Total Monthly Interest Earned			$\begin{aligned} I &= \$0.542465 + \$0.739726 + \$0.452054 \\ &= \$1.73 \end{aligned}$

For the month of March, the savings account earned a total simple interest of \$1.73, which was deposited to the account on April 1.

### Exercise: Savings Accounts

In the exercise that follow, try it on your own. Full solution is available should you get stuck.

1. Canadian Western Bank offers a Summit Savings Account with posted interest rates as indicated in the table below. Only each tier is subject to the posted rate, and interest is calculated daily based on the closing balance.

Table 4.3.3. Interest Rates for Summit Savings Account

Balance	Interest Rate
\$0 – \$5,000.00	0%
\$5,000.01 – \$1,000,000.00	1.05%
\$1,000,000.01 and up	0.80%

December's opening balance was \$550,000. Two deposits in the amount of \$600,000 each were made on December 3 and December 21. Two withdrawals in the amount of \$400,000 and \$300,000 were made on December 13 and December 24, respectively. What interest for the month of December will be deposited to the account on January 1?



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://pressbooks.nsc.ca/businessmath2500/?p=36#h5p-8>

TIERED SAVINGS ACCOUNTS

A tiered savings account pays higher rates of interest on higher balances in the account. This is very much like a graduated commission on gross earnings. For example, you might earn 0.25% interest on the first \$1,000 in your account and 0.35% for balances over \$1,000. Most of these tiered savings accounts use a portioning system. This means that if the account has \$2,500, the first \$1,000 earns the 0.25% interest rate and it is only the portion above the first \$1,000 (hence, \$1,500) that earns the higher interest rate.

HOW IT WORKS

Follow these steps to calculate the monthly interest for a tiered savings account:

- Step 1:** Identify the interest rate, opening balance, and the monthly transactions in the savings account.
- Step 2:** Set up a tiered interest rate table as illustrated below. Create a number of rows equaling the number of monthly transactions (deposits or withdrawals) in the account plus one. Adjust the number of columns to suit the number of tiered rates. Fill in the headers for each tiered rate with the balance requirements and interest rate for which the balance is eligible.

Table 4.3.4. Example of a Tiered Interest Rate Table for Step 2

Dates	Closing Balance in Account	# of Days	Tier Rate #1 Balance Requirements and Interest Rate	Tier Rate #2 Balance Requirements and Interest Rate	Tier Rate #3 Balance Requirements and Interest Rate
			Eligible P $I = Prt$	Eligible P $I = Prt$	Eligible P $I = Prt$
Total Monthly Interest Earned					

**Step 3:** For each row of the table, set up the date ranges for each transaction and calculate the balance in the account for each date range.

**Step 4:** For each row, calculate the number of days that the closing balance is maintained.

**Step 5:** Assign the closing balance to the different tiers, paying attention to whether portioning is being used. In each cell with a balance, apply simple interest formula  $I = Prt$ . Ensure that rate and time are expressed in the same units. Do not round off the resulting interest amounts (I).

**Step 6:** To calculate the Total Monthly Interest Earned, sum all interest earned amounts from all tier columns and round off to two decimals.

#### Example 4.3.2: A Rate Builder Tiered Account

The Rate Builder savings account at your local credit union pays simple interest on the daily closing balance as indicated in the table below:

**Table 4.3.5. Balance and Corresponding Interest Rate for Rate Builder Savings Account**

Balance	Interest Rate
\$0.00 to \$500.00	0% on entire balance
\$500.01 to \$2,500.00	0.5% on entire balance
\$2,500.01 to \$5,000.00	0.95% on this portion of balance only
\$5,000.01 and up	1.35% on this portion of balance only

In the month of August, the opening balance on an account was \$2,150.00. Deposits were made to the account on August 5 and August 15 in the amounts of \$3,850.00 and \$3,500.00. Withdrawals were made from the account on August 12 and August 29 in the amounts of \$5,750.00 and \$3,000.00. Calculate the simple interest earned for the month of August.

#### Solution

Calculate the total interest amount (I) for the month of August.

**Step 1:** The interest rate structure is in the table above.

The transactions and dates are also known:

August 1 opening balance = \$2,150.00

August 5 deposit = \$3,850.00

August 12 withdrawal = \$5,750.00

August 15 deposit = \$3,500.00

August 29 withdrawal = \$3,000.00

**Step 2:** Set up a tiered interest rate table with four columns for the tiered rates.

**Step 3:** Determine the date ranges for each balance throughout the month and calculate the closing balances.

**Step 4:** Calculate the number of days involved on each row of the table.

**Step 5:** Assign the closing balance to each tier accordingly. Apply Formula 4.1 to any cell containing a balance.

**Step 6:** Total up all of the interest from all cells of the table.

Table 4.3.6. Interest Calculated from Tiered Interest Table in Example 4.3.2

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: Aug 1 to Aug 5  
Closing Balance in Account: \$2,150  
# of Days: 5 - 1 = 4  
0.5% \$500.01 to \$2,500 (Entire balance)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,150$$

$$I = \$2,150(0.005) \left( \frac{4}{365} \right) = \$0.117808$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: Aug 5 to Aug 12  
Closing Balance in Account:  $\$2,150 + \$3,850 = \$6,000$   
# of Days:  $12 - 5 = 7$   
0.5% \$500.01 to \$2,500 (Entire balance)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,500$$

$$I = \$2,500(0.005) \left( \frac{7}{365} \right) = \$0.239726$$

0.95% \$2,500.01 to \$5,000 (**This portion only**)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,500$$

$$I = \$2,500(0.0095) \left( \frac{7}{365} \right) = \$0.455479$$

1.35% \$5,000.01 and up (This portion only)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$1,000$$

$$I = \$1,000(0.0135) \left( \frac{7}{365} \right) = \$0.258904$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

**Dates:** Aug 12 to Aug 15

**Closing Balance in Account:** \$6,000 - \$5,750 = \$250

**# of Days:** 15 - 12 = 3

**0% \$0 to \$500** (Entire balance)

$$P = \$250.00$$

$$I = \$0.00$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: Aug 15 to Aug 29  
Closing Balance in Account:\$250 + \$3,500= \$3,750  
# of Days: 29 – 15 = 14  
0.5% \$500.01 to \$2,500 (**Entire balance**)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$2,500$$

$$I = \$2,500(0.005) \left( \frac{14}{365} \right) = \$0.479452$$

0.95% \$2,500.01 to \$5,000 (This portion only)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$1,250$$

$$I = \$1,250(0.005) \left( \frac{14}{365} \right) = \$0.455479$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

Dates: Aug 29 to Sep 1  
Closing Balance in Account:  $\$3,750 - \$3,000 = \$750$   
# of Days:  $31 - 29 + 1 = 3$   
0.5% \$500.01 to \$2,500 (Entire balance)

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

$$P = \$750$$

$$I = \$750(0.005) \left( \frac{3}{365} \right) = \$0.030821$$

Calculations of Interest Based on Date with Total Interest Earned at the Bottom

**Total Interest Earned**

$$I = \$0.117808 + \$0.239726 + \$0.455479 + \$0.258904 + \$0.00 + \$0.479452 + \$0.455479 + \$0.030821$$
$$I = \$2.04$$

For the month of August, the tiered savings account earned a total simple interest of \$2.04, which was deposited to the account on September 1.

## SHORT-TERM GUARANTEED INVESTMENT CERTIFICATES (GICS)

A guaranteed investment certificate (GIC) is an investment that offers a guaranteed rate of interest over a fixed period of time. GICs are found mostly at commercial banks, trust companies, and credit unions. In this section, you will deal only with short-term GICs, defined as those that have a time frame of less than one year.

The table below summarizes three factors that determine the interest rate on a short-term GIC: principal, time, and redemption privileges.

Table 4.3.7. Factors Determining Interest Rates on Short-Term GICs

Factors Determining Interest Rate	Higher Interest Rates	Lower Interest Rates
Principal Amount	Large	Small
Time	Longer	Shorter
Redemption Privileges	Nonredeemable	Redeemable

1. **Amount of Principal.** Typically, a larger principal is able to realize a higher interest rate than a smaller principal.
2. **Time.** The length of time that the principal is invested affects the interest rate. Short-term GICs range from 30 days to 364 days in length. A longer term usually realizes higher interest rates.
3. **Redemption Privileges.** The two types of GICs are known as redeemable and nonredeemable. A redeemable GIC can be cashed in at any point before the maturity date, meaning that you can access your money any time you want it. A nonredeemable GIC “locks in” your money for the agreed-upon term. Accessing that money before the end of the term usually incurs a stiff financial penalty, either on the interest rate or in the form of a financial fee. Nonredeemable GICs carry a higher interest rate.

To summarize, if you want to receive the most interest it is best to invest a large sum for a long time in a nonredeemable short-term GIC.

## HOW IT WORKS

Short-term GICs involve a lump sum of money (the principal) invested for a fixed term (the time) at a guaranteed interest rate (the rate). Most commonly the only items of concern are the amount of interest earned and the maturity value. Therefore, you need the same four steps as for single payments involving simple interest shown in Section 4.2.

### Example 4.3.3: GIC Choices

Your parents have \$10,000 to invest. They can either deposit the money into a 364-day nonredeemable GIC at Assiniboine Credit Union with a posted rate of 0.75%, or they could put their money into back-to-back 182-day nonredeemable GICs with a posted rate of 0.7%. At the end of the first 182 days, they will reinvest both the principal and interest into the second GIC. The interest rate remains unchanged on the second GIC. Which option should they choose?

### Solution

For both options, calculate the future value ( $S$ ), of the investment after 364 days. The one with the higher future value is your parents' better option.

**Step 1:** Given variables:

**For the first GIC investment option:**  $P = \$10,000$ ;  $r = 0.75\%$  per year;  $t = 364$  days

**For the second GIC investment option:** Initial  $P = \$10,000$ ;  $r = 0.7\%$  per year;  $t = 182$  days each

**Step 2:** The rate is annual, the time is in days. Convert the time to an annual number. Transforming both time variables,  $t = \frac{364}{365}$  and  $t = \frac{182}{365}$

**Step 3: (1st GIC option):** Calculate the maturity value  $S_1$  of the first GIC option after its 364-day term.

$$S_1 = \$10,000 \left( 1 + (0.0075) \left( \frac{364}{365} \right) \right) = \$10,074.79$$

**Step 3: (2nd GIC option, 1st GIC):** Calculate the maturity value  $S_2$  after the first 182-day term.

$$S_2 = \$10,000 \left( 1 + (0.007) \left( \frac{182}{365} \right) \right) = \$10,034.90$$

**Step 3: (2nd GIC option, 2nd GIC):** Reinvest the first maturity value as principal for another term of 182 days and calculate the final future value  $S_3$ .

$$S_3 = \$10,034.90 \left( 1 + (0.007) \left( \frac{182}{365} \right) \right) = \$10,069.93$$

The 364-day GIC results in a maturity value of \$10,074.79, while the two back-to-back 182-day GICs result in a maturity value of \$10,069.93. Clearly, the 364-day GIC is the better option as it will earn \$4.86 more in simple interest.

### Exercises: Short Term GIC

In the exercise that follow, try it on your own. Full solution is available should you get stuck.

1. If you place \$25,500 into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?
2. Interest rates in the GIC markets are always fluctuating because of changes in the short-term financial markets. If you have \$50,000 to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
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## CHAPTER ATTRIBUTION

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## 4.4 Application of Present Value: Treasury Bills and Commercial Paper

### APPLICATION: TREASURY BILLS AND COMMERCIAL PAPER

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#### Treasury Bills: The Basics

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**Treasury bills**, also known as T-bills, are short-term financial instruments that both federal and provincial governments issue with maturities no longer than one year. Approximately 27% of the national debt is borrowed through T-bills.

Here are some of the basics about T-bills:

1. The Government of Canada regularly places T-bills up for auction every second Tuesday. Provincial governments issue them at irregular intervals.
2. The most common terms for federal and provincial T-bills are 30 days, 60 days, 90 days, 182 days, and 364 days.
3. T-bills do not earn interest. Instead, they are sold at a discount and redeemed at full value. This follows the principle of “buy low, sell high.” The percentage by which the value of the T-bill grows from sale to redemption is called the **yield** or rate of return. From a mathematical perspective, the yield is calculated in the exact same way as an interest rate is calculated, and therefore the yield is mathematically substituted as the discount rate in all simple interest formulas. Up-to-date yields on T-bills can be found at [www.bankofcanada.ca/en/rates/monmrt.html](http://www.bankofcanada.ca/en/rates/monmrt.html).
4. The **face value of a T-bill** (also called *par value*) is the maturity value, payable at the end of the term. It includes both the principal and yield together.
5. T-bills do not have to be retained by the initial investor throughout their entire term. At any point during a T-bill’s term, an investor is able to sell it to another investor through secondary financial markets. Prevailing yields on T-bills at the time of sale are used to calculate the price.

## COMMERCIAL PAPERS – THE BASICS

A **commercial paper** (or paper for short) is the same as a T-bill except that it is issued by a large corporation instead of a government. It is an alternative to short-term bank borrowing for large corporations. Most of these large companies have solid credit ratings, meaning that investors bear very little risk that the face value will not be repaid upon maturity.

Commercial papers carry the same properties as T-bills. The only fundamental differences lie in the term and the yield:

1. The terms are usually less than 270 days but can range from 30 days to 364 days. The most typical terms are 30 days, 60 days, and 90 days.
2. The yield on commercial papers tends to be slightly higher than on T-bills since corporations do carry a higher risk of default than governments.

## HOW IT WORKS

Mathematically, T-bills and commercial papers operate in the exact same way. The future value for both of these investment instruments is always known since it is the face value. Commonly, the two calculated variables are either the present value (price) or the yield (interest rate). The yield is explored later in this section. Follow these steps to calculate the price:

**Step 1:** The face value, yield, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify the following:

1. The face value ( $S$ ).
2. The yield ( $r$ ) on the date of the sale, which is always expressed annually. Remember that mathematically the yield is the same as the discount rate.
3. The number of days ( $t$ ) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually to match the annual yield.



Figure 4.4.0: General Timeline for T-Bills and Commercial Papers [Image Description]

**Step 2:** Solve for the present value. using  $P = \frac{S}{1 + rt}$ , which is the price of the T-bill or commercial paper. This price is always less than the face value.

## CONCEPT CHECK



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<https://pressbooks.nsc.ca/businessmath2500/?p=42#h5p-10>

## Example 4.4.1: Price of a Treasury Bill

A Government of Canada 182-day issue T-bill has a face value of \$100,000. Market yields on these T-bills are 1.5%. Calculate the price of the T-bill on its issue date.

**Solution**

**Step 1:** Given variables:

$S = \$100,000$ ;  $r = 1.5\%$ ;  $t = 182/365$

**Step 2:** Solve for the present value,  $P$ .

$$\begin{aligned} P &= \frac{S}{1 + rt} \\ &= \frac{\$100,000}{1 + (0.015) \left( \frac{182}{365} \right)} \\ &= \$99,257.61 \end{aligned}$$

An investor will pay \$99,257.61 for the T-bill. If the investor holds onto the T-bill until maturity, the investor realizes a yield of 1.5% and receives \$100,000.

## Example 4.4.2: Selling a Commercial Paper During Its Term

Pfizer Inc. issued a 90-day, \$250,000 commercial paper on April 18 when the market rate of return was 3.1%. The paper was sold 49 days later when the market rate of return was 3.63%. Calculate the price of the commercial paper on its date of sale.

**Solution**

Note that the historical rate of return of 3.1% is irrelevant to the price of the commercial paper today. The number of days elapsed since the date of issue is also unimportant. The number of days before maturity is the key piece of information.

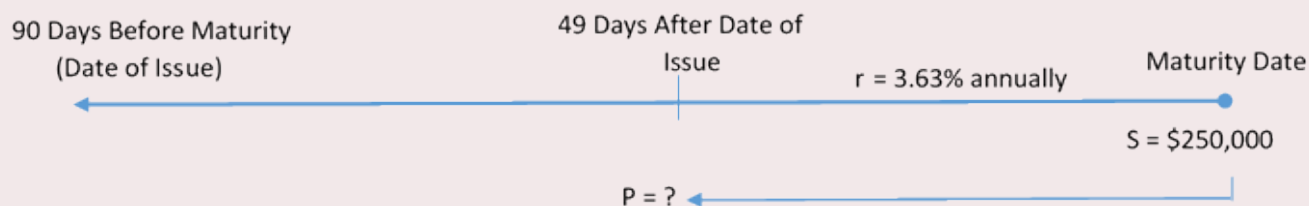


Figure 4.4.2: Timeline [Image Description]

**Step 1:** Given variables:

$S = \$250,000$ ;  $r = 3.63\%$ ;  $t = 90 - 49 = 41$  days or  $41/365$  years

**Step 2:** Solve for the present value,  $P$ .

$$\begin{aligned}
 P &= \frac{S}{1 + rt} \\
 &= \frac{\$250,000}{1 + (0.0363) \left( \frac{41}{365} \right)} \\
 &= \$248,984.76
 \end{aligned}$$

An investor pays \$248,984.76 for the commercial paper on the date of sale. If the investor holds onto the commercial paper for 41 more days (until maturity), the investor realizes a yield of 3.63% and receives \$250,000.

## How It Works

Calculating a Rate of Return: Sometimes the unknown value when working with T-bills and commercial papers is the yield, or rate of return. In these cases, follow these steps to solve the problem:

**Step 1:** The face value, price, and time before maturity must be known. Draw a timeline if necessary, as illustrated below, and identify:

1. The face value ( $S$ ).
2. The price on the date of the sale ( $P$ ).

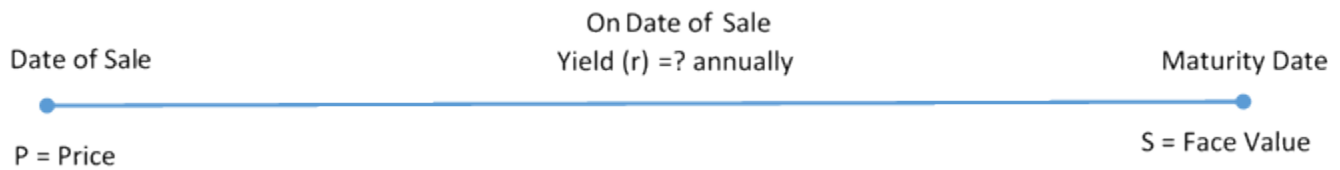


Figure 4.4.Y: General Timeline for Yield [Image Description]

3. The number of days ( $t$ ) remaining between the date of the sale and the maturity date. Count the first day but not the last day. Express the number of days annually so that the calculated yield will be annual.

**Step 2:** Apply formula  $I = S - P$ , to calculate the interest earned during the investment.

**Step 3:** Apply simple interest formula,  $I = Prt$ , rearranging for  $r$  to solve for the interest rate (or yield or rate of return).

#### Example 4.4.3: Figuring Out Rates of Return for Multiple Investors

Marlie paid \$489,027.04 on the date of issue for a \$500,000 face value T-bill with a 364-day term. Marlie received \$496,302.21 when he sold it to Josephine 217 days after the date of issue. Josephine held the T-bill until maturity. Determine the following:

- a. Marlie's actual rate of return.
- b. Josephine's actual rate of return.
- c. If Marlie held onto the T-bill for the entire 364 days instead of selling it to Josephine, what would his rate of return have been?
- d. Comment on the answers to (a) and (c).

#### Solution

Calculate three yields or rates of return ( $r$ ) involving Marlie and the sale to Josephine, Josephine herself, and Marlie without the sale to Josephine. Afterwards, comment on the rate of return for Marlie with and without the sale.

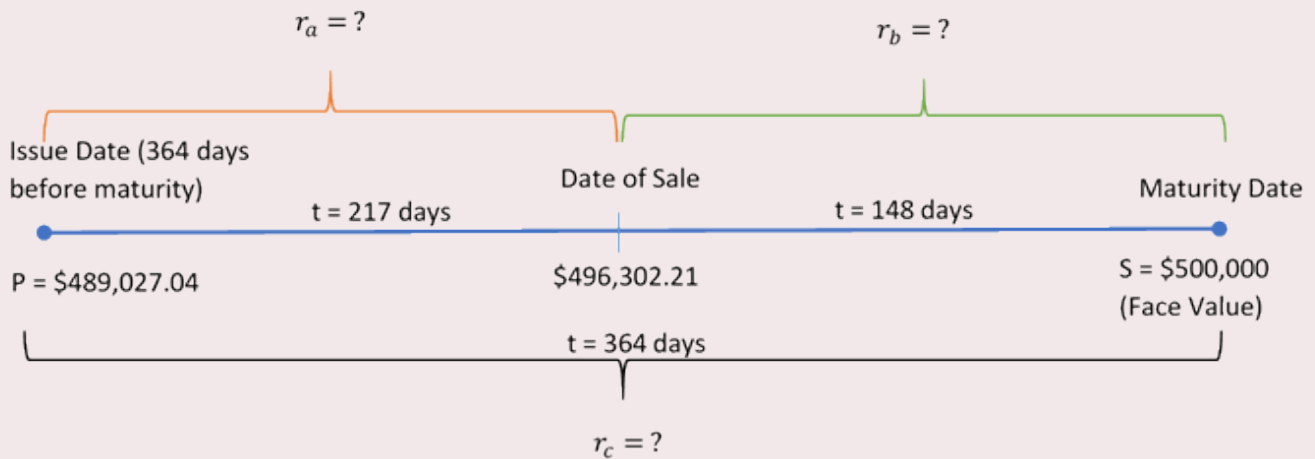


Figure 4.4.3: Timeline [Image Description]

**Step 1:** Given information:

The present values, maturity value, and terms are known.

**a) Marlie with sale:**

$P = \$489,027.04$ ;  $S = \$496,302.21$ ;  $t = 217/365$

**b) Josephine:**

$P = \$496,302.21$ ;  $S = \$500,000$ ;  $364 - 217 = 147$  days remaining;  $t = 147/365$

**c) Marlie without sale:**

$P = \$489,027.04$ ;  $S = \$500,000$ ;  $t = 364/365$

**Step 2:** For each situation, calculate the interest amount ( $I$ ).

**a) Marlie with sale to Josephine:**

$I = \$496,302.21 - \$489,027.04 = \$7,275.17$

**b) Josephine by herself:**

$I = \$500,000 - \$496,302.21 = \$3,697.79$

**c) Marlie without sale to Josephine:**

$I = \$500,000 - \$489,027.04 = \$10,972.96$

**Step 3:** For each situation, apply simple interest formula, rearranging for  $r$ .

**a) Marlie with sale to Josephine:**

$$r = \frac{\$7,275.17}{(\$489,027.04) \left( \frac{217}{365} \right)} = 2.50\%$$

**b)** Josephine by herself:

$$r = \frac{\$3,697.79}{(\$496,302.21) \left( \frac{147}{365} \right)} = 1.85\%$$

**c)** Marlie without sale to Josephine:

$$r = \frac{\$10,972.96}{(\$489,027.04) \left( \frac{364}{365} \right)} = 2.25\%$$

When Marlie sold the T-bill after holding it for 217 days, he realized a 2.50% rate of return. Josephine then held the T-bill for another 148 days to maturity, realizing a 1.85% rate of return. If Marlie hadn't sold the note to Josephine and instead held it for the entire 364 days, he would have realized a 2.25% rate of return.

**Step 4:** Compare the answers for (a) and (c) and comment.

The yield on the date of issue was 2.25%. Marlie realized a higher rate of return because the interest rates in the market decreased during the 217 days he held it (to 1.85%, which is what Josephine is able to obtain by holding it until maturity). This raises the selling price of the T-bill. If his investment of \$489,027.04 grows by 2.25% for 217 days, he has \$6,541.57 in interest. The additional \$733.60 of interest (totaling \$7,275.17) is due to the lower yield in the market, increasing his rate of return to 2.50% instead of 2.25%.

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. A 60-day, \$90,000 face value commercial paper was issued when yields were 2.09%. What was its purchase price?
2. A 90-day Province of Ontario T-bill with a \$35,000 face value matures on December 11. Farrah works for Hearthplace Industries and notices that the company temporarily has some extra cash available. If she invests the money on October 28, when the yield is 4.94%, and sells the T-bill on November 25, when the yield is 4.83%, calculate how much money Farrah earned and the rate of return she realized.
3. Philippe purchased a \$100,000 Citicorp Financial 220-day commercial paper for \$96,453.93. He sold

it 110 days later to Damien for \$98,414.58, who then held onto the commercial paper until its maturity date.

- a) What is Philippe's actual rate of return?
- b) What is Damien's actual rate of return?
- c) What is the rate of return Philippe would have realized if he had held onto the note instead of selling it to Damien?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=42#h5p-11>

## CHAPTER ATTRIBUTION

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## 4.5 Simple Interest Terminology (Interactive Activity)

**Complete the following activity.**



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<https://pressbooks.nsc.ca/businessmath2500/?p=44#h5p-12>

### CHAPTER ATTRIBUTION

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Chapter 8: Simple Interest Terminology (Interactive Activity) in Business Math: A Step-by-Step Handbook Abridged by Sanja Krajisnik, Carol Leppinen, and Jelena Loncar-Vines, Conestoga College. shared under CC BY-NC-SA license.

## 4.6 Review, Symbols, and Formulas

### KEY CONCEPTS

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#### 4.1: Principal, Rate, Time

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- Calculating the amount of simple interest either earned or charged in a simple interest environment
- Calculating the time period when specific dates or numbers of days are involved
- Calculating the simple interest amount when the interest rate is variable throughout the transaction

#### 4.2: Moving Money Involving Simple Interest

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- Putting the principal and interest together into a single calculation known as maturity value
- Altering a financial agreement and establishing equivalent payments

#### 4.3: Application: Savings Accounts and Short-Term GICs

---

- How to calculate simple interest for flat-rate and tiered savings accounts
- How to calculate simple interest on a short-term GIC

#### 4.4 Application: Treasury Bills and Commercial Papers

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- The characteristics of treasury bills
- The characteristics of commercial papers
- Calculating the price of T-Bills and commercial papers
- Calculating the yield of T-Bills and commercial papers

## THE FORMULAS YOU NEED TO KNOW

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### Symbols Used

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$S$  = Maturity value or future value in dollars

$I$  = Interest amount in dollars

$P$  = Principal or present value in dollars

$r$  = Interest rate (in decimal format)

$t$  = Time or term

### Formulas Introduced

---

Simple Interest:  $I = Prt$

Maturity Value:  $S = P(1 + rt)$

Present Value:  $P = \frac{S}{1 + rt}$

## 5 COMPOUND INTEREST

### Learning Objectives

- Differentiate between the concept of compound interest and simple interest.
- Calculate the future value and present value of investments and loans in compound interest applications.
- Calculate equivalent payments that replace another payment or a set of payments.
- Calculate the effective and equivalent interest rates for nominal interest rates.
- Calculate periodic and nominal interest rates.
- Calculate the number of compounding periods and time period of an investment or loan.

## 5.1 Compound Interest Fundamentals

Compound interest is used for most transactions lasting one year or more. In simple interest, interest is converted to principal at the end of the transaction. Therefore, all interest is based solely on the original principal amount of the transaction. **Compound interest**, by contrast, involves interest being periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

### CALCULATING THE PERIODIC INTEREST RATE

The first step in learning about investing or borrowing under compound interest is to understand the interest rate used in converting interest to principal. You commonly need to convert the posted interest rate to find the exact rate of interest earned or charged in any given time period.

### THE FORMULA

#### Formula 5.1: Periodic Interest Rate

$$i = \frac{\text{NominalRate}(I/Y)}{\text{CompoundsperYear}(C/Y)}$$

### CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=56#h5p-13>

Calculate the **periodic interest rate**,  $i$ , for the following nominal interest rates:

- a) 9% compounded monthly
- b) 6% compounded quarterly

**Solution**

**Step 1:** Given information:

- a)  $I/Y = 9\%$ ;  $C/Y = \text{monthly} = 12 \text{ times per year}$
- b)  $I/Y = 6\%$ ;  $C/Y = \text{quarterly} = 4 \text{ times per year}$

**Step 2:** For each question apply the [periodic interest formula](#).

$$\text{a) } i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{9\%}{12} = 0.75\% \text{ per month}$$

Nine percent compounded monthly is equal to a periodic interest rate of 0.75% per month. This means that interest is converted to principal 12 times throughout the year at the rate of 0.75% each time.

$$\text{b) } i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{6\%}{4} = 1.5\% \text{ per quarter}$$

Six percent compounded quarterly is equal to a periodic interest rate of 1.5% per quarter. This means that interest is converted to principal 4 times (every three months) throughout the year at the rate of 1.5% each time.

#### Example 5.1.2: The Nominal Interest Rate ( $I/Y$ )

Calculate the nominal interest rate,  $I/Y$ , for the following periodic interest rates:

- a)  $0.58\bar{3}\%$  per month
- b) 0.05% per day

**Solution**

**Step 1:** Given information:

- a)  $i = 0.58\bar{3}\%$ ;  $C/Y = \text{monthly} = 12 \text{ times per year}$
- b)  $i = 0.05\%$ ;  $C/Y = \text{daily} = 365 \text{ times per year}$

**Step 2:** For each question, apply the [periodic interest formula](#) and rearrange for the nominal rate,  $I/Y$ .

$$\text{a) } I/Y = i \times C/Y = 0.58\bar{3} \times 12 = 7\%$$

A periodic interest rate of  $0.58\bar{3}$  per month is equal to a nominal interest rate of 7% compounded monthly.

$$\text{b) } I/Y = i \times C/Y = 0.05 \times 365 = 18.25\%$$

A periodic interest rate of 0.05% per day is equal to a nominal interest rate of 18.25% compounded daily.

#### Example 5.1.3: Compounds per Year (C/Y)

Calculate the compounding frequency (C/Y) for the following nominal and periodic interest rates:

a) nominal interest rate = 6%, periodic interest rate = 3%

b) nominal interest rate = 9%, periodic interest rate = 2.25%

#### Solution

**Step 1:** Given information:

a)  $I/Y = 6\%$ ;  $i = 3\%$

b)  $I/Y = 9\%$ ;  $i = 2.25\%$

**Step 2:** For each question, apply the [periodic interest formula](#) and rearrange for the compounding frequency, C/Y.

$$\text{a) } C/Y = \frac{I/Y}{i} = \frac{6\%}{3\%} = 2 \text{ compounds per year} = \text{semi-annually}$$

For the nominal interest rate of 6% to be equal to a periodic interest rate of 3%, the compounding frequency must be twice per year, which means a compounding period of every six months, or semi-annually.

$$\text{b) } C/Y = \frac{I/Y}{i} = \frac{9\%}{2.25\%} = 4 \text{ compounds per year} = \text{quarterly}$$

For the nominal interest rate of 9% to be equal to a periodic interest rate of 2.25%, the compounding frequency must be four times per year, which means a compounded period of every three months, or quarterly.

#### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Calculate the periodic interest rate if the nominal interest rate is 7.75% compounded monthly.

2. Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%.
3. Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.
4. After a period of three months, Alese saw one interest deposit of \$176.40 for a principal of \$9,800. What nominal rate of interest is Alese earning?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=56#h5p-14>

## CHAPTER ATTRIBUTION

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## 5.2 Determining the Future (Maturity) Value

The simplest future value scenario for compound interest is for all of the variables to remain unchanged throughout the entire transaction. To understand the derivation of the formula, continue with the following scenario. If \$4000 was borrowed two years ago at 12% compounded semi-annually, then a borrower will owe two years of compound interest in addition to the original principal of \$4,000.

That means  $PV = \$4,000$ . The compounding frequency is semi-annually, or twice per year, which makes the periodic interest rate  $i = \frac{I/Y}{C/Y} = \frac{12\%}{2} = 6\%$ . Therefore, after the first six months, the borrower has 6% interest converted to principal. This a future value, or  $FV$ , calculated as follows:

Principal after one compounding period (six months) = Principal plus interest

$$\begin{aligned} FV &= PV + i(PV) \\ &= \$4,000 + 0.06(\$4,000) \\ &= \$4,000 + \$240 = \$4,240 \end{aligned}$$

Now proceed to the next six months. The future value after two compounding periods (one year) is calculated in the same way. Note that the equation  $FV = PV + i(PV)$  can be factored and rewritten as  $FV = PV(1 + i)$ .

$$\begin{aligned} FV(\text{after two compounding periods}) \\ &= PV(1 + i) = \$4,200(1 + 0.06) = \$4,240(1.06) = \$4,494.40 \end{aligned}$$

Since the  $PV = \$4,240$  is the result of the previous calculation where  $PV(1 + i) = \$4,240$ , the following algebraic substitution is possible:

$$\begin{aligned} FV(\text{after two compounding periods}) \\ &= PV(1 + i)(1 + i) = \$4,000(1.06)(1.06) = \$4,494.40 \end{aligned}$$

Simplifying algebraically, you get:

$$FV = PV(1 + i)(1 + i) = PV(1 + i)^2$$

Do you notice a pattern? With one compounding period, the formula has only one  $(1 + i)$ . With two compounding periods involved, it has two factors of  $(1 + i)$ . Each successive compounding period multiplies a further  $(1 + i)$  onto the equation. This makes the exponent on the  $(1 + i)$  exactly equal to the number of times that interest is converted to principal during the transaction.

## THE FORMULA

---

First, you need to know how many times interest is converted to principal throughout the transaction. You can then calculate the future value. Use Formula 5.2A below to determine the number of compound periods involved in the transaction.

### Formula 5.2A: Number of Compound Periods

$$n = \frac{C}{Y} \times (\text{Number of Years}) \text{ where, } \frac{C}{Y} \text{ is the number of compounding periods per year.}$$

Once you know  $n$ , substitute it into Formula 5.2B, which finds the amount of principal and interest together at the end of the transaction, or the future (maturity) value,  $FV$ .

### Formula 5.2B: Future (Maturity)

$$\text{Value : } FV = PV \times (1 + i)^n$$

- where,  $PV$  is the present value or principal. This is the starting amount upon which compound interest is calculated.
- $i$  is the periodic interest rate from Formula 5.1.
- $n$  is the number of compound periods from Formula 5.2A.

## IMPORTANT NOTES

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### Calculating the Interest Amount (I):

In any situation of lump-sum compound interest, you can isolate the interest amount using the formula

$$I = FV - PV.$$

## HOW IT WORKS

---

Follow these steps to calculate the future value of a single payment:

Step 1: Calculate the periodic interest rate ( $i$ ) using the formula

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}$$

Step 2: Calculate the total number of compound periods (n) using the formula

$$n = C/Y \times (\text{Number of years})$$

Step 3: Calculate the future value using the formula

$$FV = PV(1 + i)^n$$

You will first need to calculate i and n using steps 1 and 2.

## CASH FLOW SIGN CONVENTION

---

### Calculating FV (PV is given)

---

**For investments:** When money is invested (paid-out), this amount is considered as a cash-outflow and this amount has to be entered as a negative number for PV.

**For Loans:** When money is received (loaned), this amount is considered as a cash-inflow and this amount has to be entered as a positive number for PV.

### Calculating PV (FV is given)

---

**For investments:** When you receive your matured investment at the end of the term this is considered as a cash-inflow for you and the future value should be entered as a positive amount.

**For Loans:** When the loan is repaid at the end of the term this is considered as a cash-outflow for you and the future value should be entered as a negative amount.

#### Example 5.2.1: Making an Investment

If you invested \$5,000 for 10 years at 9% compounded quarterly, how much money would you have? What is the interest earned during the term?

**Solution**

The timeline for the investment is below.

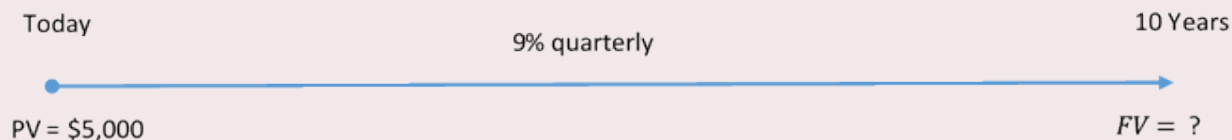


Figure 5.2.1: Timeline [Image Description]

**Step 1:** Given information:

$$PV = 5,000; I/Y = 9\%; C/Y = 4$$

**Step 2:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{9\%}{4} = 2.25\% = 0.0225$$

**Step 3:** Calculate the total number of compoundings,  $n$ .

$$n = C/Y \times (\text{Number of Years}) = 4 \times 10 = 40$$

**Step 4:** Solve for the future value,  $FV$ .

$$FV = \$5,000(1 + 0.0225)^{40} = \$12,175.94$$

**Step 5:** Find the interest earned.

$$I = FV - PV = \$12,175.94 - \$5,000 = \$7,175.94$$

After 10 years, the principal grows to \$12,175.94, which includes your \$5,000 principal and \$7,175.94 of compound interest.

## FUTURE VALUE CALCULATIONS WITH VARIABLE CHANGES

What happens if a variable such as the nominal interest rate, compounding frequency, or even the principal changes somewhere in the middle of the transaction? When any variable changes, you must break the timeline into separate time fragments at the point of the change. To arrive at the solution, you need to work from left to right one time segment at a time using the future value formula.

## HOW IT WORKS

Follow these steps when variables change in calculations of future value based on lump-sum compound interest:

**Step 1:** Read and understand the problem. Identify the present value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the length of the time segment in years.

**Step 2:** For each time segment, calculate the periodic interest rate ( $i$ ) using Formula 5.1.

**Step 3:** For each time segment, calculate the total number of compound periods ( $n$ ) using Formula 5.2A.

**Step 4:** Starting with the present value in the first time segment (starting on the left), solve for the future value using Formula 5.2B.

**Step 5:** Let the future value calculated in the previous step become the present value for the next step. If the principal changes, adjust the new present value accordingly.

**Step 6:** Using Formula 5.2B calculate the future value of the next time segment.

**Step 7:** Repeat steps 5 and 6 until you obtain the final future value from the final time segment.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=62#h5p-15>

### Example 5.2.2: Delaying a Facility Upgrade

Five years ago Coast Appliances was supposed to upgrade one of its facilities at a quoted cost of \$48,000. The upgrade was not completed, so Coast Appliances delayed the purchase until now. The construction company that provided the quote indicates that prices rose 6% compounded quarterly for the first  $1\frac{1}{2}$  years, 7% compounded semi-annually for the following  $2\frac{1}{2}$  years, and 7.5% compounded monthly for the final year. If Coast Appliances wants to perform the upgrade today, what amount of money does it need?

#### **Solution**

The timeline below shows the original quote from five years ago until today.

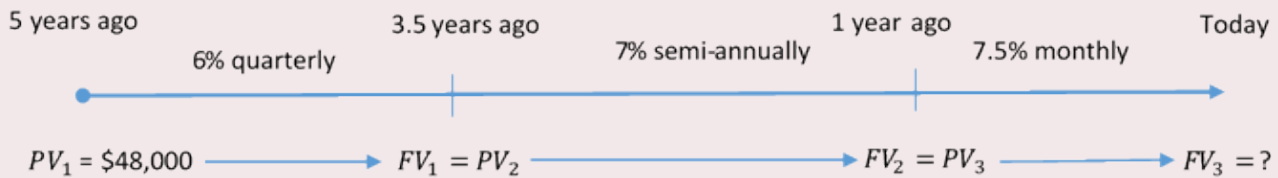


Figure 5.2.2: Timeline [Image Description]

**Step 1:** First time segment:

$$PV_1 = \$48,000; \quad I/Y = 6\%; \quad C/Y = 4; \quad \text{Years} = 2 \quad i = \frac{I/Y}{C/Y} = \frac{6\%}{4} = 1.5\%$$

$$FV_1 = PV_1(1 + i)^n$$

$$n = C/Y \times (\text{Number of Years}) = 4 \times 1.5 = 6 \quad \text{Find } FV_1$$

$$= \$48,000(1 + 0.015)^6$$

$$= \$24,500(1.015)^6$$

$$= \$52,485.27667$$

becomes  $PV_2$  for the next calculation in Step 2. **Step 2:** Second line segment:  $PV_2 = FV_1 = \$52,485.27667$ ;  $I/Y = 7\%$ ;

$$C/Y = 2; \quad \text{Years} = 2.5 \quad i = \frac{I/Y}{C/Y} = \frac{7\%}{2} = 3.5\%$$

$$n = C/Y \times (\text{Number of Years}) = 2 \times 2.5 = 5$$

$$FV_2 = PV_2(1 + i)^n$$

$$\text{Find } FV_2 = \$52,485.27667(1 + 0.035)^5 \quad \text{This becomes } PV_3 \text{ for the next calculation in Step 3.}$$

$$\text{Step 3:} \quad \text{Third line segment: } PV_3 = FV_2 = \$62,336.04435; \quad I/Y = 7.5\%; \quad C/Y = 12; \quad \text{Years} = 1$$

$$i = \frac{I/Y}{C/Y} = \frac{7.5\%}{12} = 0.625\%$$

$$n = C/Y \times (\text{Number of Years}) = 12 \times 1 = 12 \quad \text{Find } FV_3$$

$$FV_3 = PV_3(1 + i)^n$$

$$= \$62,336.04435(1 + 0.00325)^{12}$$

$$= \$67,175.35$$

The future value is \$67,175.35.

Coast Appliances requires \$67,175.35 to perform the upgrade today. This consists of \$48,000 from the original quote plus \$19,175.35 in price increases.

Two years ago Lorelei placed \$2,000 into an investment earning 6% compounded monthly. Today she makes a deposit to the investment in the amount of \$1,500. What is the maturity value of her investment three years from now?

### Solution

The timeline for the investment is below.

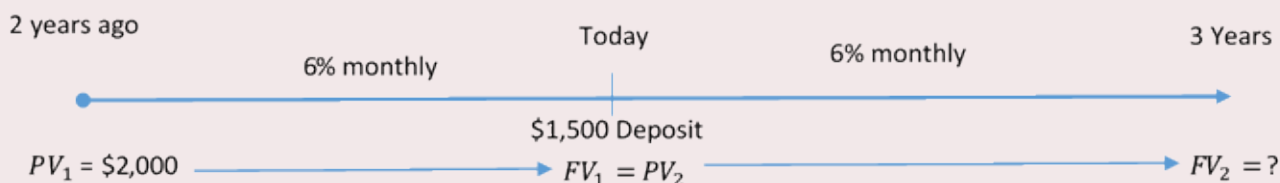


Figure 5.2.3: Timeline [Image Description]

### Step 1: First time segment:

$$PV_1 = \$2,000; I/Y = 6\%; C/Y = 12; \text{Years} = 2 \quad i = \frac{I/Y}{C/Y} = \frac{6\%}{12} = 0.5\%$$

$$\begin{aligned} n &= C/Y \times (\text{Number of Years}) = 12 \times 2 = 24 \quad \text{Find } FV_1 \\ FV_1 &= PV_1(1 + i)^n \\ &= \$2,000(1 + 0.005)^{24} \\ &= \$2,000(1.005)^{24} \\ &= \$2,254.319552 \end{aligned}$$

\$2,254.319552 + \$1,500 = \$3,754.319552 This becomes  $PV_2$  for the second line segment in Step 2.

Second line segment:  $PV_2 = FV_1 = 3,754.319552; I/Y = 6\%; C/Y = 12; \text{Years} = 3$

$$i = \frac{I/Y}{C/Y} = \frac{6\%}{12} = 0.5\%$$

$$n = C/Y \times (\text{Number of Years}) = 12 \times 3 = 36 \quad \text{Find } FV_2$$

$$\begin{aligned} FV_2 &= PV_2(1 + i)^n \\ &= \$3,754.319552(1 + 0.005)^{36} \quad \text{The future value is } \$4,492.72 \\ &= \$4,492.72 \end{aligned}$$

Three years from now Lorelei will have \$4,492.72. This represents \$3,500 of principal and \$992.72 of compound interest.

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Find the future value if \$53,000 is invested at 6% compounded monthly for 4 years and 3 months.
2. Find the future value if \$24,500 is invested at 4.1% compounded annually for 4 years; then 5.15%

compounded quarterly for 1 year, 9 months; then 5.35% compounded monthly for 1 year, 3 months.

3. Nirdosh borrowed \$9,300  $4\frac{1}{4}$  years ago at 6.35% compounded semi-annually. The interest rate changed to 6.5% compounded quarterly  $1\frac{3}{4}$  years ago. What amount of money today is required to pay off this loan?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=62#h5p-16>

Timeline for exercise 3 is included in Solution to Exercises.

## CHAPTER ATTRIBUTION

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## 5.3 Determining the Present Value

PV is the **Present Value or Principal**. This is the new unknown variable. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some earlier point in time than when the future value is known. In any case, the amount excludes the future interest. To calculate this variable, substitute the values for the other three variables into the formula and then algebraically rearrange to isolate PV.

### THE FORMULA

---

Solving for present value requires you to use the future value formula (Formula 5.2B) we introduced in section 5.2. We rearrange the future value formula to solve for P.

#### Formula 5.3A Present Value (Principal)

$$: PV = \frac{FV}{(1 + i)^n}$$

### HOW IT WORKS

---

Follow these steps to calculate the present value of a single payment:

**Step 1:** Calculate the periodic interest rate (i) using the formula

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}}$$

**Step 2:** Calculate the total number of compound periods (n) using the formula

$$n = C/Y \times (\text{Number of years})$$

**Step 3:** Calculate the present value using the present value formula

$$PV = \frac{FV}{(1 + i)^n}$$

### Example 5.3.1: Achieving a Savings Goal

Castillo's Warehouse will need to purchase a new forklift for its warehouse operations three years from now, when its new warehouse facility becomes operational. If the price of the new forklift is \$38,000 and Castillo's can invest its money at 7.25% compounded monthly, how much money should it put aside today to achieve its goal?

#### Solution

Step 1: Given variables:

$FV = 38,000$ ;  $I/Y = 7.25\%$ ;  $C/Y = 12$ ; Years = 3

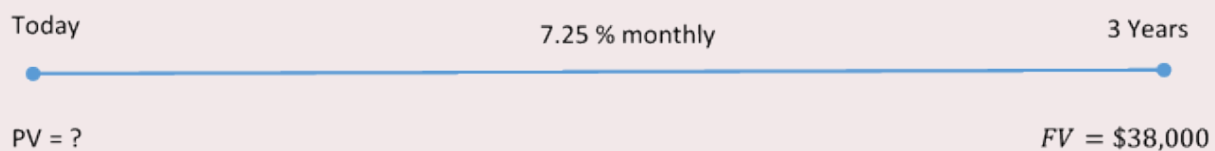


Figure 5.3.1: Timeline [Image Description]

Step 2: Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{7.25\%}{12} = 0.6041\bar{6}\% = 0.006041\bar{6}$$

Step 3: Calculate the number of compound periods,  $n$ .

$$n = C/Y \times (\text{Number of Years}) = 3 \times 12 = 36$$

$$PV = \frac{FV}{(1 + i)^n}$$

Step 4: Solve for the present value, PV.

$$= \frac{38,000}{(1 + 0.006041\bar{6})^{36}} = 30,592.06$$

If Castillo's Warehouse places \$30,592.06 into the investment, it will earn enough interest to grow to \$38,000 three years from now to purchase the forklift.

## PRESENT VALUE CALCULATIONS WITH VARIABLE CHANGES

Addressing variable changes in present value calculations follows the same techniques as future value calculations. You must break the timeline into separate time segments, each of which involves its own calculations.

Solving for the unknown PV at the left of the timeline means you must start at the right of the timeline. You must work from right to left, one time segment at a time using the formula for PV each time. Note that the present value for one time segment becomes the future value for the next time segment to the left.

## HOW IT WORKS

Follow these steps to calculate a present value involving variable changes in single payment compound interest:

**Step 1:** Read and understand the problem. Identify the future value. Draw a timeline broken into separate time segments at the point of any change. For each time segment, identify any principal changes, the nominal interest rate, the compounding frequency, and the segment's length in years.

**Step 2:** For each time segment, calculate the periodic interest rate,  $i$ .

**Step 3:** For each time segment, calculate the total number of compounding periods,  $n$ .

**Step 4:** Starting with the future value in the first time segment on the right, solve for the present value.

**Step 5:** Let the present value calculated in the previous step become the future value for the next time segment to the left. If the principal changes, adjust the new future value accordingly.

**Step 6:** Using the present value formula, calculate the present value of the next time segment.

**Step 7:** Repeat steps 5 and 6 until you obtain the present value from the leftmost time segment.

### Example 5.3.2: A Variable Rate Investment

Sebastien needs to have \$9,200 saved up three years from now. The investment he is considering pays 7% compounded semi-annually, 8% compounded quarterly, and 9% compounded monthly in successive years. To achieve his goal, how much money does he need to place into the investment today?**Solution:**

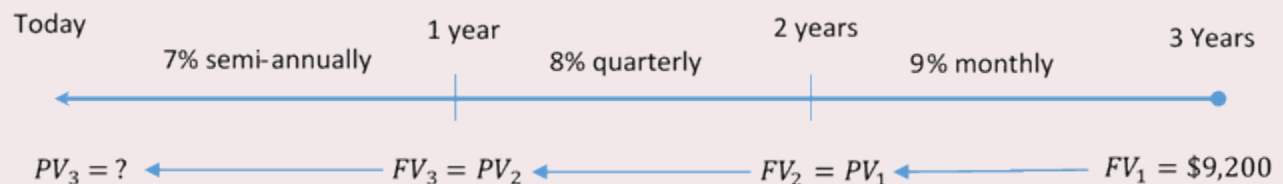


Figure 5.3.2: Timeline [Image Description]

### Solution

Starting from the right end of the timeline and working backwards:

Step 1: First time segment:

$$FV_1 = \$9,200; \quad I/Y = 9\%; \quad C/Y = 12; \quad \text{Years} = 1 \quad i = \frac{I/Y}{C/Y} = \frac{9\%}{12} = 0.75\%$$

$$n = C/Y \times (\text{Number of Years}) = 1 \times 12 = 12$$

**Find  $PV_1$**

$$\begin{aligned} PV_1 &= \frac{FV_1}{(1 + i)^n} \\ &= \frac{9,200}{(1 + 0.0075)^{12}} \\ &= 8,410.991026 \end{aligned}$$

This becomes  $FV_2$  in Step 2.

Step 2: Second line segment:

$$FV_2 = PV_1 = 8,410.991026; \quad I/Y = 8\%; \quad C/Y = 4; \quad \text{Years} = 1$$

$$i = \frac{I/Y}{C/Y} = \frac{8\%}{4} = 2\%$$

$$n = C/Y \times (\text{Number of Years}) = 1 \times 4 = 4$$

Find  $PV_2$

$$\begin{aligned} PV_2 &= \frac{FV_2}{(1 + i)^n} \\ &= \frac{8,410.991026}{(1 + 0.02)^4} \\ &= 7,770.455587 \end{aligned}$$

This becomes  $FV_3$  in Step 3.

**Step 3:** Third line segment:

$$FV_3 = PV_2 = 7,770.455587; \quad I/Y = 7\%; \quad C/Y = 2; \quad \text{Years} = 1$$

$$i = \frac{I/Y}{C/Y} = \frac{7.5\%}{12} = 0.625\%$$

$$n = C/Y \times (\text{Number of Years}) = 1 \times 2 = 2$$

**Find  $PV_3$**

$$\begin{aligned} PV_3 &= \frac{FV_3}{(1+i)^n} \\ &= \frac{7,770.455587}{(1+0.035)^2} \\ &= 7,253.80 \end{aligned}$$

The present value is \$7,253.80

Sebastien needs to place \$7,253.80 into the investment today to have \$9,200 three years from now.

When you calculate the present value of a single payment for which only the interest rate fluctuates, it is possible to find the principal amount in a single division:

$$PV = \frac{FV}{(1+i_1)^{n_1} \times (1+i_2)^{n_2} \times (1+i_3)^{n_3} \times \dots (1+i_n)^{n_n}}$$

where  $n$  represents the time segment number.

In the previous example you can calculate the same principal as follows:

$$PV = \frac{\$9,200}{(1+0.0075)^{12} \times (1+0.02)^4 \times (1+0.035)^2} = \$7,253.80$$

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. A debt of \$37,000 is owed 21 months from today. If prevailing interest rates are 6.55% compounded quarterly, what amount should the creditor be willing to accept today?
2. For the first 4½ years, a loan was charged interest at 4.5% compounded semi-annually. For the next 4 years, the rate was 3.25% compounded annually. If the maturity value was \$45,839.05 at the end of the 8½ years, what was the principal of the loan?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=66#h5p-17>

Timeline for exercise 2 is included in Solution to Exercises.

## CHAPTER ATTRIBUTION

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## 5.4 Equivalent Payments

This section explores the concept of equivalent payment streams. This involves equating two or more alternative financial streams to ensure that neither party is penalized by any choice. You then apply the concept of present value to loans and loan payments.

### THE FUNDAMENTAL CONCEPT OF EQUIVALENCY

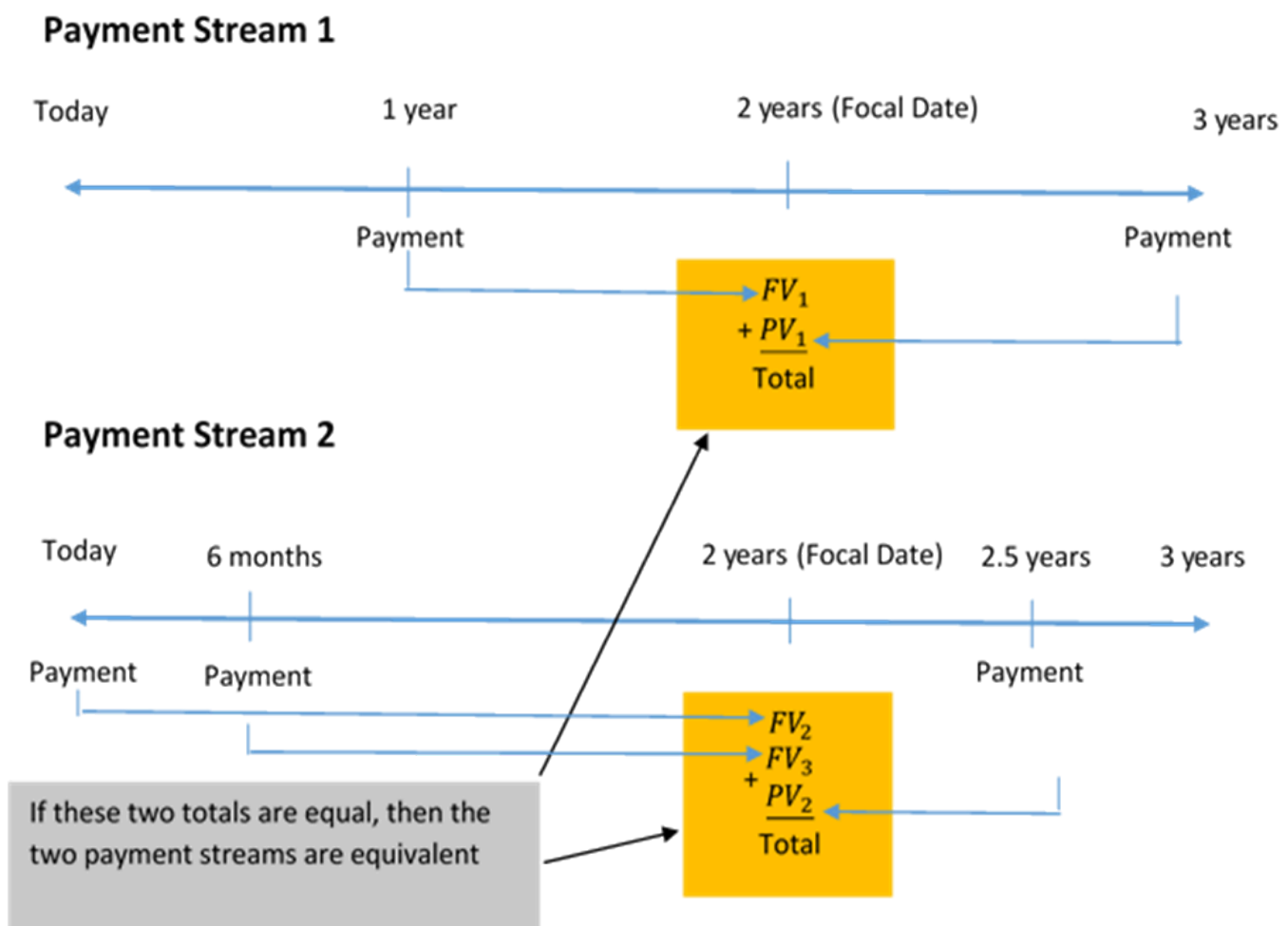


Figure 5.4.0: Two Payment Streams [Image Description]

The fundamental concept of equivalency states that two or more payment streams are equal to each other if they have the same economic value on the same focal date. As illustrated in the figure, the two alternative financial streams are equivalent if the total of Payment Stream 1 is equal to the total of Payment Stream 2 on the same focal date. Note that the monies involved in each payment stream can be summed on the focal date because of the fundamental concept of time value of money.

## HOW IT WORKS

Follow these steps to solve an equivalent payment question:

**Step 1:** Draw as many timelines as needed to illustrate each of the original and proposed agreements. Clearly indicate dates, payment amounts, and the interest rate(s).

If you draw two or more timelines, align them vertically, ensuring that all corresponding dates are in the same columns. This allows you to see which payments need to be future valued and which need to be present valued to express them in terms of the chosen focal date.

**Step 2:** Choose a focal date to which all money will be moved. You should simplify your calculations by selecting a focal date corresponding to the date of an unknown variable.

**Step 3:** Calculate all needed periodic interest rates ( $i$ ) using the formula

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compound per Year (C/Y)}}$$

**Step 4:** Calculate the total number of compounds, ( $n$ ) for each payment using the formula

$$n = (C/Y) \times (\text{Number of Years})$$

**Step 5:** Perform the appropriate time value calculation for each payment using either the formula for FV or PV.

**Step 6:** Equate the values of the original and proposed agreements on the focal date and solve for any unknowns.

### Example 5.4.1: Replacing a Payment Stream with a Single Payment

Assume you owe \$1,000 today and \$1,000 one year from now. You find yourself unable to make that payment today, so you indicate to your creditor that you want to make both payments six months from now instead. Prevailing interest rates are at 6% compounded semi-annually. What single payment six months from now (the proposed payment stream) is equivalent to the two payments (the original payment stream)?

#### Solution

**Step 1:** The timeline illustrates the scenario.

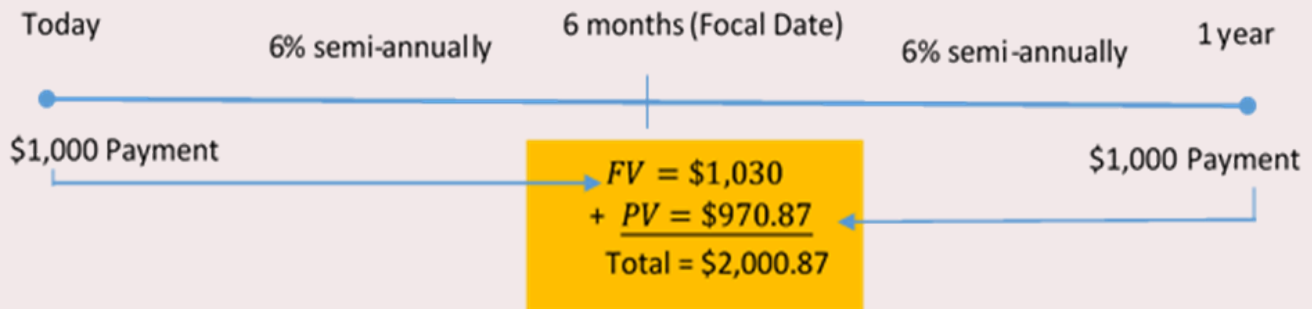


Figure 5.4.1: Timeline [Image Description]

$I/Y = 6\%$ ;  $C/Y = \text{semi-annually} = 2$

**Step 2:** Apply the fundamental concept of time value of money, moving all of the money to the same date. Since the proposed payment is for six months from now, you choose a focal date of six months.

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{6\%}{2} = 3\%.$$

**Step 4:** Calculate the total number of compounds,  $n$ .

$$n = C/Y \times (\text{Number of Years}) = 2 \times \frac{1}{2} = 1.$$

**Step 5:** Perform the appropriate time value calculations.

Moving today's payment of \$1,000 six months into the future, you have

$$FV = \$1,000(1 + 0.03)^1 = \$1,030.$$

Moving the future payment of \$1,000 six months earlier, you have

$$PV = \frac{\$1,000}{(1 + 0.03)^1} = \$970.87.$$

**Step 6:** Now that money has been moved to the same focal date you can sum the two totals to determine the equivalent payment, which is  $\$1,030 + \$970.87 = \$2,000.87$ . Note that this is financially fair to both parties. For making your \$1,000 payment six months late, the creditor is charging you \$30 of interest. Also, for making your second \$1,000 payment six months early, you are receiving a benefit of \$29.13. This leaves both parties compensated equitably: Neither party is financially better or worse off because of the change in the deal.

### Example 5.4.2: Replacing a Payment Stream with a Single Payment

Johnson's Garden Centre has recently been unprofitable and concludes that it cannot make two debt payments of \$4,500 due today and another \$6,300 due in three months. After discussions between Johnson's Garden Centre and its creditor, the two parties agree that both payments could be made nine months from today, with interest at 8.5% compounded monthly. What total payment does Johnson's Garden Centre need nine months from now to clear its debt?

#### Solution

**Step 1:** The timeline illustrates the scenario.

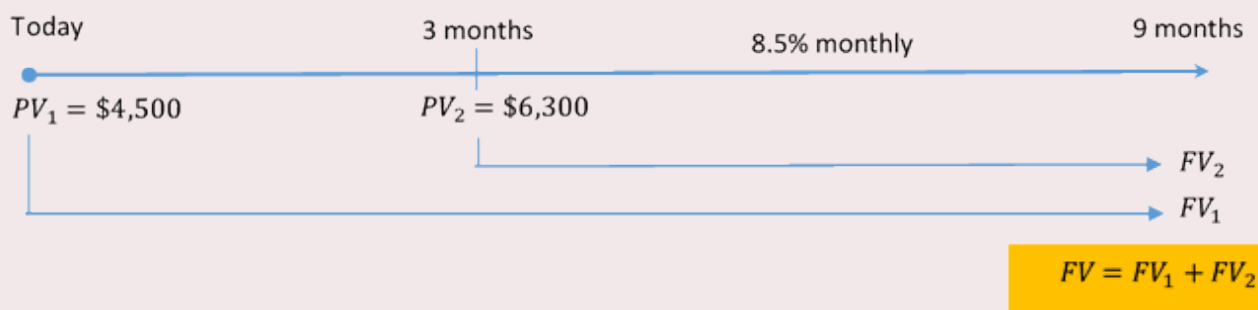


Figure 5.4.2: Timeline [Image Description]

$PV_1$  (today) = \$4,500;  $PV_2$  (3 months from now) = \$6,300;  $I/Y = 8.5\%$ ;  $C/Y = \text{monthly} = 12$

**Step 2:** Due date for all payments = 9 months from today. This is your focal date.

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{8.5\%}{12} = 0.708\bar{3}\% = 0.00708\bar{3}.$$

**Step 4:** Calculate the total number of compounds,  $n$ .

The first payment moves nine months into the future, or  $9/12$  of a year.

$$n = C/Y \times (\text{Number of Years}) = 12 \times \frac{9}{12} = 9.$$

The second payment moves six months into the future, or  $6/12$  of a year.

$$n = C/Y \times (\text{Number of Years}) = 12 \times \frac{6}{12} = 6.$$

**Step 5:** Perform the appropriate time value calculations.

Moving the first payment of \$4,500 nine months into the future, you have

$$FV_1 = \$4,500(1 + 0.00708\bar{3})^9 = \$4,795.138902.$$

Moving the second payment of \$6,300 six months into the future, you have

$$FV_2 = \$6,300(1 + 0.00708\bar{3})^6 = \$6,572.536425.$$

With interest, the two payments total \$11,367.68. This is the \$10,800 of the original principal plus \$567.68 in interest for making the late payments.

#### Example 5.4.3: Replacing a Payment Stream with Multiple Payments

You have three debts to the same creditor: \$3,000 due today, \$2,500 due in 2¼ years, and \$4,250 due in 3 years 11 months. Unable to fulfill this obligation, you arrange with your creditor to make two alternative payments: \$3,500 in nine months and a second payment due in two years. You agree upon an interest rate of 9.84% compounded monthly. What is the amount of the second payment?

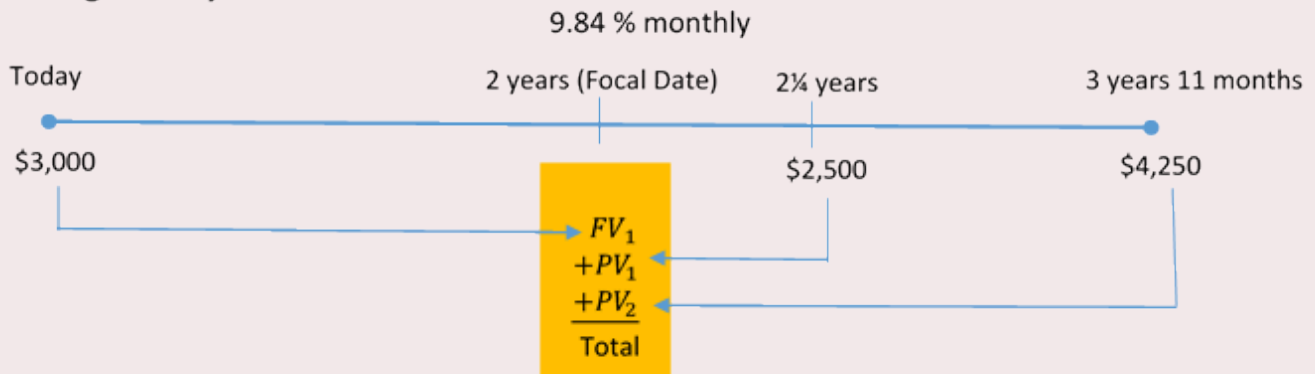
#### **Solution**

Determine the amount of the second payment that is due two years from today. Apply the fundamental concept of time value of money, moving all money from the original and proposed payment streams to a focal date positioned at the unknown payment. Once all money is moved to this focal date, apply the fundamental concept of equivalence, solving for the unknown payment, or  $X = X$ .

#### **Step 1:**

With two payment streams and multiple amounts all on different dates, visualize two timelines. The payment amounts, interest rate, and due dates for both payment streams are known.

### Original Payment Stream



### Proposed Payment Stream



Figure 5.4.3: Two Payment Streams [Image Description]

$I/Y = 9.84\%$ ;  $C/Y = \text{monthly} = 12$

**Step 2:** Focal Date = 2 years.

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{9.84\%}{12} = 0.82\% \text{ or } 0.0082$$

**Step 4:** For each payment, calculate the number of compoundings per year by applying the formula

$$n = C/Y \times (\text{Number of Years}).$$

**Step 5:** For each payment, calculate the appropriate time value calculation. Note that all payments before the two year focal date require you to calculate future values, while all payments after the two-year focal date require you to calculate present values.

**Steps 4 and 5:** Using the circled number references from the timelines:

$$\textcircled{1} n = 12 \times 2 = 24; FV_1 = \$3,000(1 + 0.0082)^{24} = \$3,649.571607$$

$$\textcircled{2} n = 12 \times \frac{1}{4} = 3; PV_1 = \frac{\$2,500}{1.0082^3} = \$2,439.494983$$

$$\textcircled{3} n = 12 \times 1\frac{11}{12} = 23; PV_2 = \frac{\$4,250}{1.0082^{23}} = \$3,522.207915$$

$$\textcircled{4} n = 12 \times 1\frac{1}{4} = 15; FV_2 = \$3,500(1 + 0.0082)^{15} = \$3,956.110749$$

**Step 6:**

$$\$3,649.571607 + \$2,439.494983 + \$3,522.207915 = x + \$3,956.110749$$

$$\$9,611.274505 = x + \$3,956.110749$$

$$\$5,655.16 = x$$

**Exercises**

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. A winning lottery ticket offers the following two options:
  - a) A single payment of \$1,000,000 today or
  - b) \$250,000 today followed by annual payments of \$300,000 for the next three years. If money can earn 9% compounded annually, which option should the winner select? How much better is that option in current dollars?
2. James is a debt collector. One of his clients has asked him to collect an outstanding debt from one of its customers. The customer has failed to pay three amounts: \$1,600 eighteen months ago, \$2,300 nine months ago, and \$5,100 three months ago. In discussions with the customer, James finds she desires to clear up this situation and proposes a payment of \$1,000 today, \$4,000 nine months from now, and a final payment two years from now. The client normally charges 16.5% compounded quarterly on all outstanding debts. What is the amount of the third payment?
3. Four years ago, Aminata borrowed \$5,000 from Randal with interest at 8% compounded quarterly to be repaid one year from today. Two years ago, Aminata borrowed another \$2,500 from Randal at 6% compounded monthly to be repaid two years from today. Aminata would like to restructure the payments so that she can pay 15 months from today and 2½ years from today. The first payment is to be twice the size of the second payment. Randal accepts an interest rate of 6.27% compounded monthly on the proposed agreement. Calculate the amounts of each payment assuming the focal

date is 15 months from today.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=72#h5p-18>

Timelines for the exercises are included in Solution to Exercises.

## CHAPTER ATTRIBUTION

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## 5.5 Determining the Interest Rate

This section shows how to calculate the nominal interest rate on single payments when you know both the future value and the present value.

### HOW IT WORKS

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Follow these steps to solve for the nominal interest rate on a single payment:

**Step 1:** Draw a timeline to help you visualize the question. Of utmost importance is identifying the values of PV and FV, the number of years involved, and the compounding for the interest rate.

**Step 2:** Calculate the number of compounds (n) using the formula

$$n = C/Y \times (\text{Number of Years}).$$

**Step 3:** Substitute known variables into the formula  $FV = PV(1+i)^n$  rearrange and solve for the periodic interest rate, i.

$$i = \left( \frac{FV}{PV} \right)^{\frac{1}{n}} - 1$$

**Step 4:** Substitute the periodic interest rate and the compounding frequency into the formula

$$i = \frac{I/Y}{C/Y} \text{ rearrange, and solve for the nominal interest rate, } I/Y.$$

$$I/Y = i \times C/Y$$

Ensure that the solution is expressed with the appropriate compounding words.

### IMPORTANT NOTES

---

Handling Decimals in Interest Rate Calculations

### Rule 1: A Clear Marginal Effect

Use this rule when it is fairly obvious how to round the interest rate. The dollar amounts used in calculating the interest rate are rounded by no more than a half penny. Therefore, the calculated interest rate should be extremely close to its true value. For example, if you calculate an I/Y of 7.999884%, notice this value would have a marginal difference of only 0.000116% from a rounded value of 8%. Most likely the correct rate is 8% and not 7.9999%. However, if you calculate an I/Y of 7.920094%, rounding to 8% would produce a difference of 0.070006%, which is quite substantial. Applying marginal rounding, the most likely correct rate is 7.92% and not 7.9201%, since the marginal impact of the rounding is only 0.000094%.

### Rule 2: An Unclear Marginal Effect

Use this rule when it is not fairly obvious how to round the interest rate. For example, if the calculated I/Y = 7.924863%, there is no clear choice of how to round the rate. In these cases or when in doubt, apply the standard rule established for this book of rounding to four decimals. Hence, I/Y = 7.9249% in this example.

It is important to stress that the above recommendations for rounding apply to final solutions. If the calculated interest rate is to be used in further calculations, then you should carry forward the unrounded interest rate.

#### Example 5.5.1: Finding the Nominal Rate

When Sandra borrowed \$7,100 from Sanchez, she agreed to reimburse him \$8,615.19 three years from now including interest compounded quarterly. What nominal quarterly compounded rate of interest is being charged?

#### Solution

**Step 1:** The present value, future value, term, and compounding are known, as illustrated in the timeline.

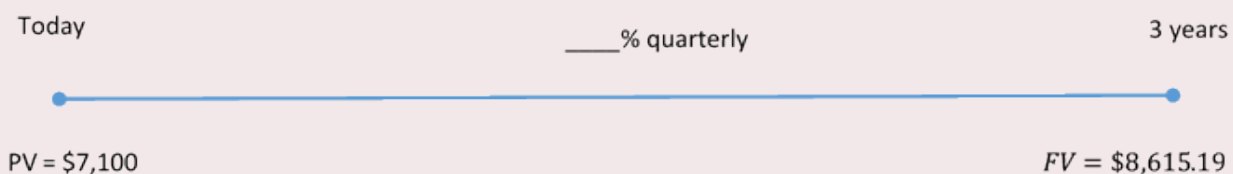


Figure 5.5.1: Timeline [Image Description]

PV = \$7,100; FV = \$8,615.19; C/Y = quarterly = 4; Term = 3 years  
**Step 2:** Calculate the total number of compoundings,  $n = C/Y \times \text{Number of Years} = 4 \times 3 = 12$   
**Step 3:** Calculate the periodic rate,  $i$ .

$$FV = PV(1 + i)^n$$

$$\$8,615.19 = \$7,100(1 + i)^{12}$$

$$1.213407 = (1 + i)^{12}$$

$$1.213407^{\frac{1}{12}} = 1 + i$$

$$1.01624996 = 1 + i$$

$$i = 0.01624996$$

$$i = \frac{I/Y}{C/Y}$$

$$0.01624996 = \frac{I/Y}{4}$$

$$I/Y = 0.06499985 = 0.065 \text{ or } 6.5\% \text{ (compounded quarterly)}$$

an interest rate of 6.5% compounded quarterly on the loan to Sandra.

**Step 4:** Calculate the nominal rate,  $I/Y$ .

Sanchez is charging

#### Example 5.5.2: Known Interest Amount

Five years ago, Taryn placed \$15,000 into an RRSP that earned \$6,799.42 of interest compounded monthly. What was the nominal interest rate for the investment?

#### Solution

**Step 1:** The present value, interest earned, term, and compounding are known, as illustrated in the timeline.

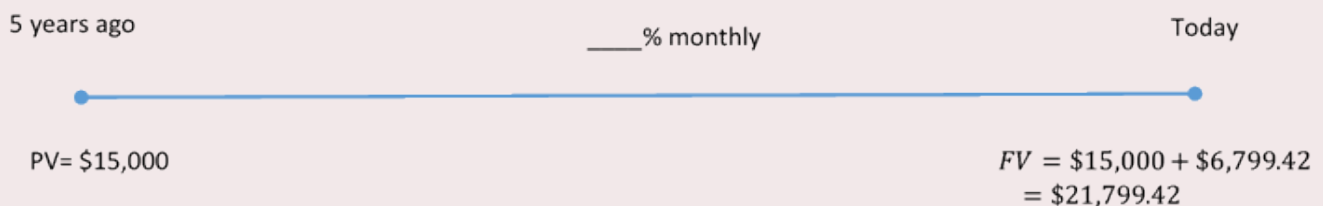


Figure 5.5.2: Timeline [Image Description]

FV = \$15,000 + \$6,799.42 = \$21,799.42; PV = \$15,000; C/Y = monthly = 12; Term = 5 years

**Step 2:** Calculate the total number of compoundings,  $n$ .

$$n = C/Y \times \text{Number of Years} = 12 \times 5 = 60$$

**Step 3:** Calculate the periodic rate,  $i$ .

$$FV = PV(1 + i)^n$$

$$\$21,799.42 = \$15,000(1 + i)^{60}$$

$$1.453294 = (1 + i)^{60}$$

$$1.453294^{\frac{1}{60}} = 1 + i$$

$$1.00625 = 1 + i$$

$$i = 0.00625$$

**Step 4:** Calculate the nominal rate,  $I/Y$ .

$$i = \frac{I/Y}{C/Y}$$

$$0.00625 = \frac{I/Y}{12}$$

$$I/Y = 0.075 \text{ or } 7.5\% \text{ (compounded monthly)}$$

Taryn's investment in his RRSP earned 7.5% compounded monthly over the five years.

## CONVERTING VARIABLE INTEREST RATES TO A FIXED INTEREST RATE

When you deal with a series of variable interest rates it is extremely difficult to determine their overall effect. This also makes it hard to choose wisely between different series. For example, assume that you could place your money into an investment earning interest rates of 2%, 2.5%, 3%, 3.5%, and 4.5% over the course of five years, or alternatively you could invest in a plan earning 1%, 1.5%, 1.75%, 3.5%, and 7% (all rates compounded semi-annually). Which plan is better? The decision is unclear. But you can make it clear by converting the variable rates on each investment option into an equivalent fixed interest rate.

## HOW IT WORKS

Follow these steps to convert variable interest rates to their equivalent fixed interest rates:

**Step 1:** Draw a timeline for the variable interest rate. Identify key elements including any known PV or FV, interest rates, compounding, and terms.

**Step 2:** For each time segment, calculate the periodic interest rate ( $i$ ) and the number of compoundings ( $n$ ).

**Step 3:** One of three situations will occur, depending on what variables are known:

**PV Is Known:** Calculate the future value at the end of the transaction in each time segment, working left to right across the timeline.

**FV Is Known:** Calculate the present value at the beginning of the transaction in each time segment, working right to left across the timeline.

**Neither PV nor FV Is Known:** Pick an arbitrary number for PV (\$10,000 is recommended) and solve for the future value in each time segment at the end of the transaction, working left to right across the timeline.

**Step 4:** Determine the compounding required on the fixed interest rate (C/Y) and calculate a new value for  $n$  to reflect the entire term of the transaction.

**Step 5:** Solve for  $i$  using the  $n$  from step 4 along with the starting PV and ending FV for the entire timeline.

**Step 6:** Solve for  $I/Y$ .

#### Example 5.5.3: Interest Rate under Variable Rate Conditions

Continue working with the two investment options mentioned previously. The choices are to place your money into a five year investment earning semi-annually compounded interest rates of either:

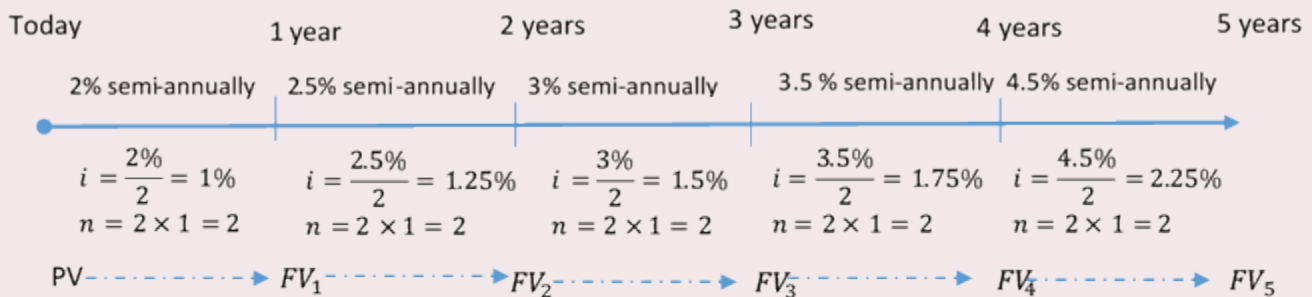
- a) 2%, 2.5%, 3%, 3.5%, and 4.5%
- b) 1%, 1.5%, 1.75%, 3.5%, and 7%

Calculate the equivalent semi-annual fixed interest rate for each plan and recommend an investment.

#### Solution

**Step 1:** Draw a timeline for each investment option, as illustrated below.

### First Investment Option



### Second Investment Option

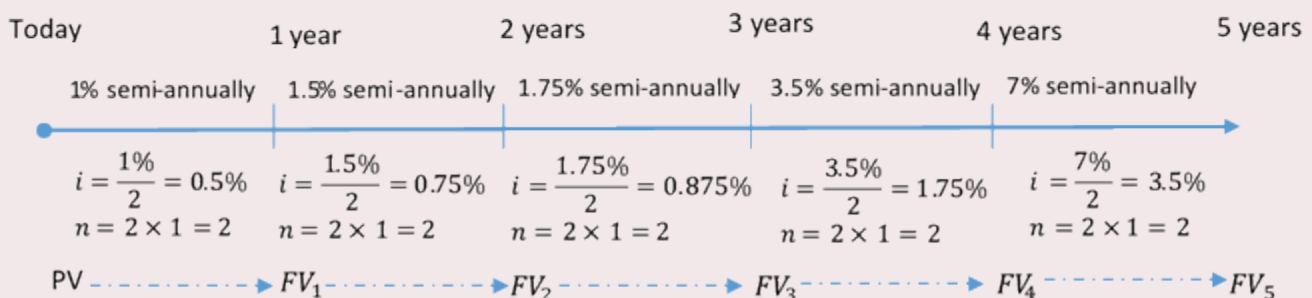


Figure 5.5.3: Timelines for Two Investments with Different Interest Rates [Image Description]

**Step 2:** For each time segment calculate  $i$  and  $n$  using the formulas  $i = \frac{I/Y}{C/Y}$

$n = C/Y \times \text{Number of Years}$  Calculations are found in the timeline figure above. **Step 3:** There is no value for PV or FV. Choose an arbitrary value of PV = \$10,000 and solve for FV. Since only the interest rate fluctuates, solve in one calculation:  $FV_5 = PV \times (1 + i_1)^{n_1} \times (1 + i_2)^{n_2} \times \dots \times (1 + i_5)^{n_5}$  **First**

$$\text{Investment: } FV_5 = 10,000(1 + 0.01)^2(1 + 0.0125)^2(1 + 0.015)^2(1 + 0.0175)^2(1 + 0.0225)^2 \quad \text{Second}$$

$$= \$11,661.65972$$

$$\text{Investment: } FV_5 = 10,000(1 + 0.005)^2(1 + 0.0075)^2(1 + 0.00875)^2(1 + 0.0175)^2(1 + 0.035)^2 \quad \text{Step 4:}$$

$$= \$11,570.14666$$

First Investment:  $n = 2 \times 5 = 10$

Second Investment:  $n = 2 \times 5 = 10$

**Step 5:**

**First Investment:**

$$\$11,661,659.72 = \$10,000(1 + i)^{10}$$

$$1.166165 = (1 + i)^{10}$$

$$1.166165^{\frac{1}{10}} = 1 + i$$

$$1.001549 = 1 + i$$

$$i = 0.001549$$

**Second Investment:**

$$\$11,570,146.66 = \$10,000(1 + i)^{10}$$

$$1.157014 = (1 + i)^{10}$$

$$1.157014^{\frac{1}{10}} = 1 + i$$

$$1.014691 = 1 + i$$

$$i = 0.014691$$

**Step 6:**

**First Investment:**

$$i = \frac{I/Y}{C/Y}$$

$$0.001549 = \frac{I/Y}{2}$$

$$I/Y = 0.030982 \text{ or } 3.0982\%$$

**Second Investment:**

$$i = \frac{I/Y}{C/Y}$$

$$0.014691 = \frac{I/Y}{2}$$

$$I/Y = 0.029382 \text{ or } 2.9382\%$$

The variable interest rates on the first investment option are equivalent to a fixed interest rate of 3.0982% compounded semi-annually. For the second option, the rates are equivalent to 2.9382% compounded semi-annually. Therefore, recommend the first investment since its rate is higher by  $3.0982\% - 2.9382\% = 0.16\%$  compounded semi-annually.

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Your company paid an invoice five months late. If the original invoice was for \$6,450 and the amount paid was \$6,948.48, what monthly compounded interest rate is your supplier charging on late payments?
2. At what monthly compounded interest rate does it take five years for an investment to double?
3. Indiana just received a maturity value of \$30,320.12 from a semi-annually compounded investment that paid 4%, 4.1%, 4.35%, 4.75%, and 5.5% in consecutive years. What amount of money did Indiana invest? What fixed quarterly compounded nominal interest rate is equivalent to the variable rate his investment earned?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=77#h5p-19>

## CHAPTER ATTRIBUTION

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## 5.6 Compound Interest Terminology (Interactive Activity)

**Complete the following activity.**



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<https://pressbooks.nsc.ca/businessmath2500/?p=84#h5p-25>

### CHAPTER ATTRIBUTION

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Chapter 9: Simple Interest Terminology (Interactive Activity) in Business Math: A Step-by-Step Handbook Abridged by Sanja Krajisnik, Carol Leppinen, and Jelena Loncar-Vines, Conestoga College. shared under CC BY-NC-SA license.

## 5.7 Review, Symbols, and Formulas

### KEY CONCEPTS

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#### 5.1: Compound Interest Fundamentals

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- How compounding works
- How to calculate the periodic interest rate

#### 5.2: Determining the Future Value

---

- The basics of taking a single payment and moving it to a future date
- Moving single payments to the future when variables change

#### 5.3: Determining the Present Value

---

- The basics of taking a single payment and moving it to an earlier date
- Moving single payments to the past when variables change

#### 5.4: Equivalent Payments

---

- The concept of equivalent payments
- The fundamental concept of time value of money
- The fundamental concept of equivalency
- Applying single payment concepts to loans and payments

#### 5.5: Determining the Interest Rate

---

- Solving for the nominal interest rate
- How to convert a variable interest rate into its equivalent fixed interest rate

## THE FORMULAS YOU NEED TO KNOW

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### Symbols Used

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$C/Y$  = Compounds per year, or compounding frequency

$C/Y_{\text{New}}$  = The new compounding frequency an interest rate is converted to

$C/Y_{\text{Old}}$  = The old compounding frequency an interest rate is converted from

$FV$  = Future value, or maturity value

$i$  = Periodic interest rate

$i_{\text{New}}$  = The new periodic interest rate after a conversion

$i_{\text{Old}}$  = The old periodic interest rate before a conversion

$I/Y$  = Nominal interest rate per year

$\ln$  = Natural logarithm

$n$  = Number of compound periods

$PV$  = Present value, or principal

### FORMULAS INTRODUCED

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Formula 5.1 Periodic Interest Rate:

$$i = \frac{I/Y}{C/Y}$$

Formula 5.2 Number of Compound Periods for Single Payments:

$$n = C/Y \times (\text{Number of Years})$$

Formula 5.3 Compound Interest for Single Payments:

$$FV = PV(1 + i)^n$$

Formula 5.4 Interest Rate Conversion:

$$i_{\text{New}} = (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1$$

## 6 COMPOUND INTEREST - APPLICATIONS & SINGLE PAYMENTS

### Learning Objectives

- Calculate the interest payment for an interest payout GIC.
- Calculate the maturity value of a fixed rate GIC.
- Calculate the maturity value of a variable rate GIC.
- Calculating the proceeds (selling price) of an interest-bearing promissory note.
- Calculating the proceeds (selling price) of a non-interest-bearing promissory note.
- Calculating the price of a strip bond.
- Calculating the nominal yield of a strip bond.

## 6.1 Application: Long Term GICs

Recall that Guaranteed Investment Certificates (GICs) are investments offering a guaranteed rate of interest over a predetermined time period. Whereas a short-term GIC involves terms less than one year, most long-term GICs range from one to five years. Though terms longer than this are available, they are not very common.

Note that the factors that determine interest rates for short-term GICs are the same factors that apply to long-term GICs: To receive the highest interest rate on a GIC, you should still invest a large principal in a non-redeemable GIC for the longest term possible.

The key difference between short- and long-term GICs lies in the compounding of interest. Long-term GICs do not wait until the end of the term for interest on them to appear and be paid out. Rather, in line with the definition of compound interest, a long-term GIC periodically converts the accrued interest into principal throughout the transaction. Although GICs come in many varieties (remember, financial institutions try to market these products attractively to investors), three structures are commonly available:

1. **Interest Payout GICs.** An **interest payout GIC** uses interest rates that by all appearances you might assume to be compounded periodically since they are listed side-by-side with compound interest rates. In practice, though (and by reading the fine print), you will find that the periodically calculated interest is never added to the principal of the GIC, and in essence the concepts of simple interest are used. Instead, the interest is paid out to the investor (perhaps into a chequing account) and does not actually compound unless the investor takes the interest payment received and invests it in another compounding investment. Interest payout GIC interest rates can take either a fixed or variable format. For example, in an online browsing of long-term GICs you may find a posted rate on a three-year GIC at 2% semi-annually. The fine print and footnotes may show that the interest is paid out on a simple interest basis at the end of each six months.
2. **Compound Interest GICs.** A **compound interest GIC** uses compound interest rates for which interest is periodically calculated and converted to the principal of the GIC for further compounding. Interest rates can be either fixed or variable.

## THE FORMULA

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Calculating the interest payment requires a simplification of Formula 5.2B<sup>1</sup>. However, in an interest payout GIC you never add the interest to the principal, so you do not need the  $1 +$  term in the formula. There is also no future value to calculate, just an interest amount. This changes Formula 5.2B from  $FV = PV(1+i)^n$  to  $I = PV \times i^n$ . Simplifying further, you calculate the interest one compound period at a time, where  $n = 1$ . This eliminates the need for the  $n$  exponent and establishes Formula 6.1.

### Formula 6.1: Periodic Interest Amount

$$I = PV \times i$$

## INTEREST PAYOUT GIC

---

Follow these steps to calculate the interest payment for an interest payout GIC:

**Step 1:** Identify the amount of principal invested (PV), the nominal interest rate (I/Y) and interest payout frequency (C/Y).

**Step 2:** Calculate the periodic interest rate (i) using the formula

$$i = \frac{I/Y}{C/Y}$$

**Step 3:** Calculate the interest amount (I) per compounding period using Formula 6.1.

$$I = PV \times i$$

## Important Notes

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If the interest rate is variable, you must apply Formula 6.1 to each of the variable interest rates in turn to calculate the interest payment amount in the corresponding time segment. For example, if a \$1,000 GIC earns 2% quarterly for the first year and 2.4% monthly for the second year, then in the first year  $I = \$1,000 \times \frac{2\%}{4} = \$5$ , and in the second year  $I = 1,000 \times \frac{2.4\%}{12} = \$2$ .

## CONCEPT CHECK

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1. Formulas contained in Unit 5.2 Determining the Future (Maturity) Value



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=97#h5p-26>

#### Example 6.1.1: An Interest Payout GIC

Jackson placed \$10,000 into a four-year interest payout GIC earning 5.5% semi-annual interest. Calculate the amount of each interest payment and the total interest earned throughout the term.

##### **Solution**

Step 1: Given information:

$PV = \$10,000$ ;  $I/Y = 5.5\%$ ;  $C/Y = 2$

**Step 2:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{5.5\%}{2} = 2.75\% = 0.0275$$

**Step 3:** Calculate the periodic interest amount.

$$I = PV \times i = \$10,000 \times 0.0275 = \$275$$

**Step 4:** Calculate the total interest.

$$n = C/Y \times (\text{Number of Years}) = 2 \times 4 = 8$$

$$\text{Total interest} = \$275 \times 8 = \$2,200.$$

Every six months, Jackson receives an interest payment of \$275. Over the course of four years, these interest payments total \$2,200.

## COMPOUND INTEREST GIC

Follow these steps to **calculate the maturity value** of the investment or the compound rate.

## Fixed Interest Rate

If the compound interest rate is fixed, then you find the maturity value using the formula  $FV = PV(1+i)^n$ . These 4 steps were introduced in Section 5.2.

## Variable Interest Rate

If the compound interest rate is variable, then you find the maturity value for each segment of the timeline. These 7 steps were also introduced in Section 5.2. Note that in step 5, no principal adjustment needs to be made since only the interest rate variable changes.

## Alternative approach

Since only the interest rate changes, expand the future value formula:

$$FV = PV(1 + i_1)^{n_1} (1 + i_2)^{n_2} (1 + i_3)^{n_3} \dots$$

## Calculating the Interest Rate

In the event that the unknown variable is the interest rate, recall the 6 steps that were introduced in Section 5.5.

### Example 6.1.2: Calculating the Future (Maturity) Value

Assume an investment of \$5,000 is made into a three-year compound interest GIC earning 5% compounded quarterly. Solve for the future (maturity) value.

#### Solution

**Step 1:** Given information:

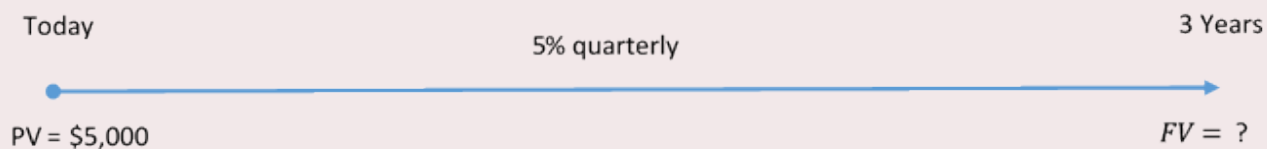


Figure 6.1.2: Timeline [Image Description]

PV = \$5,000; I/Y = 5%; C/Y = 4; Term = 3 years

**Step 2:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{5\%}{4} = 1.25\% = 0.0125$$

**Step 3:** Calculate the number of compound periods,  $n$ .

$$n = C/Y \times (\text{Number of years}) = 4 \times 3 = 12$$

**Step 4:** Calculate the maturity value,  $FV$ .

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$5,000(1 + 0.0125)^{12} \\ &= \$5,803.77 \end{aligned}$$

Hence, at maturity the GIC contains \$5,803.77, consisting of \$5,000 of principal and \$803.77 of compound interest.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=97#h5p-27>

### Example 6.1.3: How Much Do You Have at Maturity

Andrej invested \$23,500 into a three-year variable compound interest GIC. The quarterly compounded interest rate was 3.8% for the first 15 months, 3.7% for the next 12 months, and 3.65% after that. What is the maturity value of Andrej's GIC?

#### Solution

**Step 1:** The principal, terms, and interest rates are known, as shown in the timeline.

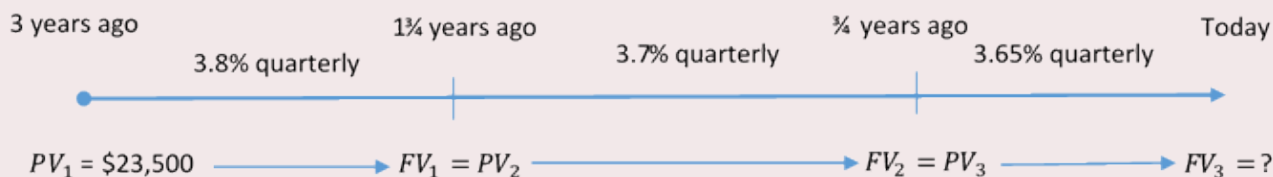


Figure 6.1.3: Timeline [Image Description]

**First time segment:**

$I/Y = 3.8\%$ ;  $C/Y = \text{quarterly} = 4$ ; Term =  $1\frac{1}{4}$  years

**Second time segment:**

$I/Y = 3.7\%$ ;  $C/Y = \text{quarterly} = 4$ ; Term = 1 year

**Third time segment:**

$I/Y = 3.65\%$ ;  $C/Y = \text{quarterly} = 4$ ; Term =  $\frac{3}{4}$  years

**Step 2:** For each time segment, calculate the periodic interest rate,  $i$ .

**First time segment:**

$$i_1 = \frac{3.8\%}{4} = 0.95\% = 0.0095$$

**Second time segment:**

$$i_2 = \frac{3.7\%}{4} = 0.925\% = 0.00925$$

**Third time segment:**

$$i_3 = \frac{3.65\%}{4} = 0.9125\% = 0.009125$$

**Step 3:** For each time segment, calculate the number of compound periods,  $n$ .

**First time segment:**

$$n_1 = 4 \times \frac{1}{4} = 5$$

**Second time segment:**

$$n_2 = 4 \times 1 = 4$$

**Third time segment:**

$$n_3 = 4 \times \frac{3}{4} = 3$$

**Step 4:** Calculate the future value for the first time segment,  $FV_1$ .

$$FV_1 = \$23,500(1 + 0.0095)^5 = \$24,637.66119$$

This becomes  $PV_2$  for the second time segment.

**Step 5:** Let  $FV_1 = PV_2$ .

**Step 6:** Calculate the future value for the second time segment,  $FV_2$ .

$$FV_2 = \$24,637.66119(1 + 0.00925)^4 = \$25,561.98119$$

This becomes  $PV_3$  for the third time segment.

**Repeat Step 5:** Let  $FV_2 = PV_3$ .

**Repeat Step 6:** Calculate the future value for the third line segment,  $FV_3$ .

$$FV_3 = \$25,561.98119(1 + 0.009125)^3 = \$26,268.15$$

**Alternative approach:**

Expand the future value formula:

$$\begin{aligned} FV &= PV(1 + i_1)^{n_1}(1 + i_2)^{n_2}(1 + i_3)^{n_3} \\ &= \$23,500(1 + 0.0095)^5(1 + 0.00925)^4(1 + 0.009125)^3 \\ &= \$26,268.15 \end{aligned}$$

At the end of the three-year term compound interest GIC, Andrey has \$26,268.15, consisting of the \$23,500 principal plus \$2,768.15 in interest.

**Example 6.1.4: Equivalent Interest Rate on the GIC**

Using the previous example, what equivalent fixed quarterly compounded interest rate did Andrey earn on his GIC?

**Solution**

**Step 1:** Given information (from the previous example):

PV = \$23,500; FV = \$26,268.15; C/Y = 4; Term = 3 years

**Step 2:** Calculate the total number of compoundings,  $n$ .

$$n = C/Y \times (\text{Number of Years}) = 4 \times 3 = 12$$

**Step 3:** Using the future value formula, rearrange and solve for  $i$ .

$$FV = PV(1 + i)^n$$

$$\$26,268.15 = \$23,500(1 + i)^{12}$$

$$1.117793 = (1 + i)^{12}$$

$$1.117793^{\frac{1}{12}} = 1 + i$$

$$1.009322 = 1 + i$$

$$i = 0.00932$$

**Step 4:** Using the formula for  $i$ , rearrange and solve for  $I/Y$ .

$$i = \frac{I/Y}{C/Y}$$

$$I/Y = i \times C/Y$$

$$= 0.009322 \times 4$$

$$= 0.037292 \text{ or } 3.7292\% \text{ compounded quarterly}$$

**Calculator instructions:**

A fixed rate of 3.7292% compounded quarterly is equivalent to the three variable interest rates that Andrej realized.

## ESCALATOR INTEREST GIC

An escalator interest GIC is a compound interest rate GIC with the following characteristics:

1. The nominal interest rate always increases with each change so that higher returns on longer terms will encourage the investor to keep the sum of money invested in this GIC.
2. The interest rates are known in advance and fixed for the duration of each time segment of the investment.
3. Each time segment is most commonly one year in length.

## HOW IT WORKS

The exact same formulas and procedures used for compound interest GICs remain applicable. The most common applications with escalator GICs involve finding one of the following:

1. The maturity value of the GIC.
2. The equivalent fixed rate of interest on the GIC so that the investor can either compare it to that of other options or just better understand the interest being earned. Recall from Chapter 9 the 6 steps you need to calculate equivalent fixed rates.

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Sanchez placed \$11,930 into a five-year GIC at 4.2% compounded monthly. Calculate the interest amount earned at the end of the five-year term.
2. TD Canada Trust is offering its five-year Stepper GIC at annually escalating rates of 1.15%, 2%, 2.75%, 3.5%, and 4.5%. All rates are compounded semi-annually. Alternatively, it is offering a five-year fixed rate GIC at 2.7% compounded monthly. What total interest amount does an \$18,000 investment earn under each option?
3. Calculate the interest earned on each of the following five-year GICs. Rank the GICs from best to worst based on the amount of interest earned on a \$15,000 investment.
  - a) An interest payout GIC earning 4.5% compounded quarterly.
  - b) A fixed rate compound interest GIC earning 4.2% compounded monthly.
  - c) A variable rate quarterly compound interest GIC earning consecutively 3.9% for 1.5 years, 4.25% for 1.75 years, 4.15% for 0.75 years, and 4.7% for 1 year.
  - d) An escalator rate GIC earning semi-annually compounded rates of 1.25%, 2%, 3.5%, 5.1%, and 7.75% in successive years.



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://pressbooks.nscc.ca/businessmath2500/?p=97#h5p-28>

Calculator instructions for the exercises are included in Solution to Exercises.

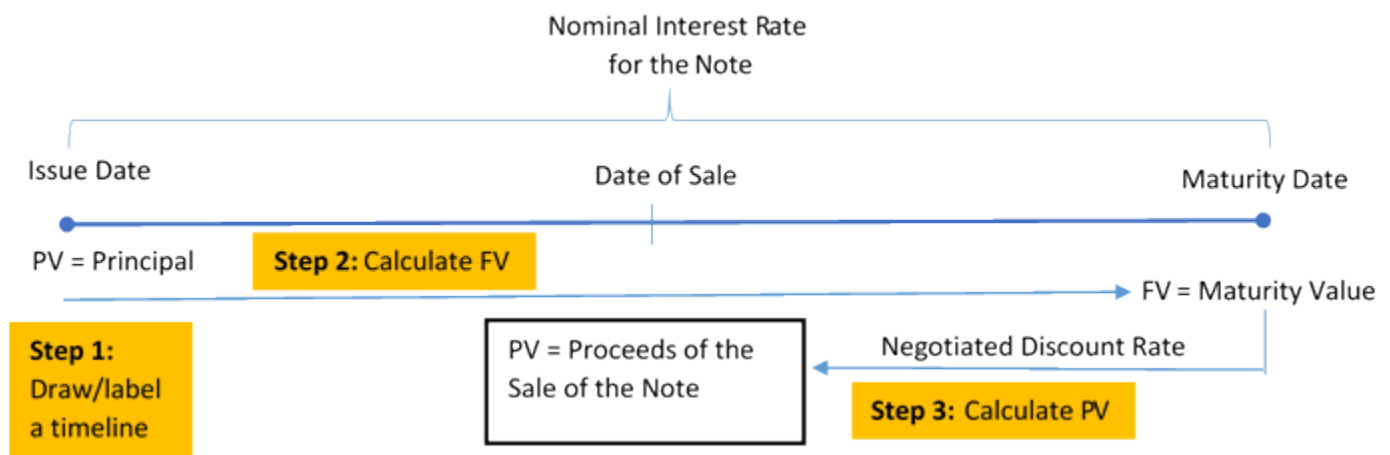
## 6.2 Application: Long Term Promissory Notes

This section introduces the mathematics behind the sale of promissory notes between companies. A *promissory note*, more commonly called a note, is a written debt instrument that details a promise made by a buyer to pay a specified amount to a seller at a predetermined and specified time. If the debt allows for interest to accumulate, then it is called an *interest-bearing promissory note*. If there is no allowance for interest, then it is called a *non-interest-bearing promissory note*. When promissory notes extend more than one year, they involve compound interest instead of simple interest.

### HOW IT WORKS

With simple interest, the sale of short-term promissory notes involves three steps. You use the same three step sequence for long-term compound interest promissory notes. On long-term promissory notes, a three-day grace period is not required, so the due date of the note is the same as the legal due date of the note.

**Step 1:** Draw a timeline detailing the original promissory note and the sale of the note.



**Step 2:** At the stated nominal interest rate, take the initial principal on the date of issue and determine the note's future value using the formula

$$FV = PV(1 + i)^n$$

**Step 3:** Using the date of sale, discount the maturity value of the note using a new negotiated discount rate of interest to determine the proceeds of the sale. This involves calculating the present value using the formula

$$PV = \frac{FV}{(1 + i)^n}$$

## TWO INTEREST RATES

The sale involves two interest rates: an interest rate tied to the note itself and an interest rate (the discount rate) used by the purchasing company to acquire the note. Do not confuse these two rates.

### Example 6.2.1: Proceeds on an Interest Bearing Note

Assume that a three-year \$5,000 promissory note with 9% compounded monthly interest is sold to a finance company 18 months before the due date at a discount rate of 16% compounded quarterly. What are the proceeds of the sale?

#### Solution

**Step 1:** The timeline below illustrates the situation.

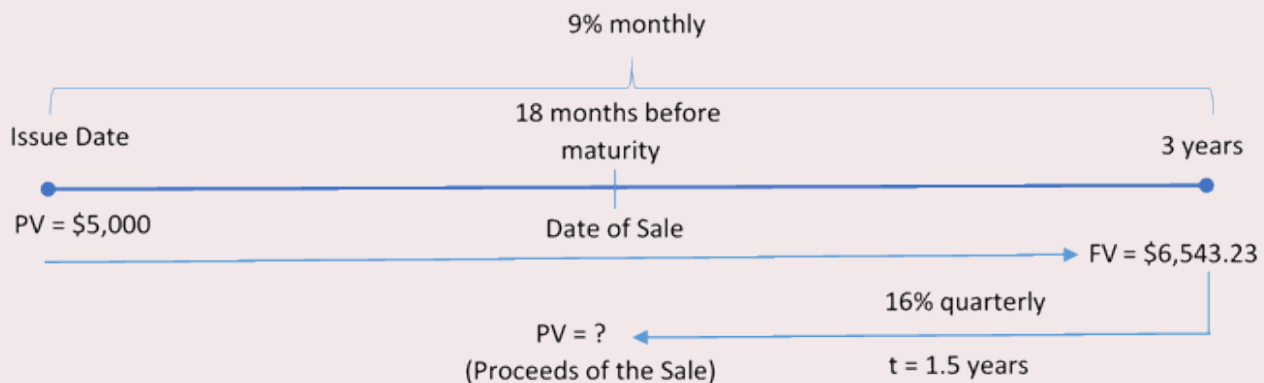


Figure 6.2.1: Timeline of a Promissory Note [Image Description]

**Step 2:** Calculate the maturity value of the note at the end of the three-year term.

**For the three-year term:**

$I/Y = 9\%$ ;  $C/Y = \text{monthly} = 12$

$$i = \frac{I/Y}{C/Y} = \frac{9\%}{12} = 0.75\% = 0.0075$$

$$n = C/Y \times (\text{Number of Years}) = 12 \times 3 = 36$$

$$FV = \$5,000(1 + 0.0075)^{36} = \$6,543.23$$

**Step 3:** Calculate the proceeds (PV) of the sale at 18 months before maturity. Discount the note back for 1.5 years.

$I/Y = 16\%$ ;  $C/Y = \text{quarterly} = 4$

$$i = \frac{I/Y}{C/Y} = \frac{16\%}{4} = 4\% = 0.04$$

$$n = C/Y \times (\text{Number of Years}) = 4 \times 1.5 = 6$$

$$\begin{aligned} PV &= \frac{FV}{(1 + i)^n} \\ &= \frac{\$6,543.23}{(1 + 0.04)^6} \\ &= \$5,171.21 \end{aligned}$$

The finance company purchases the note (invests in the note) for \$5,171.21. Eighteen months later, when the note is paid, it receives \$6,543.23.

## IMPORTANT NOTES

The assumption behind the three-step procedure for selling a long-term promissory note is that the process starts with the issuance of the note and ends with the proceeds of the sale. However, mathematically you may deal with any part of the transaction as an unknown. For example, perhaps the details of the original note are known, the finance company's offer on the date of sale is known, but the quarterly discounted rate used by the finance company needs to be calculated.

The best strategy in any of these scenarios is always to execute step 1 and create a timeline. Identify the known variables to visualize the process, then solve for any variable(s) remaining unknown.

### Example 6.2.2: Finding an Unknown Discount Rate

A \$6,825 two-year promissory note bearing interest of 12% compounded monthly is sold six months before maturity to a finance company for proceeds of \$7,950.40. What semi-annually compounded discount rate was used by the finance company?

#### **Solution**

**Step 1:** The term, principal, promissory note interest rate, date of sale, and proceeds amount are known, as shown in the timeline.

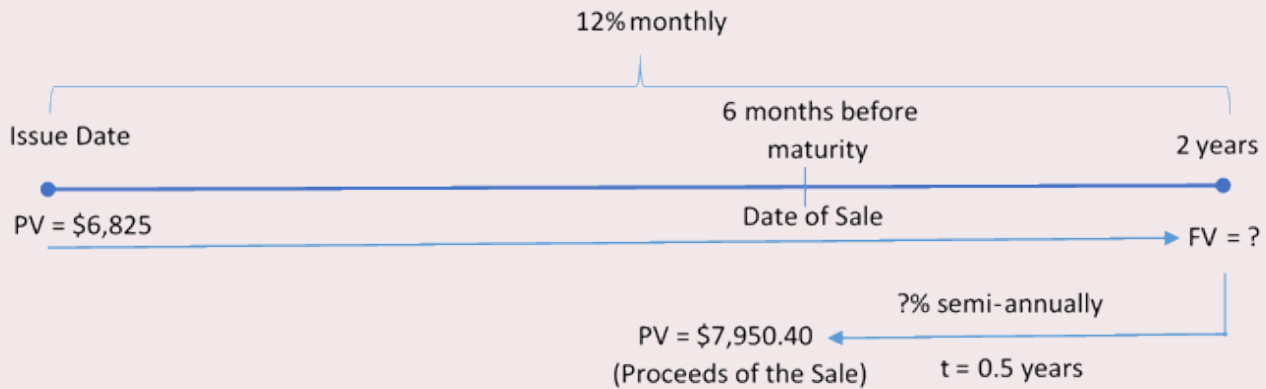


Figure 6.2.2: Timeline of a Promissory Note [Image Description]

**Step 2:** Calculate the maturity value of the note at the end of the two-year term.

**For the two-year term:**

$I/Y = 12\%$ ;  $C/Y = \text{monthly} = 12$

$$i = \frac{I/Y}{C/Y} = \frac{12\%}{12} = 1\% = 0.01$$

$$n = C/Y \times \text{Number of Years} = 12 \times 2 = 24$$

$$FV = \$6,825(1 + 0.01)^{24} = \$8,665.94$$

**Step 3:** Using the future value formula, rearrange and solve for  $i$ .

Proceeds of the sale (PV) = \$7,950.40;  $C/Y = \text{semi-annually} = 2$

$$n = C/Y \times (\text{Number of Years}) = 2 \times 0.5 = 1$$

$$FV = PV(1 + i)^n$$

$$\$8,665.94 = \$7,950.40(1 + i)^1$$

$$1.090000 = 1 + i$$

$$i = 0.090000$$

**Step 4:** Using the formula for the periodic interest rate  $i$ , rearrange and solve for  $I/Y$ .

$$i = \frac{I/Y}{C/Y}$$

$$0.09 = \frac{I/Y}{2}$$

$$I/Y = 0.090000 \times 2$$

$$= 0.18 \text{ or } 18\% \text{ compounded semi-annually}$$

The sale of the promissory note is based on a maturity value of \$8,665.94. The finance company used a discount rate of 18% compounded semi-annually to arrive at proceeds of \$7,950.40.

## NON-INTEREST-BEARING PROMISSORY NOTES

A non-interest-bearing promissory note involves either truly having 0% interest or else already including a flat fee or rate within the note's face value. Therefore, the principal amount and maturity amount of the promissory note are the same. Since the maturity value is known you can just use step 1 and step 3 to calculate the proceeds of the sale.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nscc.ca/businessmath2500/?p=102#h5p-29>

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Determine the proceeds of the sale on a six-year interest-bearing promissory note for \$5,750 at 6.9% compounded monthly, discounted two years and three months before its due date at a discount rate of 9.9% compounded quarterly.
2. A \$36,555 interest-bearing note at 5% compounded monthly is issued on October 15, 2011, for a term of 87 months. Fifty-seven months later, the note is sold to yield a discount amount of \$11,733.41. What quarterly compounded discount rate is being used?
3. A seven-year interest-bearing note for \$19,950 at 8.1% compounded quarterly is issued on January 19, 2006. Four years and 11 months later, the note is discounted at 14.55% compounded monthly. Determine the proceeds on the note and how much interest the original owner of the note realized.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=102#h5p-30>

Timelines for the exercises are included in Solution to Exercises.

## 6.3 Application: Strip Bonds

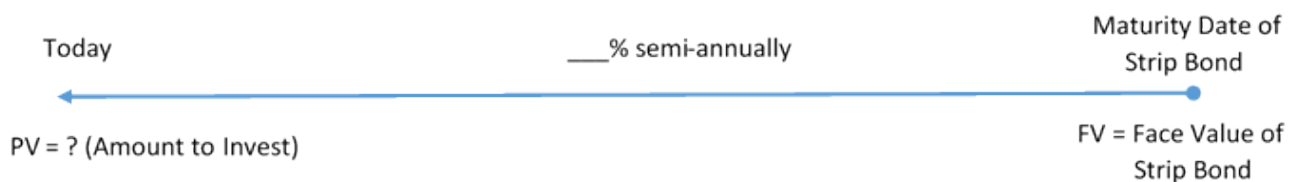
A **strip bond** is a marketable bond that has been stripped of all interest payments and is one of the many financial tools through which you may earn nontaxable income inside your Registered Retirement Savings Plan (RRSP). Mathematically, a strip bond essentially is a long-term version of the treasury bill. Whereas T-bills are found with terms of less than one year, strip bonds have longer terms because of the large sums of money that governments or large corporations require. These large sums take a long time to pay back.

### HOW IT WORKS

The future value of a strip bond is always known since it is the face value. You calculate the present value or purchase price by applying the following steps:

**Step 1:** Determine the face value (FV) of the strip bond, the years between the date of the sale and the maturity date (Years), the yield (I/Y) on the date of the sale, and the compounding frequency (C/Y). If needed, draw a timeline similar to the one below, which illustrates a typical strip bond timeline.

The compounding on strip bonds is assumed to be semi-annual unless stated otherwise. Thus,  $C/Y = 2$  in most scenarios.



**Step 2:** Calculate the periodic interest rate ( $i$ ) and the number of compounding periods ( $n$ ) using the formulas

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compoundings per Year (C/Y)}}$$

$$n = C/Y \times (\text{Number of Years})$$

**Step 3:** Calculate the present value (PV) of the strip bond using the formula

$$PV = \frac{FV}{(1 + i)^n}$$

**Step 4:** If you are interested in the actual dollar amount that the investor gains by holding onto the strip bond until maturity, use the formula for the amount of interest.

$$I = FV - PV$$

#### Example 6.3.1: Purchase Price of a Strip Bond

Johansen is considering purchasing a \$50,000 face value Government of Canada strip bond with 23½ years until maturity. The current market yield for these bonds is posted at 4.2031%. What is the price of the strip bond today, and how much money is gained if the bond is held until maturity?

#### Solution

**Step 1:** The timeline below illustrates the scenario.

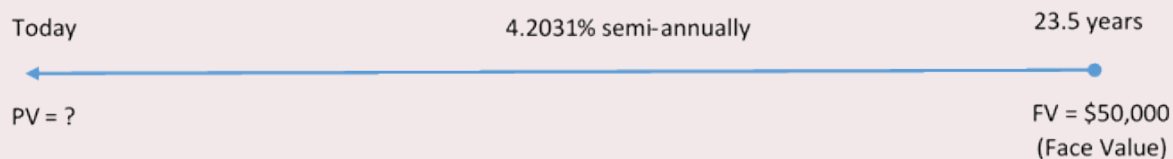


Figure 6.3.1: Timeline for a Strip Bond [Image Description]

**Step 2:** Calculate the periodic interest rate ( $i$ ) and the number of compounding periods ( $n$ ).

$$i = \frac{I/Y}{C/Y} = \frac{4.2031\%}{2} = 2.10155\% \text{ or } 0.0210155$$

$$n = C/Y \times (\text{Number of Years}) = 2 \times 23.5 = 47$$

**Step 3:** Calculate the present value, PV.

$$\begin{aligned} PV &= \frac{FV}{(1 + i)^n} \\ &= \frac{\$50,000}{(1 + 0.0210155)^{47}} \\ &= \$18,812.66 \end{aligned}$$

**Step 4:** Calculate the amount of interest.

$$I = FV - PV = \$50,000 - \$18,812.66 = \$31,187.34$$

The strip bond has a purchase price of \$18,812.66 today. If you hold onto the strip bond until maturity, you will receive a payment of \$50,000 23½ years from today. This represents a \$31,187.34 gain.

## NOMINAL YIELDS ON STRIP BONDS

To calculate nominal yields you need the same four formulas as for the strip bond's purchase price.

### HOW IT WORKS

Follow these steps to calculate the nominal yield of a strip bond:

**Step 1:** Determine the present value or purchase price of the strip bond (PV). This may already be known, or you may have to calculate this amount using the previously introduced steps for calculating the present value.

**Step 2:** Determine the future value (FV) of the strip bond. If it is the maturity of the strip bond, then the future value is the face value of the bond. If the investor is selling the strip bond prior to maturity, then this number is based on a present value calculation using the yield at the time of sale and time remaining until maturity.

**Step 3:** Determine the years between the purchase and the sale of the strip bond. Using  $CY = C/Y = 2$  (unless otherwise stated), calculate the number of compounding periods (n) using the formula

$$n = C/Y \times (\text{Number of Years})$$

**Step 4:** Calculate the periodic interest rate (i). Use Formula 9.2B, rearrange and solve for i.

**Step 5:** Calculate the nominal interest rate (I/Y).  
Use Formula 9.1, rearrange and solve for I/Y.

**Step 6:** If you are interested in the actual dollar amount that the investor gained while holding the strip bond, use the formula for the amount of interest.

$$I = FV - PV$$

Example 6.3.2: The Investor's Actual Yield (PV known)

Assume that when market yields were 4.254% an investor purchased a \$25,000 strip bond 20 years before maturity. The purchase price was \$10,772.52. The investor then sold the bond five years later, when current yields dropped to 3.195%. The selling price was \$15,539.94. The investor wants to know the gain and the actual yield realized on her investment.

**Solution**

**Step 1:**  $PV = \$10,772.52$ . This is the purchase price paid for the strip bond.

**Step 2:**  $FV = \$15,539.94$ . This is the selling price received for the strip bond.

**Step 3:**  $C/Y = 2$ ; Years = 5 (between the purchase and sale price);

$$n = C/Y \times (\text{Number of Years}) = 2 \times 5 = 10$$

**Step 4:** Calculate the periodic interest rate,  $i$ .

$$FV = PV(1 + i)^n$$

$$\$15,539.94 = \$10,772.52(1 + i)^{10}$$

$$1.442554 = (1 + i)^{10}$$

$$1.442554^{\frac{1}{10}} = 1 + i$$

$$1.037321 = 1 + i$$

$$i = 0.037321$$

**Step 5:** Calculate the nominal interest rate,  $I/Y$ .

$$i = \frac{I/Y}{C/Y}$$

$$I/Y = i \times C/Y$$

$$= 0.037321 \times 2$$

$$= 0.074642\% \text{ or } 7.4642\%$$

**Step 6:** Calculate the actual gain (interest earned).

$$I = FV - PV = \$15,539.94 - \$10,772.52 = \$4,767.42$$

The investor gained \$4,767.42 on the investment, representing an actual yield of 7.4642% compounded semiannually.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=107#h5p-31>

### Example 6.3.3: The Investor's Actual Yield (PV unknown)

Cameron invested in a \$97,450 face value strip bond through the Bank of Nova Scotia with 25 years until maturity. Market yields at that time were 9.1162%. Ten and half years later Cameron needed the money, so he sold the bond when current yields were 10.0758%. Calculate the actual yield and gain that Cameron realized on his investment.

#### Solution

The timeline below illustrates the situation.

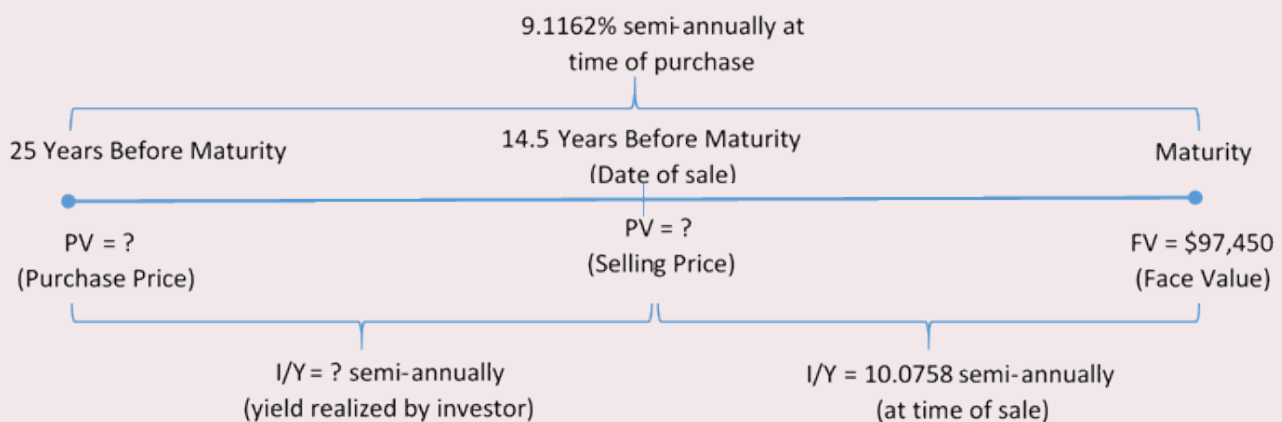


Figure 6.3.3: Timeline [Image Description]

**Step 1:** Calculate the purchase price (PV) based on the purchase date.

$I/Y = 9.1162\%$ ;  $C/Y = 2$ ; Years = 25

$$i = \frac{I/Y}{C/Y} = \frac{9.1162\%}{2} = 4.5581\%$$

$$n = C/Y \times (\text{Number of Years}) = 2 \times 25 = 50$$

$$\begin{aligned}
 PV &= \frac{\$97,450}{(1 + 0.045581)^{50}} \\
 &= \$10,492.95
 \end{aligned}$$

**Step 2:** Calculate the selling price (PV) based on the selling date.

$I/Y = 10.0758\%$ ;  $C/Y = 2$ ; Years = 14.5

$$i = \frac{I/Y}{C/Y} = \frac{10.0758\%}{2} = 5.0379\%$$

$$n = C/Y \times (\text{Number of Years}) = 2 \times 14.5 = 29$$

$$\begin{aligned}
 PV &= \frac{\$97,450}{(1 + 0.050379)^{29}} \\
 &= \$23,428.63
 \end{aligned}$$

This becomes the FV for subsequent calculations.

**Step 3:**  $C/Y = 2$ ; Years = 14.5

$$n = C/Y \times (\text{Number of Years}) = 2 \times 10.5 = 21$$

**Step 4:** Calculate the periodic interest rate,  $i$ .

$PV = \$10,492.95$ ;  $FV = \$23,428.63$

$$\begin{aligned}
 FV &= PV(1 + i)^n \\
 \$23,428.63 &= \$10,492.95(1 + i)^{21} \\
 2.232797 &= (1 + i)^{21} \\
 2.232797^{\frac{1}{21}} &= 1 + i \\
 1.038991 &= 1 + i \\
 i &= 0.038991
 \end{aligned}$$

**Step 5:** Calculate the nominal interest rate,  $I/Y$ .

$$\begin{aligned}
 I/Y &= i \times C/Y \\
 &= 0.038991 \times 2 \\
 &= 0.077982\% \text{ or } 7.7982\%
 \end{aligned}$$

**Step 6:** Calculate the actual gain (interest earned).

$$I = FV - PV = \$23,428.63 - \$10,492.95 = \$12,935.68$$

Cameron sold the strip bond when prevailing market rates were higher than the rate at the time of purchase. He has realized a lower yield of 7.7982% compounded semi-annually, which is a gain of \$12,935.68.

## Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. A \$15,000 face value Government of Manitoba strip bond has 19.5 years left until maturity. If the current market rate is posted at 6.7322% compounded semi-annually, what is the purchase price for the bond?
2. An investor purchased a \$7,500 face value strip bond for \$2,686.01 on May 29, 2006. The strip bond had been issued on May 29, 2002, with a 25-year maturity. The investor sold the strip bond on November 29, 2012, for \$3,925.28.
  - a) What was the market yield when the investor purchased the strip bond?
  - b) What was the market yield when the investor sold the strip bond?
  - c) What actual yield did the investor realize on the strip bond?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=107#h5p-32>

Timelines for the exercises are included in Solution to Exercises.

## 6.4 Review, Symbols, and Formulas

### KEY CONCEPTS

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#### Application: Long-Term GICs

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- The characteristics and calculations involved with an interest payout GIC
- The characteristics and calculations involved with compound interest GICs
- The characteristics and calculations involved with escalator GICs

#### Application: Long-Term Promissory Notes

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- The sale of interest-bearing promissory notes
- The sale of non-interest bearing promissory notes

#### Application: STRIP Bonds

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- Key characteristics of strip bonds
- Purchase price of a strip bond
- Nominal yields on strip bonds

### THE FORMULAS YOU NEED TO KNOW

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#### Symbols Used

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$I$  = Interest payment amount

$i$  = Periodic interest rate

$n$  = Number of compounding periods

$FV$  = Future or maturity value

$PV$  = Principal or present value

#### FORMULAS USED

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Periodic Interest Amount:

$$I = PV \times i$$

Periodic Interest Rate:

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compoundings per Year (C/Y)}}$$

Maturity Value with Fixed Interest Rate:

$$FV = PV(1 + i)^n$$

Present Value or Proceeds of Sale of Promissory Note:

$$PV = \frac{FV}{(1 + i)^n}$$

## 7 COMPOUND INTEREST - ANNUITIES

### Learning Objectives

- Define and distinguish between ordinary simple annuities and ordinary general annuities and annuities due.
- Calculate the future value and present value of both ordinary simple annuities and ordinary general annuities and annuities due.
- Calculate the fair market value of a cash flow stream that includes an annuity.
- Calculate the principal balance owed on a loan immediately after any payment.
- Calculate the periodic payment in ordinary simple and ordinary general annuities and annuities due.
- Calculate the number of payments in ordinary simple and ordinary general annuities and annuities due.
- Calculate the interest rate in ordinary simple and ordinary general annuities and annuities due.

## 7.1 Fundamentals of Annuities

An **annuity** is a continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation. An **annuity payment** is the dollar amount of the equal periodic payment in an annuity environment. The payments are continuous, equal, periodic, and occur over a fixed time frame. If any one of these four characteristics is not satisfied, then the financial transaction fails to meet the definition of a singular annuity and requires other techniques and formulas to solve.

There are four types of annuities, which are based on the combination of two key characteristics: timing of payments and frequency.

### FOUR TYPES OF ANNUITIES

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#### Ordinary Simple Annuity

---

An **ordinary simple annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

#### Ordinary General Annuity

---

An **ordinary general annuity** has the following characteristics:

- Payments are made at the end of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs one interval after the beginning of the annuity.
- The last payment occurs on the same date as the end of the annuity.

### Simple Annuity Due

A **simple annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are equal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

### General Annuity Due

A **general annuity due** has the following characteristics:

- Payments are made at the beginning of the payment intervals, and the payment and compounding frequencies are unequal.
- The first payment occurs on the same date as the beginning of the annuity.
- The last payment occurs one payment interval before the end of the annuity.

The table below summarizes the four types of annuities and their characteristics for easy reference.

Table 7.1.1. Four Types of Annuities and Their Characteristics				
Annuity Type	Timing of Payments in a Payment Interval	Payment Frequency and Compounding Frequency	Start of Annuity and First Payment Same Date?	End of Annuity and Last Payment Same Date?
Ordinary Simple Annuity	End	Equal	No, first payment one interval later	Yes
Ordinary General Annuity	End	Unequal	No, first payment one interval later	Yes
Simple Annuity Due	Beginning	Equal	Yes	No, last payment one interval earlier
General Annuity Due	Beginning	Unequal	Yes	No, last payment one interval earlier

One of the most challenging aspects of annuities is recognizing whether the annuity you are working with is ordinary or due. This distinction plays a critical role in formula selection later in this chapter. To help you recognize the difference, the table below summarizes some key words along with common applications in which the annuity may appear.

Table 7.1.1.2. Ordinary Versus Due Annuities

Type	Key Words or Phrases	Common Applications
Ordinary	-... payments are at the end... -... payments do not start today... -... payments are later... -... first payment next interval...	- bank loans of any type - mortgages - bonds - Canada Pension Plan (CPP)
Due	-...payments are at the beginning... -...payments start today... -... payments are in advance... -...first payment today... -...payments start now...	- any kind of lease - any kind of rental - RRSPs (usually) - membership dues - insurance

## ANNUITIES VERSUS SINGLE PAYMENTS

To go from single payments in Chapter 9 to annuities in this chapter, you need to make several adaptations:

**Annuity Payment Amount (PMT).** Annuity calculations require you to tie a value to this variable in the formulas and when you use technology such as the BAII+ calculator.

**Payment Frequency or Payments per Year (P/Y).** When you work with annuities, an actual value for P/Y is determined by the payment frequency. For simple annuities P/Y remains the same as C/Y, whereas the variables are different for general annuities.

**Cash Flow Sign Convention on the Calculator.** It now becomes critical to ensure the proper application of the cash flow sign convention on the calculator—failure to do so will result in an incorrect answer. For example, if you borrow money and then make annuity payments on it, you enter the present value (PV) as a positive (you received the money) while you enter the annuity payments as negatives (you paid the money to the bank). This results in future balances getting smaller and you owing less money. If you inadvertently enter the annuity payment as a positive number, this would mean you are borrowing more money from the bank so your future balance would increase and you would owe more money. These two answers are very different!

**Definition and Computation of  $n$ .** When you worked with single payments,  $n$  was defined as the total number of compounds throughout the term of the financial transaction. When you work with annuities, is defined as the total number of payments throughout the term of the annuity. You calculate it using Formula 11.1 below.

## The Formula

### Formula 7.1: Number of Annuity Payments

$$n = P/Y \times \text{Number of Years}$$

- where,
- $n$  is the total number of annuity payments.

- $P/Y$  is the number of payments per year.

### How It Works

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On a two-year loan with monthly payments and semi-annual compounding, the payment frequency is monthly, or 12 times per year. With a term of two years, that makes  $n = 2 \times 12 = 24$  payments. Note that the calculation of  $n$  for an annuity does not involve the compounding frequency.

### ADAPTING TIMELINES TO INCORPORATE ANNUITIES

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A good annuity timeline should illustrate the present value (PV), future value (FV), number of annuity payments ( $n$ ), nominal interest rate ( $I/Y$ ), compounding frequency ( $C/Y$ ), annuity payment (PMT), and the payment frequency ( $P/Y$ ). One of these variables will be the unknown.

## 7.2 Future Value of Annuities

The **future value of any annuity** equals the sum of all the future values for all of the annuity payments when they are moved to the end of the last payment interval. For example, assume you will make \$1,000 contributions at the end of every year for the next three years to an investment earning 10% compounded annually. This is an ordinary simple annuity since payments are at the end of the intervals, and the compounding and payment frequencies are the same. If you wanted to know how much money you have in your investment after the three years, you would apply the fundamental concept of the time value of money to move each payment amount to the future date (the focal date) and sum the values to arrive at the future value.

### THE FORMULA

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#### Formula 7.2A Ordinary Annuity Future Value

$$FV_{ORD} = PMT \left[ \frac{(1 + i)^n - 1}{i} \right]$$

where,

- PMT is the regular payment amount.
- $i$  is the periodic interest rate.
- $n$  is the total number of payments.

### CALCULATING THE PERIODIC INTEREST RATE ( $i$ )

---

For **ordinary simple annuities** where the compounding interval equals the payment interval ( $P/Y = C/Y$ ) you calculate the periodic rate ( $i$ ) using the formula

$$i = \frac{I/Y}{C/Y}$$

For **ordinary general annuities** where the compounding interval does not equal the payment interval ( $P/Y \neq C/Y$ ) you need to calculate the equivalent periodic rate ( $i_{eq}$ ) per payment interval using the formula

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

where,

- $i$  is the given periodic rate.
- $C/Y$  is the number of compounds per year.
- $P/Y$  is the number of payments per year.

The future value formula for ordinary general annuities becomes

$$FV_{ORD} = PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right]$$

## HOW IT WORKS

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There is a five-step process for calculating the future value of any ordinary annuity:

**Step 1:** Identify the annuity type (simple or general). Draw a timeline to visualize the question.

**Step 2:** Identify the known variables, including PV,  $I/Y$ ,  $C/Y$ , PMT,  $P/Y$ , and Years.

**Step 3:** Calculate the periodic interest rate ( $i$ ). See Important Notes above.

Calculate the number of payments ( $n$ ) using the formula

$$n = P/Y \times (\text{Number of Years})$$

**Step 4:** If  $PV = \$0$ , proceed to step 5. If there is a nonzero value for PV, treat it like a single payment and calculate the future value using Formula 5.2.

$$FV_{(1)} = PV(1 + i)^n$$

**Step 5:** Apply Formula 7.2A to calculate the future value.

If you calculated a future value in step 4, combine the future values from steps 4 and 5 to arrive at the total future value.

## CALCULATING THE INTEREST AMOUNT

---

For investment annuities, if you are interested in knowing how much of the future value is principal and how much is interest, you can use the formula

$$I = FV - \text{Total Contributions} = FV - (n \times PMT + PV)$$

where PV is the initial contribution.

## YOUR BAI PLUS CALCULATOR

---

Adapting your calculator skills to suit annuities requires three important changes:

1. Enter your values for PV (if known) and PMT. Be sure to enter it with the correct cash flow sign convention. When you invest, the payment has the same sign as the PV. When you borrow, the sign of the payment is opposite that of PV.
2. Enter your values for P/Y and C/Y separately. Access the function by pressing 2nd P/Y to find the following entry fields, through which you can scroll using your arrow buttons. To enter any information into one of these fields, scroll to the field on your screen, key in the data, and press Enter. When you enter a value into the P/Y field, the calculator will automatically copy the value into the C/Y field for you. If in fact the C/Y is different, you can change the number manually. To exit the P/Y window, press 2nd Quit.

## CONCEPT CHECK

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An interactive H5P element has been excluded from this version of the text. You can view it online here:

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### Example 7.2.1: Future Value of an Investment Account

A financial adviser is reviewing one of her client's accounts. The client has been investing \$1,000 at the end of every quarter for the past 11 years in a fund that has averaged 7.3% compounded quarterly. How much money does the client have today in his account?

#### **Solution**

**Step 1:** The payments are at the end of the payment intervals, and both the compounding period and the payment intervals are the same. This is an ordinary simple annuity. Calculate its value at the end, which is its future value, or  $FV_{ORD}$ .

The timeline shows the client's account.

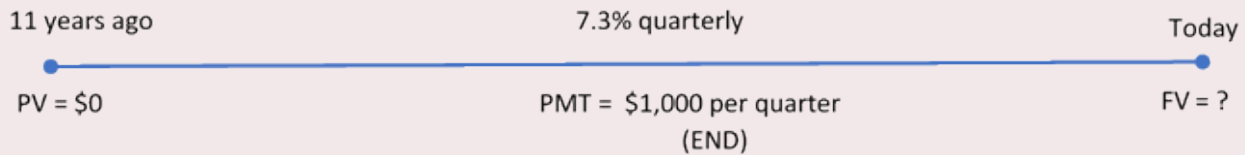


Figure 7.2.1: Timeline [Image Description]

**Step 2:** PV = \$0; I/Y = 7.3%; C/Y = 4; PMT = \$1,000; P/Y = 4; Years = 11

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{7.3\%}{4} = 1.825\%$$

**Step 4:** Skip this step since PV=\$0.

**Step 5:** Apply Formula 7.2A to calculate the future value,  $FV_{ORD}$ .

$$n = P/Y \times (\text{Number of Years}) = 4 \times 11 = 44$$

$$\begin{aligned} FV_{ORD} &= PMT \left[ \frac{(1+i)^n - 1}{i} \right] \\ &= \$1,000 \left[ \frac{(1+0.01825)^{44} - 1}{0.01825} \right] \\ &= \$66,637.03 \end{aligned}$$

Therefore,  $FV_{ORD} = FV = \$66,637.03$

**Calculator instructions:**

Table 7.2.1. Calculator Instructions for Example 11.2.1

N	I/Y	PV	PMT	FV	P/Y	C/Y
44	7.3	0	-1,000	Answer: 66,637.03449	4	4

The figure shows how much principal and interest make up the final balance. After 11 years of \$1,000 quarterly contributions, the client has \$66,637.03 in the account.

### Example 7.2.2: Future Value of a Savings Annuity (PV given)

A savings annuity already contains \$10,000. If an additional \$250 is invested at the end of every month at 9% compounded semi-annually for a term of 20 years, what will be the maturity value of the investment?

**Solution**

**Step 1:** The payments are at the end of the payment intervals, and the compounding period and payment intervals are different  $P/Y \neq C/Y$ . This is an ordinary general annuity. Calculate its value at the end, which is its future value, or  $FV_{ORD}$ .

The timeline is shown below.

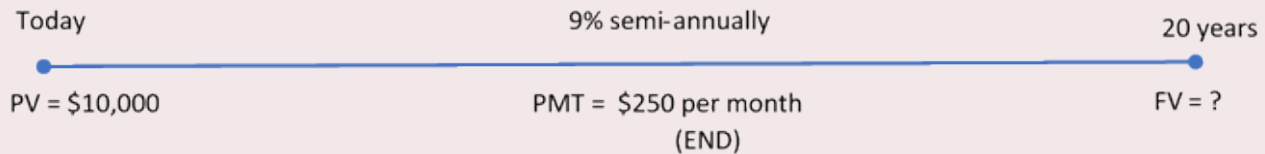


Figure 7.2.2: Timeline [Image Description]

**Step 2:**  $PV = 10,000$ ;  $I/Y = 9\%$ ;  $C/Y = 2$ ;  $PMT = \$250$ ;  $P/Y = 12$ ; Years = 20

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate,  $i_{eq}$ .

$$i = \frac{I/Y}{C/Y} = \frac{9\%}{2} = 4.5\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.045)^{\frac{2}{12}} - 1 = 0.007363123$$

**Step 4:** Apply Formula 7.2A to calculate the future value,  $FV_1$ .

Given information:  $PV = \$10,000$ ;  $n = P/Y \times (\text{Number of Years}) = 12 \times 20 = 240$

$$FV_1 = \$10,000(1 + 0.007363123)^{240} = \$58,163.64538$$

**Step 5:** Apply Formula 7.2A to calculate the future value of the payments,  $FV_{ORD}$ .

$$\begin{aligned} FV_{ORD} &= PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right] \\ &= \$250 \left[ \frac{(1 + 0.007363123)^{240} - 1}{0.007363123} \right] \\ &= \$163,529.9486 \end{aligned}$$

Combine steps 4 and 5 to calculate the total future value,  $FV$ .

$$FV = \$58,163.64538 + \$163,529.9486 = \$221,693.59$$

**Calculator instructions:**

Table 7.2.2. Calculator Instructions for Example 11.2.2

N	I/Y	PV	PMT	FV	P/Y	C/Y
240	9	-10,000	-250	Answer: 221,693.5946	12	2

The figure shows how much principal and interest make up the final balance. The savings annuity will have a balance of \$221,693.59 after the 20 years.

## IMPORTANT NOTES

If any of the variables, including I/Y, C/Y, PMT, or P/Y change between the start and end point of the annuity, or if any additional single payment deposit or withdrawal is made, a new time segment is created and must be treated separately. There will then be multiple time segments that require you to work left to right by repeating steps 3 through 5 in the procedure. The future value at the end of one time segment becomes the present value in the next time segment.

Pay extra attention when the variable that changes between time segments is the payment frequency (P/Y). When inputted into a BAII+ calculator, the P/Y automatically copies across to the compounding frequency (C/Y). Unless your C/Y also changed to the same frequency, this means that you must scroll down to the C/Y window and re-enter the correct value for this variable, even if it didn't change. The following example illustrates this concept.

### Example 7.2.3: Saving Up for a Vacation

Genevieve has decided to start saving up for a vacation in two years, when she graduates from university. She already has \$1,000 saved today. For the first year, she plans on making end-of-month contributions of \$300 and then switching to end-of-quarter contributions of \$1,000 in the second year. If the account can earn 5% compounded semi-annually in the first year and 6% compounded quarterly in the second year, how much money will she have saved when she graduates?

#### Solution

**Step 1:** There is a change of variables after one year. As a result, you need a Year 1 time segment and a Year 2 time segment. In both segments, payments are at the end of the period. In Year 1, the compounding period and payment intervals are different. In Year 2, the compounding period and payment intervals are the same. This is an ordinary general annuity followed by an ordinary simple annuity. You aim to calculate the future value,  $FV_{\text{ORD}}$ .

The timeline for her vacation saving appears below.

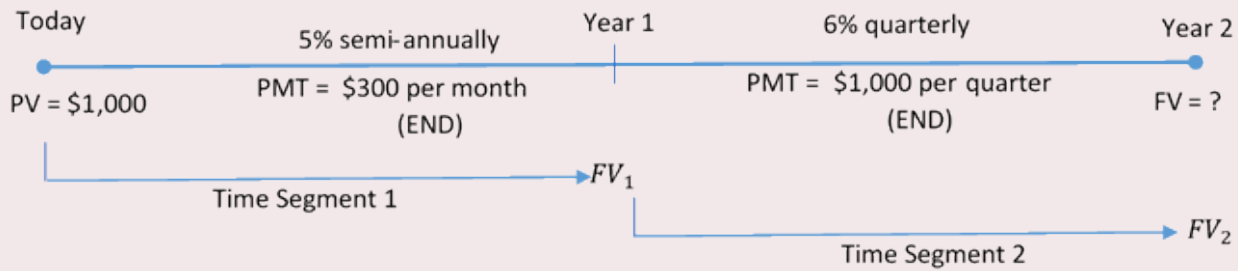


Figure 7.2.3: Timeline [Image Description]

### Step 2:

**Time Segment 1:** PV = \$1,000; I/Y = 5%; C/Y = 2; PMT = \$300; P/Y = 12; Years = 1

**Time Segment 2:** PV = FV<sub>1</sub>; I/Y = 6%; C/Y = 4; PMT = \$1,000; P/Y = 4; Years = 1

**For the first time segment:**

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate,  $i_{eq}$ .

$$i = \frac{I/Y}{C/Y} = \frac{5\%}{2} = 2.5\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.025)^{\frac{2}{12}} - 1 = 0.004123915$$

**Step 4:** Apply Formula 5.2A to calculate the future value,  $FV_{(1)}$ .

Given information: PV = \$1,000;  $n = P/Y \times (\text{Number of Years}) = 12 \times 1 = 12$

$$FV_{(1)} = \$1,000(1 + 0.004123915)^{12} = \$1,050.625$$

**Step 5:** Apply Formula 7.2A to calculate the future value of the payments.

$$\begin{aligned} FV_{ORD_1} &= PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right] \\ &= \$300 \left[ \frac{(1 + 0.004123915)^{12} - 1}{0.004123915} \right] \\ &= \$3,682.786451 \end{aligned}$$

Calculate the total future value,  $FV_1$ .

$$FV_1 = \$1,050.625 + \$3,682.786451 = \$4,733.411451$$

This becomes PV for the second time segment.

**For the second time segment:**

**Step 3:** Since  $P/Y = C/Y$ , calculate the periodic rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{6\%}{4} = 1.5\%$$

**Step 4:** Apply Formula 5.2A to calculate the future value,  $FV_{(2)}$ .

$PV = \$4,733.411442 = FV_1$ ;  $n = P/Y \times (\text{Number of Years}) = 4 \times 1 = 4$

$$FV_{(2)} = PV(1 + i)^n = \$4,733.411442(1 + 0.015)^4 = \$5,023.870384$$

**Step 5:** Apply Formula 7.2A to calculate the future value of the payments.

$$\begin{aligned} FV_{ORD_2} &= PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \\ &= \$1,000 \left[ \frac{(1 + 0.015)^4 - 1}{0.015} \right] \\ &= \$4,090.903375 \end{aligned}$$

Combine steps 4 and 5 to calculate the total future value,  $FV_2$ .

$$FV_2 = \$5,023.870384 + \$4,090.903375 = \$9,114.77$$

Therefore,  $FV_2 = FV = \$9,114.77$

**Calculator instructions:**

Table 7.2.3. Calculator Instructions for Example 7.2.3

Time segment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	12	5	-1,000	-300	Answer: 4,733.411451	12	2
2	4	6	-4,733.411451	-1,000	Answer: 9,114.773759	4	4

The figure shows how much principal and interest make up the final balance. When Genevieve graduates she will have saved \$9,114.77 toward her vacation.

## ANNUITIES DUE

An annuity due occurs when payments are made at the beginning of the payment interval.

## THE FORMULA

### Formula 7.2B: Annuity Due Future Value

$$FV_{DUE} = PMT \left[ \frac{(1+i)^n - 1}{i} \right] \times (1+i)$$

We see that the future value of an annuity due is simply  $(1+i)$  times the future value of an ordinary annuity.

## HOW IT WORKS

The steps required to solve for the future value of an annuity due are almost identical to those you use for the ordinary annuity. The only difference lies in step 5, where you use Formula 7.2B instead of Formula 7.2A.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=126#h5p-34>

### Example 7.2.4: Lottery Winnings

The Set for Life instant scratch n' win ticket offers players a chance to win \$1,000 per week for the next 25 years starting immediately upon validation. If a winner was to invest all of his money into an account earning 5% compounded annually, how much money would he have at the end of his 25-year term? Assume each year has exactly 52 weeks.

#### Solution

**Step 1:** The payments start immediately, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate its value at the end, which is its future value, or  $FV_{DUE}$ .

The timeline for the lottery savings is below.

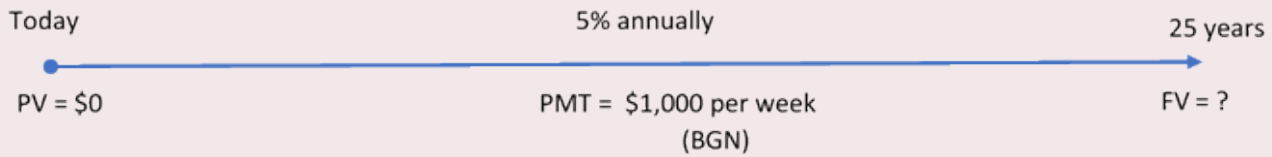


Figure 7.2.4: Timeline [Image Description]

**Step 2:** PV = \$0; I/Y = 5%; C/Y = 1; PMT = \$1,000; P/Y = 52; Years = 25

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate ( $i_{eq}$ ) that matches the payment interval.

$$i = \frac{I/Y}{C/Y} = \frac{5\%}{1} = 5\%$$

$$i_{eq} = \left(1 + i\right)^{\frac{C/Y}{P/Y}} - 1 = \left(1 + 0.05\right)^{\frac{1}{52}} - 1 = 0.000938713 \text{ per week}$$

**Step 4:** Since there is no PV, skip this step.

**Step 5:** Apply Formula 7.2B to calculate the future value of the payments,  $FV_{DUE}$ .

$$n = P/Y \times (\text{Number of Years}) = 52 \times 25 = 1300$$

$$\begin{aligned} FV_{DUE} &= PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right] \times (1 + i_{eq}) \\ &= \$1,000 \left[ \frac{(1 + 0.000938713)^{1300} - 1}{0.000938713} \right] (1 + 0.000938713) \\ &= \$2,544,543.218 \end{aligned}$$

Therefore,  $FV_{DUE} = FV = \$2,544,543.22$

#### Calculator instructions:

Table 7.2.4. Calculator Instructions for Example 11.2.4

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	1300	5	0	-1000	Answer: 2,544,543.218	52	1

The figure shows how much principal and interest make up the final balance. If the winner was to invest all of his lottery prize money, he would have \$2,544,543.22 after 25 years.

#### Example 7.2.5: Saving into a Trust with a Variable Change

When Roberto's son was born, Roberto started making payments of \$1,000 at the beginning of every six months to a trust fund earning 5.75% compounded monthly. After five years, he changed his contributions and started depositing \$500 at the beginning of every quarter. How much money will be in his son's trust fund when his son turns 18?

### Solution

**Step 1:** There is a change of variables after five years. As a result, you need two time segments. In both segments, payments are at the beginning of the period and the compounding periods and payment intervals are different. Therefore, Roberto has two consecutive general annuities due. Combined, calculate the future value, or  $FV_{DUE}$ .

The timeline for the trust fund is shown below.

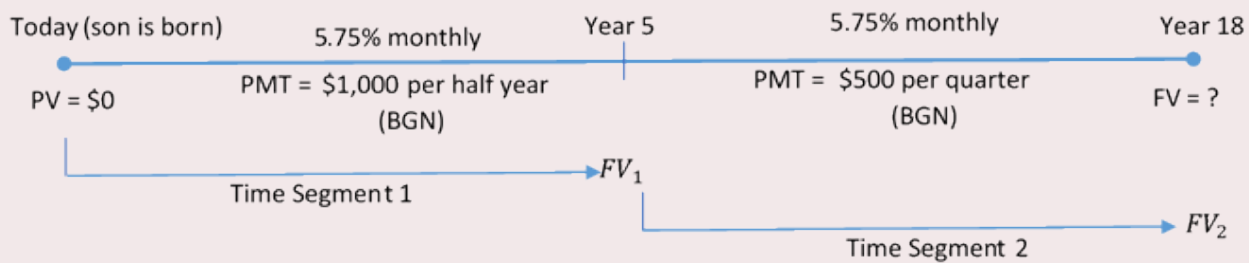


Figure 7.2.5: Timeline [Image Description]

### Step 2:

**Time segment 1:**  $PV = \$0$ ;  $I/Y = 5.75\%$ ;  $C/Y = 12$ ;  $PMT = \$1,000$ ;  $P/Y = 2$ ; Years = 5

**Time segment 2:**  $PV = FV_1$ ;  $I/Y = 5.75\%$ ;  $C/Y = 12$ ;  $PMT = \$500$ ;  $P/Y = 4$ ; Years = 13

**For the first time segment:**

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate,  $i_{eq}$ .

$$i = \frac{I/Y}{C/Y} = \frac{5.75\%}{12} = 0.4719\bar{6}\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.004719\bar{6})^{\frac{12}{2}} - 1 = 0.029096609$$

**Step 4:** Since there is no PV skip this step.

**Step 5:** Apply Formula 7.2B to calculate the future value.

$$n = P/Y \times (\text{Number of Years}) = 2 \times 5 = 10$$

$$\begin{aligned}
 FV_{DUE_1} &= PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right] \times (1 + i_{eq}) \\
 &= \$1,000 \left[ \frac{(1 + 0.029096609)^{10} - 1}{0.029096609} \right] \times (1 + 0.029096609) \\
 &= \$11,748.47466 = FV_1
 \end{aligned}$$

This becomes PV for the second time segment.

**For the second time segment:**

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate,  $i_{eq}$ .

$i$  remains unchanged  $= 0.4719\bar{6}\%$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.004719\bar{6})^{\frac{12}{4}} - 1 = 0.01444399$$

**Step 4:** Apply Formula 5.2A to calculate the future value,  $FV_{(2)}$ .

Given information:  $PV = \$11,748.47466 = FV_1$ ;

$n = P/Y \times (\text{Number of Years}) = 4 \times 13 = 52$

$$FV_{(2)} = PV(1 + i_{eq})^n = \$11,748.47466(1 + 0.01444399)^{52} = \$24,765.17$$

**Step 5:** Apply Formula 7.2B to calculate the future value of the payments.

$$\begin{aligned}
 FV_{DUE_2} &= PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right] \times (1 + i_{eq}) \\
 &= \$500 \left[ \frac{(1 + 0.01444399)^{52} - 1}{0.01444399} \right] \times (1 + 0.01444399) \\
 &= \$38,907.21529
 \end{aligned}$$

Combine steps 4 and 5 to calculate the total future value,  $FV_2$ .

$$FV_2 = \$24,765.17 + \$38,907.21529 = \$63,672.38529$$

Therefore,  $FV_2 = FV = \$63,672.39$

**Calculator instructions:**

Table 7.2.5. Calculator Instructions for Example 11.2.5

Time segment	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	10	5.75	0	-1,000	Answer: 11,748.47466	2	12
2	$\sqrt{\text{">BGN}}$	52	$\sqrt{\text{">5.75}}$	-11,748.47466	-500	Answer: 63,672.385 29	4	12

The figure shows how much principal and interest make up the final balance. When Roberto's son turns 18, the trust fund will have a balance of \$63,672.39.

## Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. You are a financial adviser. Your client is thinking of investing \$600 at the end of every six months for the next six years with the invested funds earning 6.4% compounded semi-annually. Your client wants to know how much money she will have after six years. What do you tell your client?
2. McDonald's major distribution partner, The Martin-Brower Company, needs at least \$1 million to build a new warehouse in Medicine Hat two years from today. To date, it has invested \$500,000. If it continues to invest \$50,000 at the end of every quarter into a fund earning 6% quarterly, will it have enough money to build the warehouse two years from now? Show calculations to support your answer.
3. The Saskatchewan Roughriders started a rainy-day savings fund three-and-a-half years ago to help pay for stadium improvements. At the beginning of every quarter the team has deposited \$20,000 into the fund, which has been earning 4.85% compounded semi-annually. How much money is in the fund today?
4. Carlyle plans to make month-end contributions of \$400 to his RRSP from age 20 to age 40. From age 40 to age 65, he plans to make no further contributions to his RRSP. The RRSP can earn 9% compounded annually from age 20 to age 60, and then 5% compounded annually from age 60 to age 65. Under this plan, what is the maturity value of his RRSP when he turns 65?



An interactive H5P element has been excluded from this version of the text. You can view it online here: <https://pressbooks.nsc.ca/businessmath2500/?p=126#h5p-35>

Timeline for exercise 4 is included in Solution to Exercises.

## 7.3 Present Value of Annuities

The present value of any annuity is equal to the sum of all of the present values of all of the annuity payments when they are moved to the beginning of the first payment interval. For example, assume you will receive \$1,000 annual payments at the end of every payment interval for the next three years from an investment earning 10% compounded annually. How much money needs to be in the annuity at the start to make this happen? In this case, you have an ordinary simple annuity.

With an annuity due, the first payment occurs at the beginning of the first period. The key difference is that the annuity due has one less compound of interest to remove.

### THE FORMULA

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#### Formula 7.3A: Ordinary Annuity Present Value:

$$PV_{ORD} = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right]$$

#### Formula 7.3B: Annuity Due Present Value

$$PV_{DUE} = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \times (1 + i)$$

where,

- PMT is the regular payment amount.
- i is the periodic interest rate.
- n is the total number of payments.

### HOW TO USE THE PRESENT VALUE FORMULA FOR ORDINARY ANNUITIES AND ANNUITIES DUE

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For **simple annuities** you need to calculate the periodic interest rate, i.

$$i = \frac{I/Y}{C/Y}$$

For **general annuities** you need to calculate the equivalent periodic rate ( $i_{eq}$ ) that matches the payment interval using the formula

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1$$

where,

- $i$  is the given periodic rate.
- $C/Y$  is the number of compounds per year.
- $P/Y$  is the number of payments per year.

The present value formula for **general annuities** then becomes

$$PV_{ORD} = PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right]$$

$$PV_{DUE} = PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq})$$

## HOW IT WORKS

There is a five-step process for calculating the present value of any ordinary annuity or annuity due.

**Step 1:** Identify the annuity type. Draw a timeline to visualize the question.

**Step 2:** Identify the known variables, including FV,  $I/Y$ ,  $C/Y$ , PMT,  $P/Y$ , and Years.

**Step 3:** Calculate the periodic interest rate ( $i$ ). See Important Notes above.

Calculate the number of payments ( $n$ ) using the formula

$$n = P/Y \times (\text{Number of Years})$$

**Step 4:** If FV = \$0, proceed to step 5. If there is a nonzero value for FV, treat it like a single payment and calculate the present value  $PV_1$  using Formula 5.3A.

$$PV_1 = \frac{FV}{(1 + i)^n}$$

**Step 5:** Apply Formulas 7.3A or 7.3B to calculate the present value of the payments.

If you calculated a present value in step 4, combine the present values from steps 4 and 5 to arrive at the total present value.

## CALCULATING THE INTEREST AMOUNT

If you are interested in knowing how much interest was removed in the calculation of the present value you can use the formula

$$I = (n \times PMT + FV) - PV$$

where,

- $n$  is the number of payments made.
- $PMT$  is the amount of payment.
- $FV$  is the balance still owing.
- $PV$  is the principal amount or present value.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=135#h5p-36>

### Example 7.3.1: Amount Needed at Time of Retirement

Rodriguez is planning on having an annual gross income of \$50,000 at the end of every year when he retires at age 65. He is planning for the account to be emptied by age 78, which is the average life expectancy for a Canadian man. If the account earns 5.1% compounded annually, what amount of funds needs to be in the account when he retires?

#### Solution

**Step 1:** The payments are at the end of the payment intervals, and the compounding period and payment intervals are the same. This is, therefore, an ordinary simple annuity. Calculate its value at the start, which is its present value, or  $PV_{ORD}$ .

The timeline for the client’s account appears below.

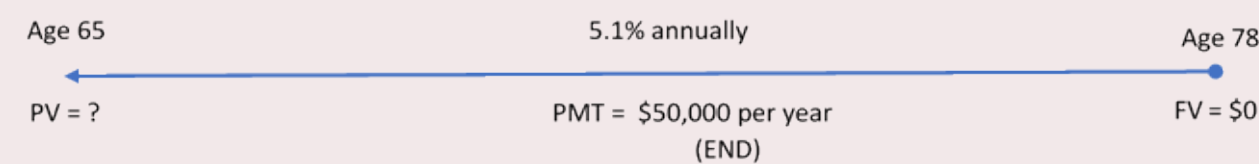


Figure 7.3.1: Timeline [Image Description]

**Step 2:** Given information:

FV = \$0; I/Y = 5.1%; C/Y = 1; PMT = \$50,000; P/Y = 1; Years = 13

**Step 3:** Calculate the periodic interest rate, *i*.

$$i = \frac{I/Y}{C/Y} = \frac{5.1\%}{1} = 5.1\%$$

**Step 4:** Since FV=\$0, skip this step.

**Step 5:** Apply Formula 9.3A to calculate the present value of the payments, PV<sub>ORD</sub>.

$$n = P/Y \times (\text{Number of Years}) = 1 \times 13 = 13$$

$$\begin{aligned} PV_{ORD} &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\ &= \$50,000 \left[ \frac{1 - (1 + 0.051)^{-13}}{0.051} \right] \\ &= \$50,000 \left[ \frac{0.476201}{0.051} \right] \\ &= \$466,863.69 \end{aligned}$$

Therefore, PV<sub>ORD</sub> = PV = \$466,863.69

**Calculator instructions:**

Table 7.3.1. Calculator Instructions for Example 7.3.1

N	I/Y	PV	PMT	FV	P/Y	C/Y
13	5.1	Answer: -466,863.694	50,000	0	1	1

The figure shows how much principal and interest make up the payments. Rodriguez will need to have \$466,863.69 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments.

### Example 7.3.2: Leaving an Inheritance

Recalculate Example 7.3.1 applying three changes:

1. Rodriguez wants to leave a \$100,000 inheritance for his children (assuming he dies at age 78).
2. Payments are at the beginning of the year.
3. His interest rate is 5.1% compounded semi-annually.

Calculate the present.

#### Solution

**Step 1:** The payments are made at the beginning of the payment intervals, and the compounding period (semiannually) and payment intervals (annually) are different. This is now a general annuity due. Calculate its value at the start, which is its present value, or  $PV_{DUE}$ .

The timeline for the client's account appears below.

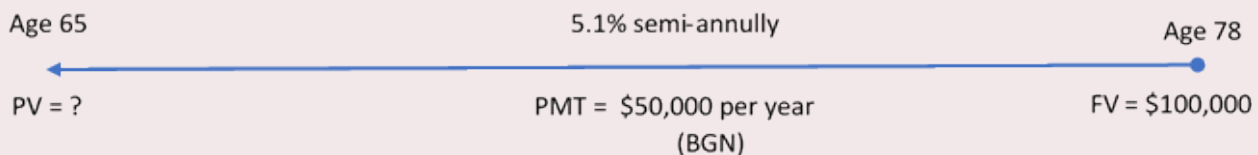


Figure 7.3.2: Timeline [Image Description]

**Step 2:** Given information:

$FV = \$100,000$ ;  $I/Y = 5.1\%$ ;  $C/Y = 2$ ;  $PMT = \$50,000$ ;  $P/Y = 1$ ; Years = 1

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate ( $i_{eq}$ ) that matches the payment interval.

$$i = \frac{I/Y}{C/Y} = \frac{5.1\%}{2} = 2.55\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.0255)^{\frac{2}{1}} - 1 = 0.05165025 \text{ per year}$$

**Step 4:** Apply Formula 5.3A to calculate the present value,  $PV_1$ .

$FV = \$100,000$ ;

$$n = P/Y \times (\text{Number of Years}) = 1 \times 13 = 13$$

$$\begin{aligned}
 PV_1 &= \frac{FV}{(1 + i_{eq})^n} \\
 &= \frac{\$100,000}{(1 + 0.05165025)^{13}} \\
 &= \$51,960.42776
 \end{aligned}$$

**Step 5:** Apply Formula 7.3A to calculate the present value of the payments,  $PV_{DUE}$ .

$$\begin{aligned}
 PV_{DUE} &= PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq}) \\
 &= \$50,000 \left[ \frac{1 - (1 + 0.05165025)^{-13}}{0.05165025} \right] \times (1 + 0.05165025) \\
 &= \$489,066.6372
 \end{aligned}$$

Combine steps 4 and 5 to calculate the total present value, PV.

$$PV = \$51,960.42776 + \$489,066.6372 = \$541,027.07$$

**Step 6:** Calculate the amount of interest.

$$\begin{aligned}
 I &= (n \times PMT + FV) - PV \\
 &= (13 \times \$50,000 + \$100,000) - \$541,027.07 \\
 &= \$750,000 - \$541,027.07 \\
 &= \$208,972.93
 \end{aligned}$$

**Calculator instructions:**

Table 7.3.2. Calculator Instructions for Example 7.3.2

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	13	5.1	Answer: -541,027.065	50,000	100,000	1	2

The figure shows how much principal and interest make up the payments. Rodriguez will require more money, needing to have \$541,027.07 in his account when he turns 65 if he wants to receive 13 years of \$50,000 payments while leaving a \$100,000 inheritance for his children. His account will earn \$208,972.93 over the time frame.

Continuing with the previous two examples, Rodriguez realizes that during his retirement he needs to make some type of adjustment to his annual gross income to account for the rising cost of living. Consequently, he will take \$50,000 at the beginning of each year for six years, then increase it to \$60,000 for the balance. Assume his interest rate is still 5.1% semiannually and that he still wants to leave a \$100,000 inheritance for his children. How much money needs to be in his retirement fund at age 65?

### Solution

**Step 1:** There is a change of variables after six years. As a result, you need two time segments. In both segments, payments are made at the beginning of the period, and the compounding periods and payment intervals are different. These are two consecutive general annuities due. You need to calculate the resulting present value, or  $PV_{DUE}$ .

The timeline for the client's account appears below.

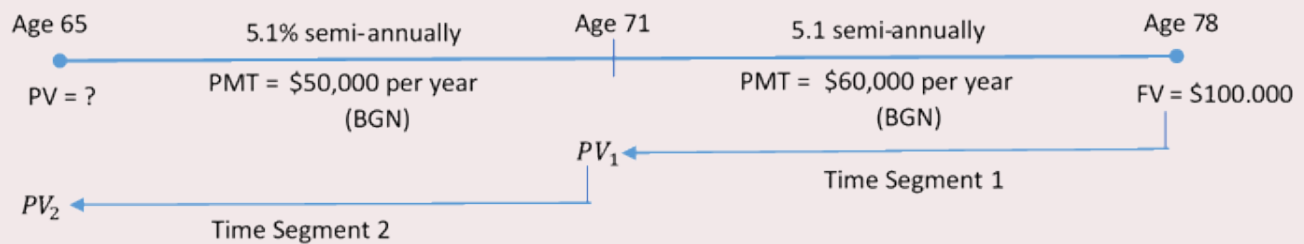


Figure 7.3.3: Timeline [Image Description]

**Step 2:** Given information:

**Time Segment 1:**  $FV = \$100,000$ ;  $I/Y = 5.1\%$ ;  $C/Y = 2$ ;  $PMT = \$60,000$ ;  $P/Y = 1$ ; Years = 7

**Time Segment 2:**  $FV = PV_1$ ;  $I/Y = 5.1\%$ ;  $C/Y = 2$ ;  $PMT = \$50,000$ ;  $P/Y = 1$ ; Years = 6

**For the first time segment:**

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate ( $i_{eq}$ ) that matches the payment interval.

$$i = \frac{I/Y}{C/Y} = \frac{5.1\%}{2} = 2.55\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.0255)^{\frac{2}{1}} - 1 = 0.05165025 \text{ per year}$$

**Step 4:** Apply Formula 9.3A to calculate the present value,  $PV_{(1)}$ .

$FV = \$100,000$ ;

$$n = P/Y \times (\text{Number of Years}) = 1 \times 7 = 7$$

$$\begin{aligned}
 PV_{(1)} &= \frac{FV}{(1 + i_{eq})^n} \\
 &= \frac{\$100,000}{(1 + 0.05165025)^7} \\
 &= \$70,291.15736
 \end{aligned}$$

**Step 5:** Apply Formula 7.3B to calculate the present value of the payments.

$$\begin{aligned}
 PV_{DUE_1} &= PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq}) \\
 &= \$60,000 \left[ \frac{1 - (1 + 0.05165025)^{-7}}{0.05165025} \right] \times (1 + 0.05165025) \\
 &= \$362,940.8778
 \end{aligned}$$

Combine steps 4 and 5 to calculate the total present value  $PV_1$ .

$$PV_1 = \$70,291.15736 + \$362,940.8778 = \$433,232.0352 = FV_1$$

This becomes the future value for the second time segment.

**For the second time segment:**

**Step 3:**  $i_{eq} = 0.05165025$  remains the same.

**Step 4:** Calculate the present value  $PV_{(2)}$  of step 4 in the first time segment.

$$PV_1 = \$433,232.0352 = FV_1;$$

$$n = P/Y \times \text{Number of Years} = 2 \times 6 = 6$$

$$\begin{aligned}
 PV_{(2)} &= \frac{FV_1}{(1 + i_{eq})^n} \\
 &= \frac{\$433,232.0352}{(1 + 0.05165025)^6} \\
 &= \$320,252.5426
 \end{aligned}$$

**Step 5:** Apply Formula 7.3B to calculate the present value of the payments.

$$\begin{aligned}
 PV_{DUE_2} &= PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq}) \\
 &= \$50,000 \left[ \frac{1 - (1 + 0.05165025)^{-6}}{0.05165025} \right] \times (1 + 0.05165025) \\
 &= \$265,489.8749
 \end{aligned}$$

Combine steps 4 and 5 to calculate the total present value,  $PV_2$ .

$$PV_2 = \$320,252.5426 + \$265,489.8749 = \$585,742.42$$

Therefore,  $PV_2 = PV = \$585,742.42$

#### Calculator instructions:

Table 7.3.3. Calculator Instructions for Example 7.3.3

Time Segment	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	BGN	7	5.1	Answer: -433,232.0352	60,000	100,000	1	2
2	✓ ">BGN	6	✓ ">5.1	Answer: -585,742.4175	50,000	433,232.0352	✓ ">1	✓ ">2

The figure shows how much principal and interest make up the payments. To have his retirement income increased by \$10,000 after six years, Rodriguez needs to have \$585,742.42 invested in his retirement fund at age 65.

## WORKING WITH LOANS

Solving for a future loan balance is a future value annuity calculation. Therefore, you use the same steps as discussed in Section 7.2. However, you need to modify your interpretation of these steps for loan balances. The figure below helps you understand these differences.

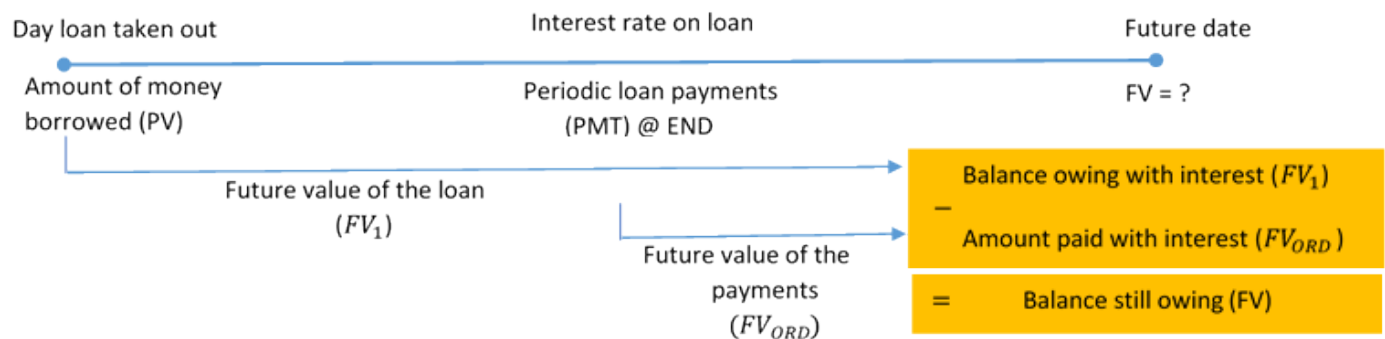


Figure 7.3.L: Timeline for Future Balance of Loans [Image Description]

$FV_1$  represents the total amount owing on the loan with interest as if no payments had been made.

$FV_{ORD}$  represents the total amount paid against the loan with interest.

With both the  $FV_1$  and  $FV_{ORD}$  on the same focal date, the fundamental concept of the time value of money allows you to then take the  $FV_1$  and subtract the  $FV_{ORD}$  to produce the balance owing on the loan.

## YOUR BAI PLUS CALCULATOR

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Proper application of the cash flow sign convention for the present value and annuity payment will automatically result in a future value that nets out the loan principal and the payments. Assuming you are the borrower, you enter the present value (PV) as a positive number since you are receiving the money. You enter the annuity payment (PMT) as a negative number since you are paying the money. When you calculate the future value (FV), it displays a negative number, indicating that it is a balance owing.

## CONCEPT CHECK

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An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=135#h5p-37>

### Example 7.3.4: Balance Owing on a New Truck

Two years ago, Jillian purchased a new Ford F-250 for \$71,482.08 with a \$5,000 down payment and the remainder financed through her Ford dealership at 5.9% compounded monthly. She has been making monthly payments of \$1,282.20. What is her balance owing today? How much interest has she paid to date?

#### **Solution**

**Step 1:** The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple ordinary annuity. Calculate its value two years after its start, which is its future value, or  $FV_{ORD}$ . Once you know the  $FV_{ORD}$ , you can determine the amount of interest, or  $I$ .

The timeline for the savings annuity appears below.

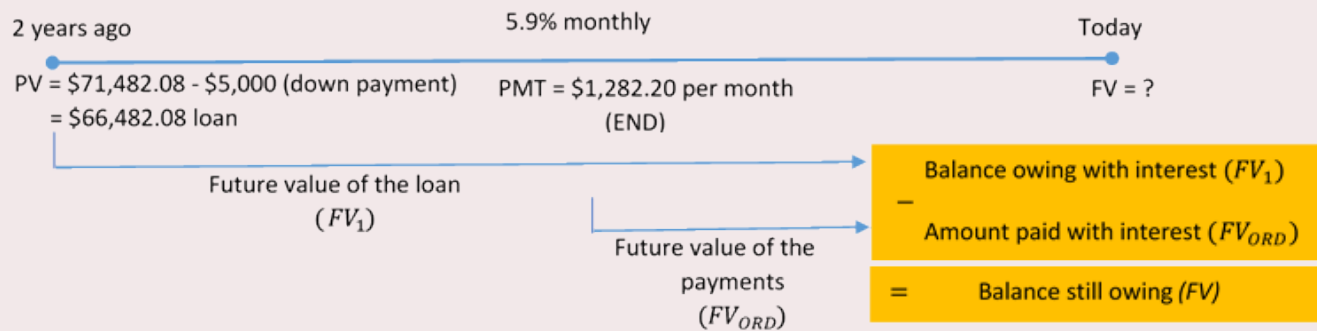


Figure 7.3.4: Timeline [Image Description]

**Step 2:** Given information.

PV = \$71,482.08 – \$5,000 = \$66,482.08; I/Y = 5.9%; C/Y = 12; PMT = \$1,282.20; P/Y = 12; Years = 2

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{5.9\%}{12} = 0.491\bar{6}\%$$

**Step 4:** Calculate the balance owing with interest, FV<sub>1</sub>.

PV = \$66,482.08;

$$n = P/Y \times (\text{Number of Years}) = 12 \times 2 = 24$$

$$FV_1 = PV(1 + i)^n = \$66,482.08(1 + 0.00491\bar{6})^{24} = \$74,786.94231$$

**Step 5:** Calculate the amount paid with interest, FV<sub>ORD</sub>.

$$\begin{aligned} FV_{ORD} &= PMT \left[ \frac{(1 + i)^n - 1}{i} \right] \\ &= \$1,282.20 \left[ \frac{(1 + 0.00491\bar{6})^{24} - 1}{0.00491\bar{6}} \right] \\ &= \$32,577.13179 \end{aligned}$$

Subtract step 5 from step 4 to calculate the balance still owing, FV.

$$FV = \$74,786.94231 - \$32,577.13179 = \$42,209.81$$

**Step 6:** Calculate the amount of interest.

$$\begin{aligned}
 I &= (n \times PMT + FV) - PV \\
 &= (24 \times \$1,282.20 + \$42,209.81) - \$66,482.08 \\
 &= \$72,982.61 - \$66,482.08 \\
 &= \$6,500.53
 \end{aligned}$$

#### Calculator instructions:

Table 7.3.4. Calculator Instructions for Example 7.3.4

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	5.9	66482.08	-1282.2	Answer: -42,209.81052	12	12

The figure shows how much principal and interest make up the payments. After two years of making monthly payments, Jillian has a balance owing on the Ford F-250 of \$42,209.81. Altogether, she has made \$30,772.80 in payments, of which \$6,500.53 went toward the interest on her loan.

## SELLING A LOAN CONTRACT

Thus, the selling of a loan contract needs to calculate the present value of all *remaining* annuity payments in the term.

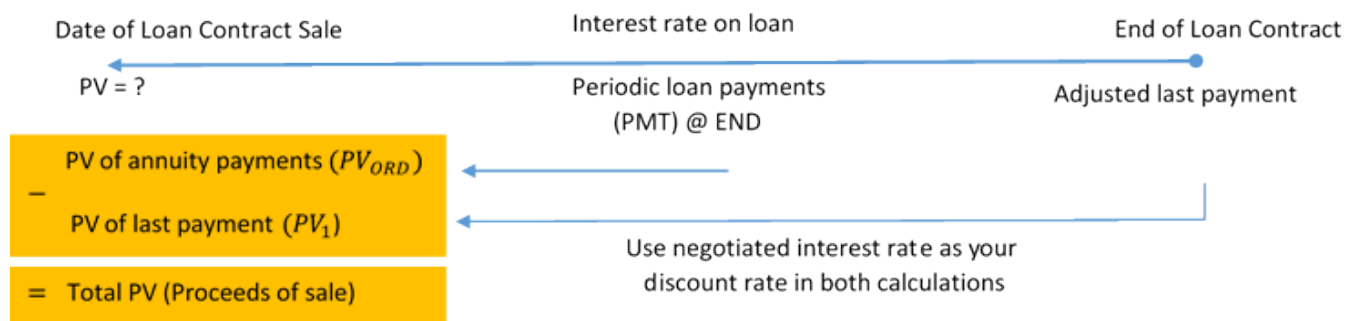


Figure 7.3.PVL: Timeline for Present Value of Loan Contract [Image Description]

**Note:** If the final payment is the same as the regular periodic payment you only need to calculate the present value of annuity payments, or  $PV_{ORD}$  to find the selling price of a loan contract.

#### Example 7.3.5: Ford Sells the Truck Contract

Continuing with Jillian's Ford F-250 purchase, recall that Jillian's monthly payments are fixed at \$1,282.20 for five years. Assume that after two years Ford wants to sell the contract to another finance company, which agrees to a discount rate of 10.8% compounded semi-annually. Jillian's final payment is known at \$1,282.49. What are the proceeds of the sale?

#### Solution

**Step 1:** The payments are made at the end of the payment intervals, and the compounding period (semi-annually) and payment intervals (monthly) are different. Therefore, this is an ordinary general annuity. Calculate its value on the date of sale, which is its present value, or  $PV_{ORD}$ , plus the present value of the final payment, or  $PV_1$ .

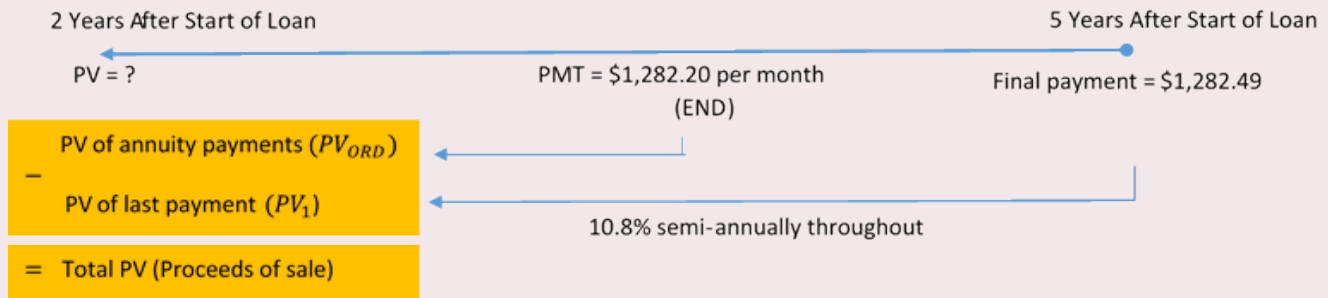


Figure 7.3.5: Timeline [Image Description]

**Step 2:** Given information.

$FV = \$1,282.49$ ;  $I/Y = 10.8\%$ ;  $C/Y = 2$ ;  $PMT = \$1,282.20$ ;  $P/Y = 12$ ; Years = 3

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent periodic rate ( $i_{eq}$ ) that matches the payment interval.

$$i = \frac{I/Y}{C/Y} = \frac{10.8\%}{2} = 5.4\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.054)^{\frac{2}{12}} - 1 = 0.008803937 \text{ per month}$$

**Step 4:** Calculate the present value,  $PV_1$ , of the last payment.

$FV = \$1,282.49$ ;

$$n = P/Y \times (\text{Number of Years}) = 12 \times 3 = 36$$

$$\begin{aligned} PV_1 &= \frac{FV}{(1 + i_{eq})^n} \\ &= \frac{\$1,282.49}{(1 + 0.008803937)^{36}} \\ &= \$935.427906 \end{aligned}$$

**Step 5:** Apply Formula 7.3A to calculate the present value of the payments,  $PV_{ORD}$ .

$$n = P/Y \times (\text{Number of Years}) = 12 \times 3 - 1 = 35$$

$$\begin{aligned}
 PV_{ORD} &= PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \\
 &= \$1,282.20 \left[ \frac{1 - (1 + 0.008803937)^{-35}}{0.008803937} \right] \\
 &= \$38,477.10711
 \end{aligned}$$

Combine steps 4 and 5 to calculate proceeds of the sale, PV.

$$PV = \$935.427906 + \$38,477.10711 = \$39,412.51$$

#### Calculator instructions:

Table 7.3.5. Calculator Instructions for Example 7.3.5

Element	N	I/Y	PV	PMT	FV	P/Y	C/Y
Final Payment	6	10.8	Answer: -935.427906	0	1,282.49	2	2
Annuity	35	$\sqrt{>10.8}$	Answer: -38,477.10711	1282.2	0	12	2

The figure shows the present value and interest amounts in the transaction. The finance company will pay \$39,412.54 for the contract. In return, it receives 35 payments of \$1,282.20 and one payment of \$1,282.49 for a nominal total of \$46,159.49.

#### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. When Sinbad retires, he expects his RRSP to pay him \$2,000 at the end of every month for 25 years. If his retirement annuity earns 3.8% compounded quarterly, how much money does he need to have in his RRSP when he retires?
2. Sandy's parents would like to have an annuity pay her \$500 at the beginning of every month from September 1, 2012, to April 1, 2017, to help with her university tuition and living expenses. On May 1, 2017, they would like to give her a graduation gift of \$5,000. If the annuity can earn 6.15% compounded quarterly, how much money must be in the account on September 1, 2012? (Use years and months in the calculations)

**Note:** Solution to exercises are demonstrated using the calculator only.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=135#h5p-38>

Timelines for exercises 1 and 2 are included in Solutions to Exercises.

## 7.4 Annuity Payment Amounts

You need to calculate an annuity payment in many situations:

- Figuring out loan or mortgage payments
- Determining membership or product payment plans
- Calculating lease payments
- Determining the periodic payment necessary to achieve a savings goal
- Determining the maximum payment that an investment annuity can sustain over a period of time

### THE FORMULA

---

Recall that the annuity payment amount,  $PMT$ , is one of the variables in Formula 7.2A, Formula 7.2B, Formula 7.3A, and Formula 7.3B. Calculating this amount then requires you to substitute the known variables and rearrange the formula for  $PMT$ . The most difficult part of this process is figuring out which of the four formulas to use. Your selection depends on the two questions stated below.

1. Are the annuity payments to be made at the beginning or at the end of the payment interval? In other words, do you have an annuity due or an ordinary annuity?
2. Do you know the amount that the annuity starts or ends with? In other words, do you know the present value or future value of the annuity?

### HOW IT WORKS

---

Follow these steps to solve for any annuity payment amount:

**Step 1:** Identify the annuity type. Draw a timeline to visualize the question.

**Step 2:** Identify the variables that you know, including  $I/Y$ ,  $C/Y$ ,  $P/Y$ , and Years. You must also identify a value for one of  $PV_{ORD}$ ,  $PV_{DUE}$ ,  $FV_{ORD}$ , or  $FV_{DUE}$ .

**Step 3:** Depending on the type of an annuity, calculate the periodic interest rate.

For **simple annuities** you need to calculate the periodic interest rate ( $i$ ) using the formula

$$i = \frac{I/Y}{C/Y}$$

For **general annuities** you need to calculate the equivalent periodic rate,  $i_{eq}$ , that matches the payment interval using the formula

$$i_{eq} = \left(1 + i\right)^{\frac{C/Y}{P/Y}} - 1$$

**Step 4:** You may or may not have a value for FV or PV. If a single payment PV or FV is known, move it to the other end of the time segment using appropriate formula. When you move the amount to the same focal date as the present or future value of the annuity, either add this number to the annuity value or subtract it as the situation demands. Example 7.4.3 later in this section will illustrate this practice.

**Step 5:** Apply the correct annuity payment formula that matches your annuity type and known present or future value. Select from Formula 7.2A, Formula 7.2B, Formula 7.3A, or Formula 7.3B then rearrange for the annuity payment amount, PMT. If you performed step 4 above, be sure to use the adjusted future or present value of your annuity in the formula.

## CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=141#h5p-39>

### Example 7.4.1: Payments on a Loan

Morgan wants to consolidate a lot of smaller debts into a single three-year loan for \$25,000. If the loan is charged interest at 7.8% compounded monthly, what is her payment amount at the end of every month?

#### Solution

The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is an ordinary simple annuity. Calculate the monthly amount of her loan payment, or PMT.

The timeline for the loan appears below.

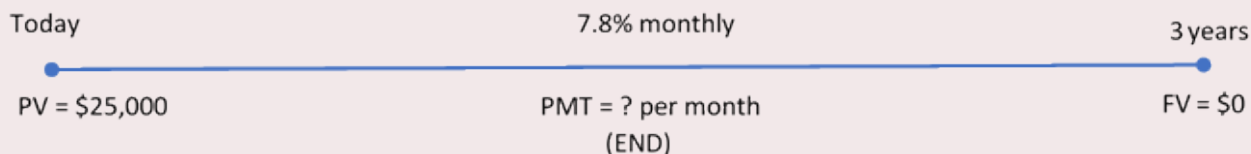


Figure 7.4.1: Timeline [Image Description]

**Step 2:** Given information:

PV = \$25,000; I/Y = 7.8%; C/Y = 12; P/Y = 12; Years = 3; FV = \$0

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{7.8\%}{12} = 0.65\%$$

**Step 4:** Since FV=\$0, skip this step.

**Step 5:** Use the formula for  $PV_{ORD}$  and solve for PMT.

$$n = P/Y \times (\text{Number of Years}) = 12 \times 3 = 36$$

$$PV_{ORD} = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right]$$

$$\$25,000 = PMT \left[ \frac{1 - (1 + 0.0065)^{-36}}{0.0065} \right]$$

$$PMT = \frac{\$25,000}{32.005957} = \$781.10$$

**Calculator instructions:**

Table 7.4.1. Calculator Instructions for Example 7.4.1

N	I/Y	PV	PMT	FV	P/Y	C/Y
36	7.8	25,000	Answer: -781.104587	0	12	12

The figure shows how much principal and interest make up the payments. To pay off her consolidated loan, Morgan's month-end payments for the next three years will be \$781.10.

Franco has placed \$10,000 into an investment fund with the goal of receiving equal amounts at the beginning of every month for the next year while he backpacks across Europe. If the investment fund can earn 5.25% compounded quarterly, how much money can Franco expect to receive each month?

### Solution

**Step 1:** The payments are at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate the monthly amount he can receive, or PMT.

The timeline for the vacation money appears below.

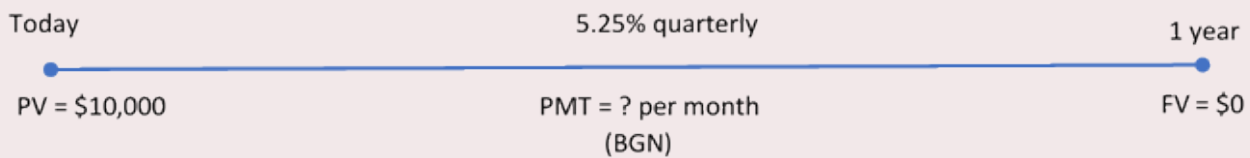


Figure 7.4.2: Timeline [Image Description]

**Step 2:**  $PV_{DUE} = \$10,000$ ;  $I/Y = 5.25\%$ ;  $C/Y = 4$ ;  $P/Y = 12$ ; Years = 1;  $FV = \$0$

**Step 3:** Since  $P/Y \neq C/Y$ , calculate the equivalent interest rate ( $i_{eq}$ ) that matches the payment interval.

$$i = \frac{I/Y}{C/Y} = \frac{5.25\%}{4} = 1.3125\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.013125)^{\frac{4}{12}} - 1 = 0.004355998 \text{ per month}$$

**Step 4:** Since  $FV = \$0$ , skip this step.

**Step 5:** Calculate the payment amount, PMT, using the formula for  $PV_{DUE}$ .

$$n = P/Y \times (\text{Number of Years}) = 12 \times 1 = 12$$

$$PV_{DUE} = PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq})$$

$$\$10,000 = PMT \left[ \frac{1 - (1 + 0.004355998)^{-12}}{0.004355998} \right] \times (1 + 0.004355998)$$

$$PMT = \frac{\$10,000}{11.717849} = \$853.40$$

**Calculator instructions:**

Table 7.4.2. Calculator Instructions for Example 7.4.2

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	12	5.25	-10,000	Answer: 853.398928	0	12	4

The figure shows how much principal and interest make up the payments. While backpacking across Europe, Franco will have his annuity pay him \$853.40 at the beginning of every month.

### Example 7.4.3: Planning RRSP Contributions

Kingsley's financial adviser has determined that when he reaches age 65, he will need \$1.7 million in his RRSP to fund his retirement. Kingsley is currently 22 years old and has saved up \$10,000 already. His adviser thinks that his RRSP will average 9% compounded annually throughout the years. To meet his RRSP goal, how much does Kingsley need to invest every month starting today?

#### Solution

**Step 1:** The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Calculate the monthly amount Kingsley needs to contribute, or PMT.

The timeline for Kingsley's RRSP contributions appears below.

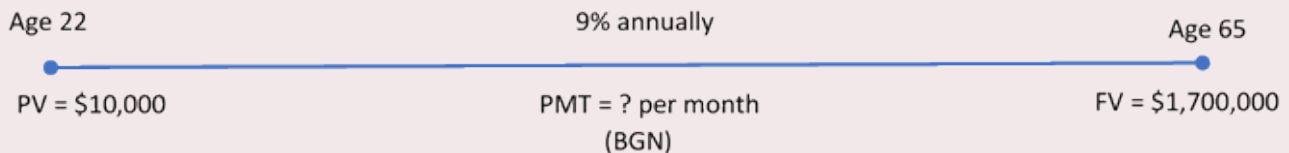


Figure 7.4.3: Timeline [Image Description]

**Step 2:**  $PV = \$10,000$ ;  $FV_{DUE} = \$1,700,000$ ;  $I/Y = 9\%$ ;  $C/Y = 1$ ;  $P/Y = 12$ ; Years = 43

**Step 3:** Calculate the equivalent interest rate ( $i_{eq}$ ) that matches the payment interval.

$$i = \frac{I/Y}{C/Y} = \frac{9\%}{1} = 9\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.09)^{\frac{1}{12}} - 1 = 0.007207323 \text{ per month}$$

**Step 4:** You need to move the present value to Kingsley's age 65, the same date as  $FV_{DUE}$ . Calculate the future value, FV. The FV is money the annuity does not have to save, so you subtract it from  $FV_{DUE}$ .

$$n = P/Y \times (\text{Number of Years}) = 12 \times 43 = 516$$

$$FV = PV \times (1 + i_{eq})^n = \$10,000 \times (1 + 0.007207323)^{516} = \$406,761.0984$$

$$\text{New } FV_{DUE} = \$1,700,000 - \$406,761.0984 = \$1,293,238.902$$

**Step 5:** Calculate the payment amount, PMT, using the formula for  $FV_{DUE}$ .

$$FV_{DUE} = PMT \left[ \frac{(1 + i_{eq})^n - 1}{i_{eq}} \right] \times (1 + i_{eq})$$

$$\$1,293,238.902 = PMT \left[ \frac{(1 + 0.007207323)^{516} - 1}{0.007207323} \right] \times (1 + 0.007207323)$$

$$PMT = \frac{\$1,293,238.902}{5,544.647665} = \$233.24$$

**Calculator instructions:**

Table 7.4.3. Calculator Instructions for Example 7.4.3

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	516	9	-10,000	Answer: -233.240952	1700000	12	1

The figure shows how much principal and interest make up the final balance. To meet his retirement goals, Kingsley needs to invest \$233.24 at the beginning of every month for the next 43 years. In doing so, he will achieve a \$1.7 million balance in his account at age 65. (Note: Similar to loan payments, the last payment in actuality is required to be a slightly higher amount since the annuity payment was rounded downwards. However, the last payment is treated equally at this time for purposes of all calculations.)

#### Example 7.4.4: Purchasing New Production Line Machinery

The production department just informed the finance department that in five years' time the robotic systems on the production line will need to be replaced. The estimated cost of the replacement is \$10 million. To prepare for this purchase, the finance department immediately deposits \$1,000,000 into a savings annuity earning 6.15% compounded semi-annually, and it plans to make semi-annual contributions starting in six months. How large do those contributions need to be?

**Solution**

**Step 1:**

The payments are made at the end of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is an ordinary simple annuity. Calculate the monthly amount the finance department needs to contribute, or PMT.

The timeline for the machinery fund appears below.

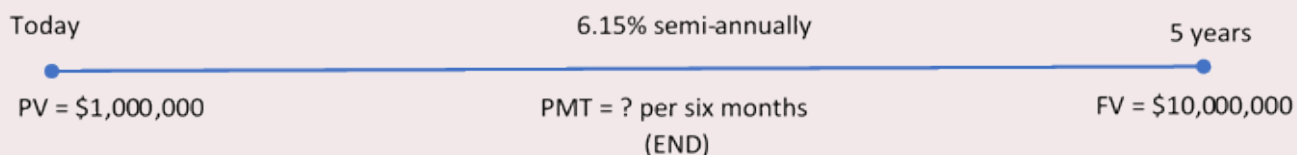


Figure 7.4.4: Timeline [Image Description]

**Step 2:** Give information:

PV = \$1,000,000; FV<sub>ORD</sub> = \$10,000,000; I/Y = 6.15%; C/Y = 2; P/Y = 2; Years = 5

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{6.15\%}{2} = 3.075\%$$

**Step 4:** The present value must be moved to the five-year date, the same date as FV<sub>ORD</sub>. Apply Formula 9.2A. This FV is money the annuity does not have to save, so it is subtracted from FV<sub>ORD</sub> to arrive at the amount the annuity must generate.

$$n = P/Y \times (\text{Number of Years}) = 2 \times 5 = 10$$

$$FV = PV \times (1 + i)^n = \$1,000,000 \times (1 + 0.03075)^{10} = \$1,353,734.306$$

$$\text{New } FV_{ORD} = \$10,000,000 - \$1,353,734.306 = \$8,646,265.694$$

**Step 5:** Use the formula for FV<sub>ORD</sub> and solve for PMT.

$$FV_{ORD} = PMT \left[ \frac{(1 + i)^n - 1}{i} \right]$$

$$\$8,646,265.694 = PMT \left[ \frac{(1 + 0.03075)^{10} - 1}{0.03075} \right]$$

$$PMT = \frac{\$8,646,265.694}{11.503554} = \$751,616.87$$

**Calculator instructions:**

Table 7.4.4. Calculator Instructions for Example 7.4.4

N	I/Y	PV	PMT	FV	P/Y	C/Y
10	6.15	-1,000,000	Answer: -751,616.8656	10,000,000	2	2

The figure shows how much principal and interest make up the final balance. To have adequate funding for the production line machinery replacement five years from now, the finance department needs to deposit \$751,616.87 into the fund every six months. (Note: Similar to loan payments, the last payment in actuality is required to be a slightly lower amount since the annuity payment was rounded upwards. However, the last payment is treated equally at this time for purposes of all calculations.)

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. To save approximately \$30,000 for a down payment on a home four years from today, what amount needs to be invested at the end of every month at 4.5% compounded semi-annually?
2. Sinclair does not believe in debt and will only pay cash for all purchases. He has already saved up \$140,000 toward the purchase of a new home with an estimated cost of \$300,000. Suppose his investments earn 7.5% compounded monthly. How much does he need to contribute at the beginning of each quarter if he wants to purchase his home in five years?
3. The Kowalskis' only child is eight years old. They want to start saving into an RESP such that their son will be able to receive \$5,000 at the end of every quarter for four years once he turns 18 and starts attending postsecondary school. When the annuity is paying out, it is forecast to earn 4% compounded monthly. While they make contributions at the end of every month to the RESP, it will earn 8% compounded semi-annually. Additionally, at the end of every year of contributions the government places a \$500 grant into the RESP. What is the monthly contribution payment by the Kowalskis?

**Note:** Solution to exercises are demonstrated using the calculator only.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=141#h5p-40>

Timelines for exercises are included in Solutions to Exercises.

## 7.5 Number of Annuity Payments

How long do you require to fulfill the goal of your annuity? It all depends on your annuity payment, interest rate, and the amount of money involved.

You must calculate the number of annuity payments in a variety of scenarios:

- Savings planning
- Debt extinguishment
- Sustaining withdrawals from an investment

### THE FORMULA

---

Recall that the number of annuity payments ( $n$ ) is one of the variables in Formula 7.2A, Formula 7.2B, Formula 7.3A, and Formula 7.3B. Calculating this amount then requires you to substitute the known variables and rearrange the formula to solve for  $n$ . Since  $n$  is located in the exponent, the rearrangement and isolation demands the usage of natural logarithms. The most difficult part is figuring out which formula you need to use. Choose the formula using the same decision criteria explained in Section 7.4.

### HOW IT WORKS

---

Follow these steps, to solve for the number of annuity payments or the annuity term:

**Step 1:** Identify the annuity type. Draw a timeline to visualize the question.

**Step 2:** Identify the variables that always appear, including PMT,  $I/Y$ ,  $C/Y$ , and  $P/Y$ . You must also identify one of the known values of  $PV_{ORD}$ ,  $PV_{DUE}$ ,  $FV_{ORD}$ , or  $FV_{DUE}$ .

**Step 3:** Calculate the periodic interest rate ( $i$ ) using the formula

$$i = \frac{I/Y}{C/Y}$$

**Step 4:** Substitute into the correct annuity payment formula that matches your annuity type and known present or future value. Select from Formula 7.2A, Formula 7.2B, Formula 7.3A, or Formula 7.3B. Rearrange and solve for  $n$ .

**Step 5:** To convert  $n$  back to a more commonly expressed format, such as years and months, take Formula 7.1 and rearrange it for Years. **If Years is a decimal number, recall the steps required for converting in .. unit not used.**

## DEALING WITH DECIMALS IN $n$

---

At the end of step 4, the calculated value of  $n$  may have decimals. Apply the rounding rules as described below.

If the *three decimals are zeroes*, then the decimals are most likely a result of a rounded FV, PV, or PMT, so treat the  $n$  like an integer (ignoring the decimals).

If the *three decimals are not all zeroes*, then  $n$  must always be rounded up to the next integer regardless of the decimal value. You must never round it down, because the calculated value of  $n$  represents the minimum payments required.

### Interpreting $n$

---

Since  $n$  represents the number of annuity payments, to assign its meaning you need to look at the  $P/Y$  value. For example, if  $n = 8$  and  $P/Y = 4$  then  $n$  represents 8 quarterly payments which means the term is two years ( $8/4=2$ ).

### The Term and the Last Payment on Annuities Due

---

One of the characteristics of an annuity due is that the last payment occurs one payment interval before the end of the term of the annuity. When you calculate  $n$ , you have calculated the term of the annuity. The last payment occurs  $n - 1$  intervals from the start of the annuity. For example, assume payments are monthly and you calculate  $n = 12$  months. This means the term of the annuity is 12 months. The last payment on the annuity due is  $12 - 1 = 11$  months from the start. Be sure to recognize whether you are looking for the end of the term or when the last payment is made. This problem does not occur for ordinary annuities since the last payment and the end of the term are on the same date.

### CONCEPT CHECK

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An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=146#h5p-41>

#### Example 7.5.1: How Long Until Retirement Savings are Depleted?

Samia has \$500,000 accumulated in her retirement savings when she decides to retire at age 60. If she wants to receive beginning-of-month payments of \$3,000 and her retirement annuity can earn 5.2% compounded monthly, how old is Samia when the fund is depleted?

##### Solution

**Step 1:** The payments are at the beginning of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple annuity due. In looking for how old Samia will be when the fund is depleted, calculate the number of annuity payments, or  $n$ , that her retirement annuity can sustain.

The timeline for the retirement annuity appears below.

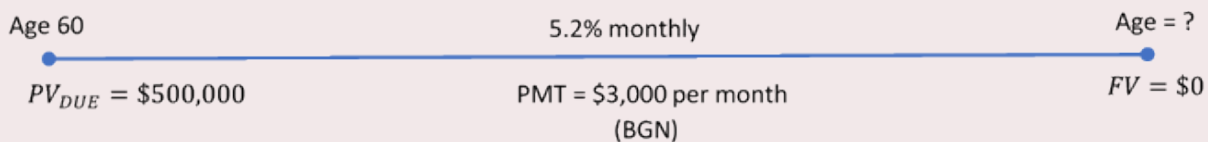


Figure 7.5.1: Timeline [Image Description]

**Step 2:** Given information:

$PV_{DUE} = \$500,000$ ;  $I/Y = 5.2\%$ ;  $C/Y = 12$ ;  $PMT = \$3,000$ ;  $P/Y = 12$ ;  $FV = \$0$

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{5.2\%}{12} = 0.4\bar{3}\%$$

**Step 4:** Substitute into Formula 7.3B, rearranging for  $n$ .

$$PV_{DUE} = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \times (1 + i)$$

$$\$500,000 = \$3,000 \left[ \frac{1 - (1 + 0.004\bar{3})^{-n}}{0.004\bar{3}} \right] \times (1 + 0.004\bar{3})$$

$$165.947560 = \frac{1 - 0.995685^n}{0.0043}$$

$$0.995685^n = 0.280893$$

$$n \times \ln(0.995685) = \ln(0.280893)$$

$$n = \frac{-1.2697323}{-0.004323}$$

$$n = 293.660180$$

Rounding up to 294 monthly payments consisting of 293 regular payments plus one additional smaller payment.

**Step 5:** Convert  $n$  to years and months.

$$\begin{aligned} \text{Years} &= \frac{n}{P/Y} \\ &= \frac{294}{12} \\ &= 24.5 \text{ years or } 24 \text{ years and } 0.5 \times 12 = 6 \text{ months} \end{aligned}$$

24 years, 6 months

#### Calculator Instructions:

Table 7.5.1. Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	Answer: 293.660180	5.2	-500,000	300	0	12	12

The figure shows how much principal and interest make up the payments. If Samia is currently 60 years old and the annuity endures for 24 years and six months, then she will be 84.5 years old when the annuity is depleted. Note that Samia will receive 293 payments of \$3,000 along with a smaller final payment that is approximated by taking  $66.018\% \times \$3,000 = \$1,980.54$

Brendan is purchasing a brand new Mazda MX-5 GT. Including all options, accessories, and fees, the total amount he needs to finance is \$47,604.41 at the dealer's special interest financing of 2.4% compounded monthly. If he makes payments of \$1,000 at the end of every month, how long will it take to pay off his car loan?

### Solution

**Step 1:** The payments are at the end of the payment intervals with a monthly compounding period and monthly payment intervals. Therefore, this is an ordinary simple annuity. Calculate the number of monthly payments, or  $n$ , to figure out the length of time required to pay off the loan.

The timeline for the car payments appears below.

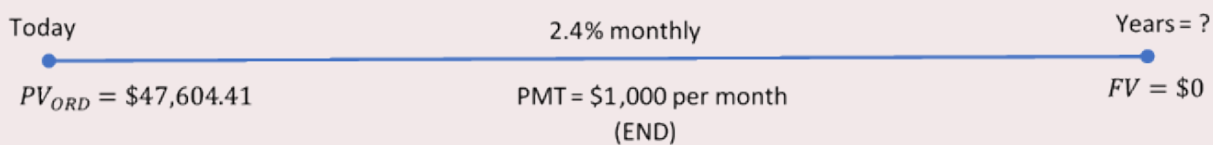


Figure 7.5.2: Timeline [Image Description]

**Step 2:** Given information:

$PV_{ORD} = \$47,604.41$ ;  $I/Y = 2.4\%$ ;  $C/Y = 4$ ;  $PMT = \$1,000$ ;  $P/Y = 12$ ;  $FV = \$0$

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{2.4\%}{12} = 0.2\%$$

**Step 4:** Substitute into Formula 7.3A, rearranging for  $n$ .

$$PV_{ORD} = PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right]$$

$$\$47,604.41 = \$1,000 \left[ \frac{1 - (1 + 0.002)^{-n}}{0.002} \right]$$

$$0.095208 = 1 - 0.998003^n$$

$$0.998003^n = 0.904791$$

$$n \times \ln(0.998003) = \ln(0.904791)$$

$$n = \frac{-0.100051}{-0.001998}$$

$$= 50.075560$$

Rounded up to 51 monthly payments.

**Step 5:** Convert  $n$  to years and months.

$$\begin{aligned}\text{Years} &= \frac{n}{P/Y} \\ &= \frac{51}{12} \\ &= 4.25 \text{ years or } 4 \text{ years and } 0.25 \times 12 = 3 \text{ months}\end{aligned}$$

4 years, 3 months

### Calculator Instructions

Table 7.5.2. Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
Answer: 50.075560	2.4	47,604.41	-1,000	0	12	12

The figure shows how much principal and interest make up the payments. To own his vehicle, Brendan will make payments for four-and-a-quarter years. This consists of 50 payments of \$1,000 and a smaller final payment that is approximated by taking  $7.556\% \times \$1,000 = \$75.56$ .

### Example 7.5.3: The Importance of the Annuity Type

Trevor wants to save up \$1,000,000. He will contribute \$5,000 annually to an investment earning 10% compounded monthly. What is the time difference between his last payments (not the end of the annuities) if he makes his contributions at the end of the year instead of at the beginning of the year?

#### Solution

**Step 1:** In this question you are being asked to compare two annuities that differ in their payment intervals and their compounding periods. One annuity makes contributions at the beginning of the interval, while the other makes contributions at the end. Therefore, you must contrast one general annuity due with one ordinary general annuity. To determine the time difference, calculate  $n$  for each annuity and compare when the last payment is made.

A combined timeline for the two annuities appears below.

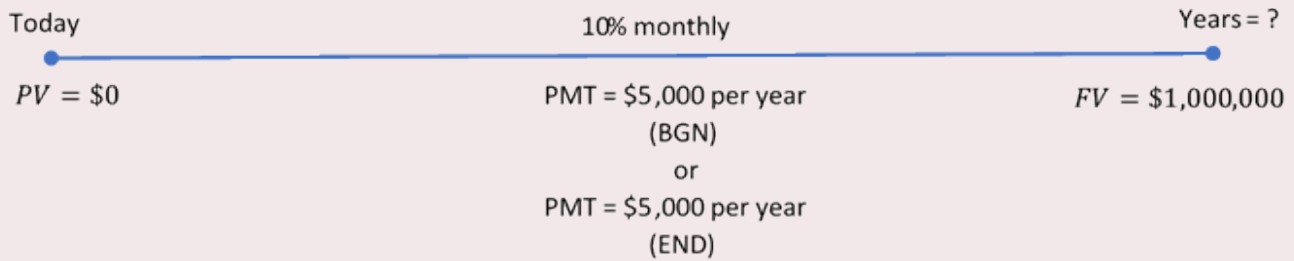


Figure 7.5.3: Timeline [Image Description]

**Step 2:** Given information:

Ordinary general annuity:  $FV_{ORD} = \$1,000,000$ ;  $I/Y = 10\%$ ;  $C/Y = 12$ ;  $PMT = \$5,000$ ;  $P/Y = 1$

General annuity due:  $FV_{DUE} = \$1,000,000$ ;  $I/Y = 10\%$ ;  $C/Y = 12$ ;  $PMT = \$5,000$ ;  $P/Y = 1$

**Step 3:** Calculate the periodic interest rate,  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{10\%}{12} = 0.8\bar{3}\%$$

**Step 4:** Apply Formula 7.2A and Formula 7.2B, rearranging for  $n$ .

**Ordinary general annuity:**

$$FV_{ORD} = PMT \left[ \frac{(1 + i)^n - 1}{i} \right]$$

$$\$1,000,000 = \$5,000 \left[ \frac{(1 + 0.008\bar{3})^n - 1}{0.008\bar{3}} \right]$$

$$20.942613 = 1.104713^n - 1$$

$$21.942613 = 1.104713^n$$

$$\ln(21.942613) = n \times \ln(1.104713)$$

$$n = \frac{21.942613}{1.104713}$$

$$= 31.012812 \text{ or } 32 \text{ years}$$

**General annuity due:**

$$FV_{DUE} = PMT \left[ \frac{(1+i)^n - 1}{i} \right] \times (1+i)$$

$$\$1,000,000 = \$5,000 \left[ \frac{(1+0.0088\bar{3})^n - 1}{0.0088\bar{3}} \right] \times (1+0.0088\bar{3})$$

$$18.957514 = 1.104713^n - 1$$

$$19.957514 = 1.104713^n$$

$$\ln(19.957514) = n \times \ln(1.104713)$$

$$n = \frac{19.957514}{1.104713}$$

$$= 30.060618 \text{ or } 31 \text{ years}$$

**Step 5:** The ordinary general annuity last payment is 32 years from today. The general annuity due has a term of 31 years, but the last payment is  $31 - 1 = 30$  years from today. The difference between the last payments is  $32 - 30 = 2$  years sooner.

#### Calculator instructions:

Table 7.5.3. Calculator Instructions

Type	Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
Ordinary		Answer: 30.060618	10	0	-5,000	1,000,000	1	12
Due	BGN	Answer: 31.012812	10	0	-5,000	1,000,000	1	12

In order for Trevor to reach his goal, if he were to make his \$5,000 contributions at the beginning of the year (that is, under the annuity due) instead of the end of the year (under the ordinary annuity), his last payment would be two years sooner. Do not confuse this with the terms of the two annuities, which end only one year apart (31 years from now and 32 years from now).

#### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Amarjit wants to save up for a down payment on his first home. A typical starter home in his area sells for \$250,000 and the bank requires a 10% down payment. If he starts making \$300 month-end contributions to an investment earning 4.75% compounded monthly, how long will it take for Amarjit to have the necessary down payment?
2. Hi-Tec Electronics is selling a 52" LG HDTV during a special "no sales tax" event for \$1,995 with end of month payments of \$100 including interest at 15% compounded semi-annually. How long will it take a consumer to pay off her new television?

3. Most financial institutions tout the benefits of “topping up” your mortgage payments—that is, increasing from the required amount to any higher amount. Assume a 25-year mortgage for \$200,000 at a fixed rate of 5% compounded semi-annually.
- a) How many fewer payments does it take to pay off your mortgage if you increased your monthly payments by 10%?
  - b) How much money is saved by “topping up” the payments? Assume that all payments are equal amounts in your calculations.

**Note:** Solution to exercises are demonstrated using the calculator only.



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=146#h5p-42>

## 7.6 Annuity Interest Rate

You must calculate the interest rate on an annuity in a variety of situations:

- To determine the interest rate being charged on any debt
- To determine the interest rate that an investment is earning
- To calculate the required interest rate for savings to reach a goal within a certain time period
- To calculate the required interest rate needed for a series of payments to be sustained over a certain time period

### THE FORMULA

---

A most interesting circumstance arises when you attempt to solve any of the future value or present value annuity formula for the periodic interest rate ( $i$ ). The periodic interest rate ( $i$ ) appears in the formulas twice and there is no algebraic way to isolate the periodic interest rate.

So how do you solve for the periodic interest rate ( $i$ )?

We will use technology (BAII+ Calculator). All problems in this section, therefore, assume you have access to technology, so no algebraic solutions appear in the formula calculations.

### HOW IT WORKS

---

Follow these steps to solve for any nominal interest rate,  $I/Y$ :

**Step 1:** Identify the annuity type. Draw a timeline to visualize the question.

**Step 2:** Identify the variables that you know, including  $C/Y$ ,  $PMT$ ,  $P/Y$ , and Years. You must also identify a value for one of  $PV_{ORD}$ ,  $PV_{DUE}$ ,  $FV_{ORD}$ , or  $FV_{DUE}$ . You may or may not have a value for  $FV$  or  $PV$ .

**Step 3:** Use Formula 11.1 to calculate the number of payments ( $n$ ). Input all six of the known variables into your calculator and solve for the nominal interest rate,  $I/Y$ , which is compounded according to the value you entered into the  $C/Y$  variable.

**Note:** If you want to solve for the effective interest rate in any situation, you can change the  $C/Y$  to a value of one ( $C/Y = 1$ ) and recalculate the  $I/Y$ .

CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=152#h5p-43>

Example: 7.6.1: Rent to Own

Smartchoice, a rent-to-own store, offers a Dell 10" Mini Inspiron Netbook for a cash n’ carry price of \$399. Alternatively, under its rent-to-own plan you could make \$59.88 monthly payments in advance and own the laptop after one year. What interest rate is effectively being charged on the rent-to-own plan?

Solution

**Step 1:** The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is a general annuity due. Solve for the effective interest rate, which is the nominal interest rate,  $IY ">I/Y$ , that has a compounding frequency of one.

The timeline for the retirement annuity appears below.

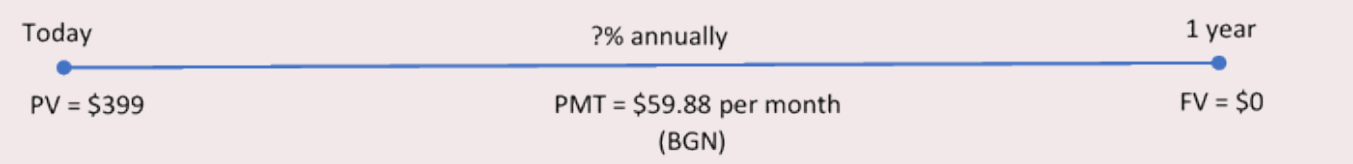


Figure 7.6.1: Timeline [Image Description]

**Step 2:** Given information:

$PV_{DUE} = \$399; FV = \$0; C/Y = 1; PMT = \$59.88; P/Y = 12; Years = 1$

**Step 3:** Enter the information into the calculator and solve for  $I/Y$ .

Calculator instructions:

Table 7.6.1. Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	12	Answer: 337.975924	399	-59.88	0	12	1

The figure shows the effective interest and the laptop’s value that make up the payments. The effective interest rate being charged under the rent-to-own payment plan is 337.9759%.

Example: 7.6.2: Interest Rate Required to Allow for Capital Project Savings

Cubonic Industries deposits \$30,000 at the end of every quarter to save up \$550,000 for a capital project in four years. To achieve its goal, what nominal interest rate compounded quarterly does Cubonic Industries require on its investment? What is the effective rate?

Solution

**Step 1:** There are two questions here. The first question about the nominal interest rate involves an ordinary simple annuity. Solve this for  $I/Y$  when  $C/Y = 4$ . The second question about the effective rate involves an ordinary general annuity. Solve this for  $I/Y$  when  $C/Y = 1$ .

The timeline for both questions appears below.

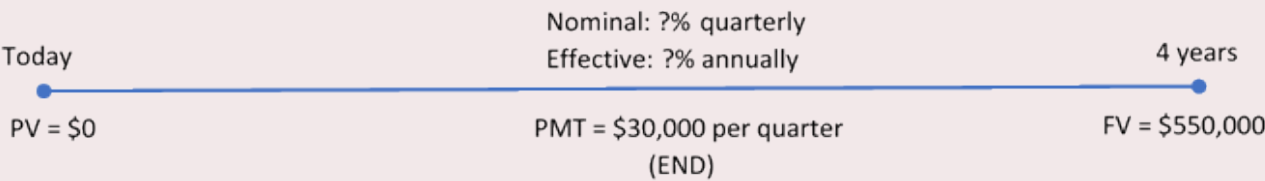


Figure 7.6.2: Timeline [Image Description]

**Step 2:** Given information:

Ordinary simple annuity:  $FV_{ORD} = \$550,000$ ;  $PV = \$0$ ;  $C/Y = 4$ ;  $PMT = \$30,000$ ;  $P/Y = 4$ ; Years = 4

Ordinary general annuity: All the same except  $C/Y = 1$

**Step 3:** Enter the information into the calculator and solve for  $I/Y$ .

Ordinary simple annuity: Enter the information into the calculator and solve for  $I/Y$ .

Ordinary general annuity: Change the  $C/Y$  to 1 (for the effective rate) and recalculate  $I/Y$ .

Calculator instructions:

Table 7.6.2. Calculator Instructions

Type	N	I/Y	PV	PMT	FV	P/Y	C/Y
Simple	16	Answer: 7.145908	0	-30,000	550,000	4	4
General	16	Answer: 7.339689	0	-30,000	550,000	4	1

The figure shows how much principal and interest make up the final balance. For Cubonic Industries to achieve its savings goal, the savings annuity must earn 7.1459% compounded quarterly, or 7.3397% effectively.

Example 7.6.3: Interest Rate Required to Achieve RRSP Goal

Amadeus has already saved \$5,000 in his RRSP today. Suppose he continues to make \$250 contributions at the beginning of each month for the next 14 years. For him to achieve his goal of having \$100,000, what monthly nominal rate of return must his investment earn?

Solution

**Step 1:** The payments are made at the beginning of the payment intervals, and the compounding period and payment intervals are the same. Therefore, this is a simple annuity due. Solve for the monthly nominal interest rate, I/Y.

The timeline for RRSP contributions appears below.

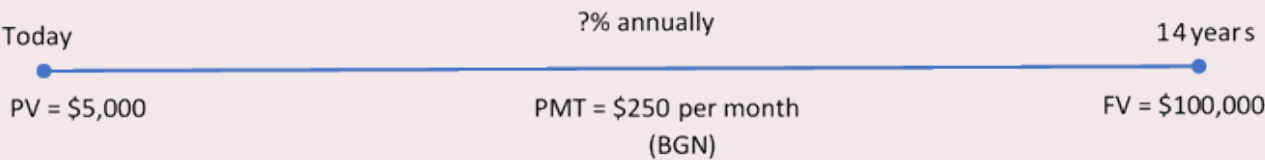


Figure 7.6.3: Timeline [Image Description]

**Step 2:** Given information:

$FV_{DUE} = \$100,000$ ;  $PV = \$5,000$ ;  $C/Y = 12$ ;  $PMT = \$250$ ;  $P/Y = 12$ ; Years = 14

**Step 3:** Enter the information into the calculator and solve for I/Y.

Calculator instructions:

Table 7.6.3. Calculator Instructions

Mode	N	I/Y	PV	PMT	FV	P/Y	C/Y
BGN	168	Answer: 8.808808	-5,000	-250	100,000	12	12

The figure shows how much principal and interest make up the final balance. The nominal rate of interest that Amadeus must earn on his investment is 8.8088% compounded monthly.

Example 7.6.4: Ordinary General Annuity with Present Value and a Balance Owing in Future

Gemma is looking to purchase a new Nissan Pathfinder for \$54,904.64 including all fees and sales taxes. She can afford to pay no more than \$1,500 at the end of every month, and she wants to have the balance owing reduced to \$30,000 after two years, when she can pay off the vehicle with her trust fund. What is the maximum effective rate of interest she could be charged on the car loan to meet her goals?

Solution

**Step 1:** The payments are made at the end of the payment intervals, and the compounding period and payment intervals are different. Therefore, this is an ordinary general annuity. Solve for the effective interest rate, I/Y.

The timeline for the car loan appears below.

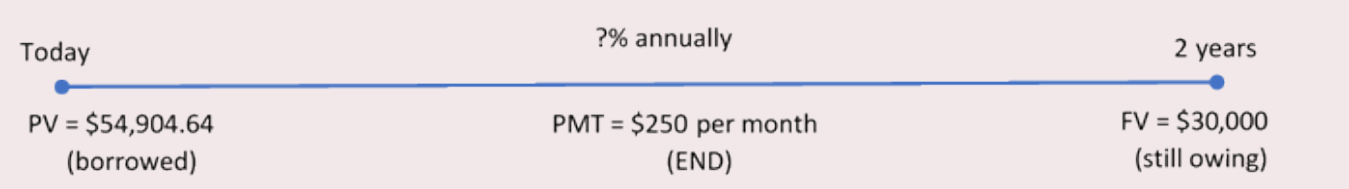


Figure 7.6.4: Timeline [Image Description]

**Step 2:** Given information:

PV<sub>ORD</sub> = \$54,904.64; FV = \$30,000; C/Y = 1; PMT = \$1,500; P/Y = 12; Years = 12

**Step 3:** Enter the information into the calculator and solve for I/Y.

Calculator instructions:

Table 7.6.4. Calculator Instructions

N	I/Y	PV	PMT	FV	P/Y	C/Y
24	Answer: 13.527019	54,904.64	-1,500	-30,000	12	1

The figure shows how much principal and interest make up the payments. Gemma will be able to purchase the car if she can obtain a car loan that has an effective interest rate lower than 13.527%.

Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. Francisco just changed occupations. Unfortunately, he is not able to transfer his company pension

with him to his new company. The administrators of the pension plan offer him the choice of a lump-sum payout of \$103,075 today or beginning-of-month payments of \$535 for the next 25 years. What semi-annually compounded rate of return are the pension administrators using in their calculations?

2. An investment today requires \$1,125.51 to purchase. In return, the investment pays out \$30 after every six months for the next 20 years, along with an additional final lump-sum payout of \$1,000. What semi-annually compounded interest rate is being earned on the investment?
3. When you buy a car, a cash rebate is usually available if you finance the vehicle through your bank instead of the dealership; if you finance the vehicle through the dealership, you are not eligible for the cash rebate. Assume you can purchase a vehicle for \$24,960 and finance it for four years with month-end payments at 0% through the dealership. Alternatively, you could get a loan from a bank and pay cash for your vehicle, which would entitle you to receive a \$3,500 cash rebate. What monthly compounded interest rate would the bank have to charge to arrive at the same monthly payment as the dealership alternative? What decision rule can you create from this calculation?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=152#h5p-44>

Timeline for exercise 2 is included in Solutions to Exercises.

## 7.7 Annuities Terminology (Interactive Activity)

**Complete the following activity.**



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=154#h5p-45>

## 7.8 Review, Symbols, and Formulas

### KEY CONCEPTS

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#### Fundamentals of Annuities

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- Understanding what an annuity is
- The four different types of annuities
- The difference between annuities and single payments
- The annuity timeline format

#### Future Value of Annuities

---

- The future value of ordinary annuities
- Variable changes in future value annuity calculations
- The future value of annuities due

#### Present Value of Annuities

---

- The present value of both ordinary annuities and annuities due
- Variable changes in present value annuity calculations
- Applying both future value and present value calculations to loans
- Determining loan balances
- Selling loan contracts between companies

#### Annuity Payment Amounts

---

- Calculating the annuity payment amount for both ordinary annuities and annuities due

## Number of Annuity Payments

---

- Calculating the number of annuity payments (term) for both ordinary annuities and annuities due
- What to do when  $N$  has decimals

## Annuity Interest Rates

---

- Calculating the interest rate for both ordinary annuities and annuities due

## THE FORMULAS YOU NEED TO KNOW

---

### Symbols Used

---

$C/Y$  = compounding per year or compounding frequency

$FV_{Due}$  = future value of annuity due

$FV_{Ord}$  = future value of an ordinary annuity

$I/Y$  = nominal interest rate

$i$  = periodic interest rate

$i$  = number of annuity payments

$P/Y$  = payments per year or payment frequency

$PMT$  = annuity payment amount

$PV_{Due}$  = present value of annuity due

$PV_{Ord}$  = present value of ordinary annuity

$Years$  = the term of the annuity

## FORMULAS INTRODUCED

---

Number of Annuity Payments:

$$n = P/Y \times (\text{Number of Years})$$

Ordinary Annuity Future Value:

$$FV_{ORD} = PMT \left[ \frac{(1+i)^n - 1}{i} \right]$$

Annuity Due Future Value:

$$FV_{DUE} = PMT \left[ \frac{(1+i)^n - 1}{i} \right] \times (1+i)$$

Ordinary Annuity Present Value:

$$PV_{ORD} = PMT \left[ \frac{1 - (1+i)^{-n}}{i} \right]$$

Annuity Due Present Value:

$$PV_{DUE} = PMT \left[ \frac{1 - (1+i)^{-n}}{i} \right] \times (1+i)$$

## 8 COMPOUND INTEREST - SPECIAL APPLICATIONS

### Learning Objectives

- Understand how to determine the current worth of future payments and identify the duration of deferral (delay) before an annuity starts to pay out.
- Calculate the present value, payment, and period of deferral for a deferred annuity.

## 8.1 Deferred Annuities

### WHAT IS A DEFERRED ANNUITY?

A **deferred annuity** is a financial transaction where annuity payments are delayed until a certain period of time has elapsed. Usually, the annuity has two stages, as illustrated in this figure.

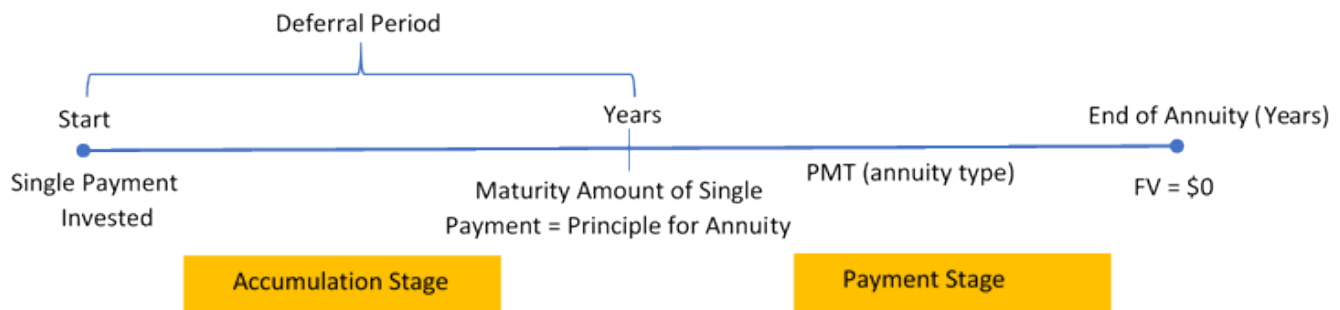


Figure 8.1.0: Timeline for a Deferred Annuity [Image Description]

1. **Accumulation Stage.** A single payment is allowed to earn interest for a specified duration. There are no annuity payments during this period of time, which is commonly referred to as the **period of deferral**.
2. **Payments Stage.** The annuity takes the form of any of the four annuity types and starts at the beginning of this stage as per the financial contract.

Note that the maturity value of the accumulation stage is the same as the principal for the payments stage, i.e. the starting amount or original sum for the phase when the annuity starts making regular payouts.

The interest rate on deferred annuities can be either variable or fixed. However, since deferred annuities are commonly used to meet a financial goal, fixed interest rates are more prevalent since they allow for certainty in the calculations.

THE FORMULA

For a deferred annuity, you apply a combination of formulas that you have already used throughout this book. The accumulation stage is not an annuity, so it uses the various single payment compound interest formulas from Chapter 5. The payments stage is an annuity, so it uses the various annuity formulas from Chapter 7.

HOW IT WORKS

For deferred annuities, the most common unknown variables are either the present value, the length of the period of deferral, the annuity payment amount, or the number of annuity payments that can be made before the funds are fully used up. Follow this sequence of steps for each of these variables:

Table 8.1.1 Steps to Solve for Various Variables in Deferred Annuities

Solving for the Present Value	Solving for the Period of Deferral	Solving for the Annuity Payment Amount	Solving for the Number of Annuity Payments
<b>Step 1:</b> Draw a timeline and identify the variables that you know, along with the annuity type.	<b>Step 1:</b> Draw a timeline and identify the variables that you know, along with the annuity type.	<b>Step 1:</b> Draw a timeline and identify the variables that you know, along with the annuity type.	<b>Step 1:</b> Draw a timeline and identify the variables that you know, along with the annuity type.
<b>Step 2:</b> Starting at the end of your timeline, calculate the present value of the annuity using the steps from Section 7.3 (Formulas 7.4 or 7.5). Round your answer to two decimals.	<b>Step 2:</b> Starting at the end of your timeline, calculate the present value of the annuity using the steps from Section 7.3 (Formulas 7.4 or 7.5). Round your answer to two decimals.	<b>Step 2:</b> Starting at the beginning of the timeline, calculate the future value of the single payment using the steps from Section 5.2 (Formula 5.3). Round your answer to two decimals.	<b>Step 2:</b> Starting at the beginning of the timeline, calculate the future value of the single payment using the steps from Section 5.2 (Formula 5.3). Round your answer to two decimals.
<b>Step 3:</b> Take the principal of the annuity, and using the steps from Section 5.3 (Formula 5.3) calculate the present value for the single amount.	<b>Step 3:</b> Solve for the number of compounding periods using the applicable steps from Section 5.7 (Formula 5.3). The single payment investment is the present value, and the principal of the annuity is the future value.	<b>Step 3:</b> Calculate the annuity payment amount using steps from Section 7.4 (Formula 7.4 or 7.5).	<b>Step 3:</b> Calculate the number of annuity payments using steps from Section 7.5 (Formula 7.4 or 7.5).

ROUNDING

The maturity value of the single payment or the present value of the annuity is always rounded to two decimals.

Combining the Deferral Period and the Annuity Term.

It is an error to treat the period of deferral and the term of the annuity as simultaneous time periods. For example, if a deferred annuity has a three-year period of deferral and a 10-year annuity term, this is sometimes interpreted, mistakenly, as an annuity ending 10 years from today. These time segments

are separate and consecutive on the timeline! The correct interpretation is that the annuity term ends 13 years from today, since the 10-year term does not start until the three-year deferral terminates.

### Incorrect Timing between Stages

A common mistake is to determine incorrectly when the period of deferral ends and the annuity starts. This error usually results from forgetting that the payments on ordinary annuities start one payment interval after the annuity starts, whereas annuity due payments start immediately. Thus, if the first quarterly payment on an ordinary annuity is to be paid  $6\frac{3}{4}$  years from today, then the period of deferral is  $6\frac{1}{2}$  years. If the deferral is for an annuity due, then the period of deferral is  $6\frac{3}{4}$  years.

### Confusing $n$

- A deferred annuity requires different calculations of  $n$  using either Formula 5.2A or Formula 7.1.
- In the *accumulation* stage, recall that  $n$  must represent the number of *compound* periods calculated by Formula 5.2A.
- In the *payments* stage, the  $n$  must represent the number of *annuity payments* calculated by Formula 7.1.

### CONCEPT CHECK



An interactive H5P element has been excluded from this version of the text. You can view it online here:

<https://pressbooks.nsc.ca/businessmath2500/?p=170#h5p-46>

#### Example 8.1.1: Investing an Inheritance for Your Retirement

Frasier is 33 years old and just received an inheritance from his parents' estate. He wants to invest an amount of money today such that he can receive \$5,000 at the end of every month for 15 years when he retires at age 65. If he can earn 9% compounded annually until age 65 and then 5% compounded annually when the fund is paying out, how much money must he invest today?

#### **Solution**

Calculate the single payment that must be invested today. This is the present value (PV) of the deferred annuity.

**Step 1:** The deferred annuity has monthly payments at the end with an annual interest rate. Therefore, this is an ordinary general annuity.

The timeline for the deferred annuity appears below.

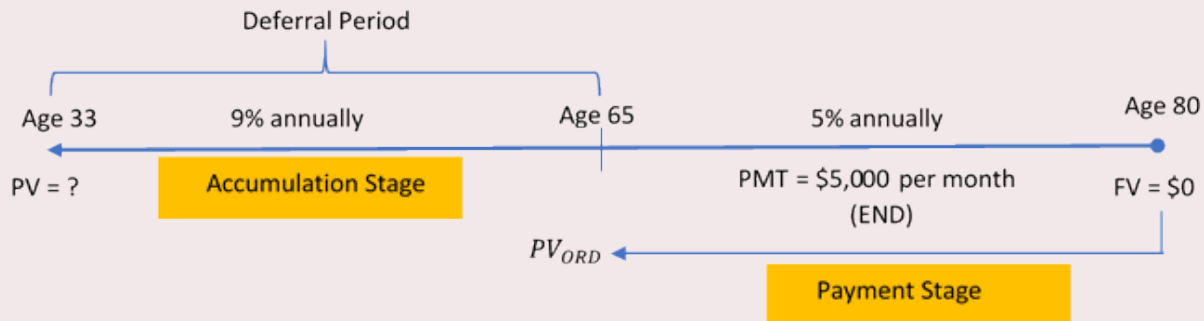


Figure 8.1.1: Timeline [Image Description]

**Ordinary General Annuity (Payment Stage):**

FV = \$0; I/Y = 5%; C/Y = 1; PMT = \$5,000; P/Y = 12; Years = 15

**Period of Deferral (Accumulation Stage):**

FV = PV<sub>ORD</sub>; I/Y = 9%; C/Y = 1; Years = 32

**Step 2:** Ordinary General Annuity (Payment Stage):

Calculate the equivalent periodic interest rate that matches the payment interval ( $i_{eq}$ , Formula 5.6), number of annuity payments ( $n$ , Formula 7.1), and present value of the ordinary general annuity (PV<sub>ORD</sub>, Formula 7.3A).

$$i = \frac{I/Y}{C/Y} = \frac{5\%}{1} = 5\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.05)^{\frac{1}{12}} - 1 = 0.004074124 \text{ per month}$$

$$n = P/Y \times (\text{Number of Years}) = 12 \times 15 = 180 \text{ payments}$$

$$\begin{aligned}
 PV_{ORD} &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\
 &= \$5,000 \left[ \frac{1 - (1 + 0.004074124)^{-180}}{0.004074124} \right] \\
 &= \$5,000 \left[ \frac{0.518982921}{0.004074124} \right] \\
 &= \$636,925.79
 \end{aligned}$$

**Step 3:** Deferral Period (Accumulation Stage):

Discount the principal of the annuity ( $PV_{ORD}$ ) back to today (Age 33). Calculate the periodic interest rate ( $i$ , [Formula 5.1](#)), number of single payment compound periods ( $n$ , [Formula 5.2A](#)), and present value of a single payment ( $PV$ , [Formula 5.2B](#)), rearranged.

$$i = \frac{I/Y}{C/Y} = 9\%1 = 9\%$$

$$n = C/Y \times (\text{Number of Years}) = 1 \times 32 = 32 \text{ compounds}$$

$$\begin{aligned}
 PV &= \frac{FV}{(1 + i)^n} \\
 &= \frac{\$636,925.79}{(1 + 0.09)^{32}} \\
 &= \$40,405.54
 \end{aligned}$$

If Frasier invests \$40,405.54 today, he will have enough money to sustain 180 withdrawals of \$5,000 in retirement.

**Example 8.1.2: Planning the Deferral Period**

Bashir wants an annuity earning 4.3% compounded semi-annually to pay him \$2,500 at the beginning of every month for 10 years. To achieve his goal, how far in advance of the start of the annuity does Bashir need to invest \$50,000 at 8.25% compounded quarterly? Assume 91 days in a quarter.

**Solution**

Calculate the amount of time between today and the start of the annuity. This is the period of deferral, or  $n$ .

**Step 1:** The deferred annuity has monthly payments at the beginning with a semi-annual interest rate. Therefore, this is a general annuity due.

The timeline for the deferred annuity appears below.

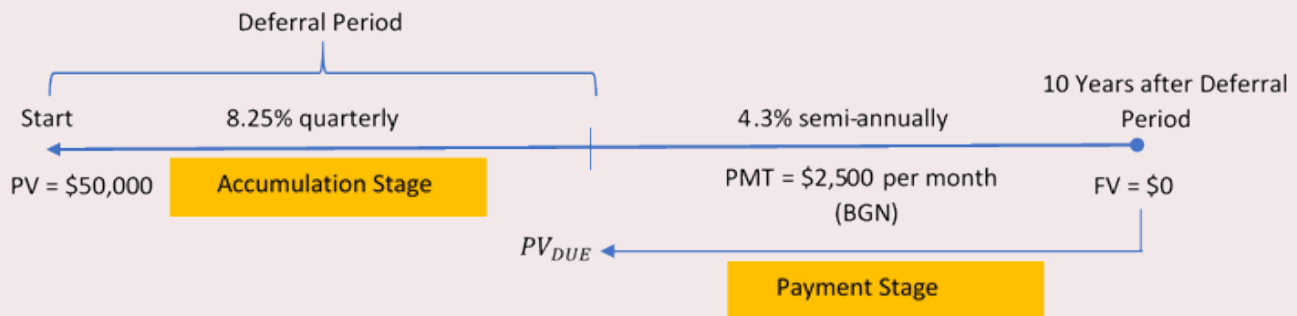


Figure 8.1.2: Timeline [Image Description]

### Ordinary General Annuity (Payment Stage):

FV = \$0; I/Y = 4.3%; C/Y = 2; PMT = \$2,500; P/Y = 12; Years = 10

### Period of Deferral (Accumulation Stage):

PV = \$50,000; FV = PV<sub>DUE</sub>; I/Y = 8.25%, C/Y = 4

### Step 2: Ordinary General Annuity (Payment stage):

Calculate the equivalent periodic interest rate that matches the payment interval ( $i_{eq}$ , Formula 5.6), number of annuity payments ( $n$ , Formula 7.1), and present value of the annuity due (PV<sub>DUE</sub>, Formula 7.3B).

$$i = \frac{I/Y}{C/Y} = \frac{4.3\%}{2} = 2.15\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.0205)^{\frac{2}{12}} - 1 = 0.003551648 \text{ per month}$$

$$n = P/Y \times (\text{Number of Years}) = 12 \times 10 = 120 \text{ payments}$$

$$\begin{aligned} PV_{DUE} &= PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq}) \\ &= \$2,500 \left[ \frac{1 - (1 + 0.003551648)^{-120}}{0.003551648} \right] \times (1 + 0.003551648) \\ &= \$244,780.93 \end{aligned}$$

This becomes FV for the deferral period in step 3.

### Step 3: Deferral Period (Accumulation stage):

$$i = \frac{I/Y}{C/Y} = \frac{8.25\%}{4} = 2.0625\%$$

$$FV = PV(1 + i)^n$$

$$\$244,780.93 = \$50,000(1 + 0.020625)^n$$

$$4.895618 = 1.020625^n$$

$$\ln(4.895618) = n \times \ln(1.020625)$$

$$n = \frac{1.588340}{0.020415}$$

$$= 77.801923 \text{ quarterly compounds}$$

Convert  $n$  to years, months and days.

$$\begin{aligned} \text{Years} &= \frac{77}{4} \\ &= 19.25 = 19 \text{ years, 3 months} \end{aligned}$$

$$0.801923 \times 91 \text{ days} = 72.975105 = 73 \text{ days}$$

19 years, 3 months, 73 days

To achieve his goal, Bashir needs to invest the \$50,000 19 years, 3 months and 73 days before the annuity starts.

#### Example: 8.1.3: How Much Income will It Provide

On the day of their granddaughter's birth, Henri and Frances deposited \$3,000 into a trust fund for her future education. The fund earns 6% compounded monthly. When she turns 18, they then want it to make payments at the end of every quarter for five years. If the income annuity can earn 4.5% compounded quarterly, what is the amount of each annuity payment to the granddaughter?

#### Solution

**Step 1:** The deferred annuity has quarterly payments at the end with a quarterly interest rate. Therefore, this is an ordinary simple annuity.

The timeline for the deferred annuity appears below.

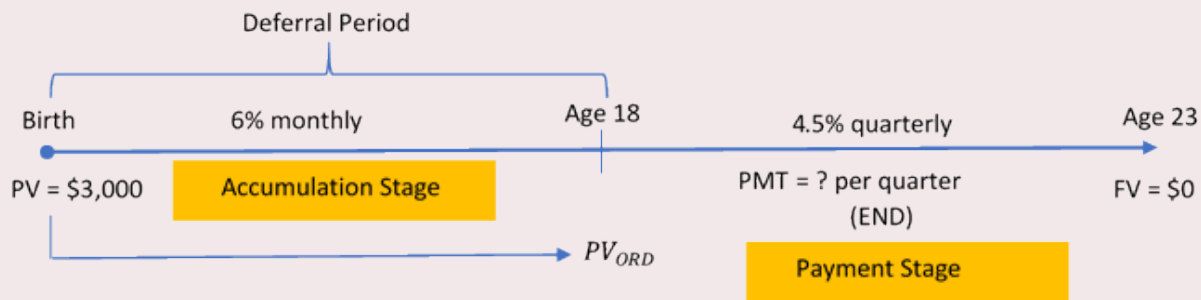


Figure 8.1.3: Timeline [Image Description]

**Period of Deferral (Accumulation Stage):**

PV = \$3,000; I/Y = 6%, C/Y = 12; Years = 18

**Ordinary Simple Annuity (Payment Stage):**

PV<sub>ORD</sub> = FV (after deferral); FV = \$0; I/Y = 4.5%; C/Y = 4; P/Y = 4; Years = 5

**Step 2:** Calculate the future value of the single payment investment. Calculate the periodic interest rate (i, [Formula 5.1](#)), number of single payment compound periods (n, [Formula 5.2A](#)), and future value of a single payment amount (FV, [Formula 5.2B](#)).

$$i = \frac{I/Y}{C/Y} = \frac{6\%}{12} = 0.5\%$$

$$n = C/Y \times (\text{Number of Years}) = 12 \times 18 = 216 \text{ compounds}$$

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$3,000(1 + 0.005)^{216} \\ &= \$8,810.30 \end{aligned}$$

**Step 3:** Work with the ordinary simple annuity. First, calculate the periodic interest rate (i, [Formula 5.1](#)), number of annuity payments (n, [Formula 7.1](#)), and finally the annuity payment amount (PMT, [Formula 7.3A](#)).

$$i = \frac{I/Y}{C/Y} = \frac{4.5\%}{4} = 1.125\%$$

$$n = P/Y \times (\text{Number of Years}) = 4 \times 5 = 20 \text{ payments}$$

$$\begin{aligned}
 PV_{ORD} &= PMT \left[ \frac{1 - (1 + i)^{-n}}{i} \right] \\
 \$8,810.30 &= PMT \left[ \frac{1 - (1 + 0.01125)^{-20}}{0.01125} \right] \\
 \$8,810.30 &= PMT \left[ \frac{0.200480}{0.01125} \right] \\
 \$8,810.30 &= PMT(17.820448) \\
 PMT &= \frac{\$8,810.30}{17.820448} \\
 &= \$494.39
 \end{aligned}$$

The granddaughter will receive \$494.39 at the end of every quarter for five years starting when she turns 18. Since the payment is rounded, the very last payment is a slightly different amount.

#### Example 8.1.4: How Long Can the Annuity be Sustained

Emile received a \$25,000 one-time bonus from his employer today, and he immediately invested it at 8% compounded annually. Fourteen years from now, he plans to withdraw \$2,300 at the beginning of every month to use as his retirement income. If the income annuity can earn 3.25% compounded semi-annually, what is the term of the annuity before it is depleted (including the smaller final payment)?

#### Solution

Figure out how long the income annuity is able to sustain the income payments. This requires you to calculate the number of annuity payments, or  $n$ .

**Step 1:** The deferred annuity has monthly payments at the beginning with a semi-annual interest rate. Therefore, this is a general annuity due.

The timeline for the deferred annuity appears below.

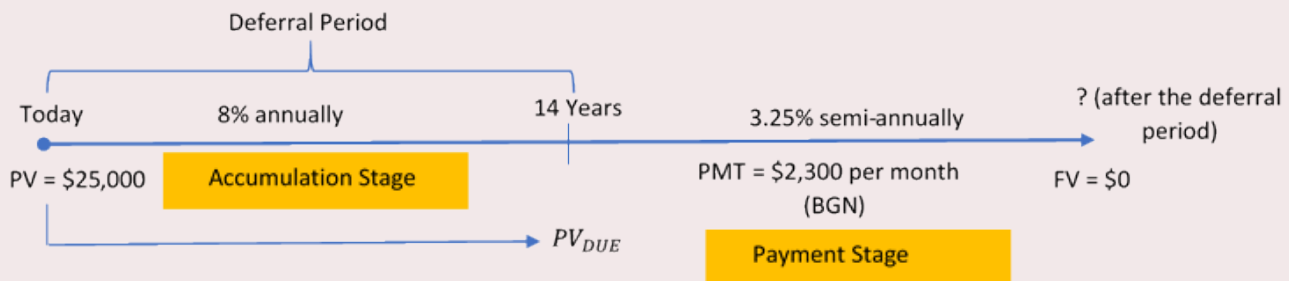


Figure 8.1.4: Timeline [Image Description]

Period of Deferral (Accumulation Stage):

PV = \$25,000; I/Y = 8%, C/Y = 1; Years = 14

**General Annuity Due (Payment Stage):**

PV<sub>DUE</sub> = FV (after deferral); FV = \$0; I/Y = 3.25%; C/Y = 2; PMT = \$2,300; P/Y = 12

**Step 2:** Calculate the future value of the single deposit. Calculate the periodic interest rate ( $i$ , Formula 5.1), number of single payment compound periods ( $n$ , Formula 5.2A), and future value of a single payment amount (FV, Formula 5.2B).

$$i = \frac{I/Y}{C/Y} = \frac{8\%}{1} = 8\%$$

$$n = C/Y \times (\text{Number of Years}) = 1 \times 14 = 14 \text{ compounds}$$

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$25,000(1 + 0.08)^{14} \\ &= \$73,429.84 \end{aligned}$$

**Step 3:** Work with the general annuity due. Calculate the equivalent periodic interest rate ( $i_{eq}$ , Formula 5.6) that matches the payment interval and the number of annuity payments ( $n$ , Formula 7.3B rearranged for  $n$ ). Finally substitute into the annuity payments Formula 7.1 to solve for Years.

$$i = \frac{I/Y}{C/Y} = \frac{3.25\%}{2} = 1.625\%$$

$$i_{eq} = (1 + i)^{\frac{C/Y}{P/Y}} - 1 = (1 + 0.01625)^{\frac{2}{12}} - 1 = 0.002690 \text{ per month}$$

$$PV_{DUE} = PMT \left[ \frac{1 - (1 + i_{eq})^{-n}}{i_{eq}} \right] \times (1 + i_{eq})$$

$$73,429.84 = \$2,300 \left[ \frac{1 - (1 + 0.002690)^{-n}}{0.002690} \right] \times (1 + 0.002690)$$

$$31.840361 = \frac{1 - 0.997317^n}{0.002690}$$

$$0.085656 = 1 - 0.99730$$

$$\ln(0.997317) = n \times \ln(0.997317)$$

$$n = 33.332019 = 34 \text{ payments}$$

$$\begin{aligned}
 \text{Years} &= \frac{34}{12} \\
 &= 2.\overline{83} \\
 &= 2 \text{ years and } 0.8\overline{3} \times 12 = 10 \text{ months}
 \end{aligned}$$

The annuity will last 2 years and 10 months before being depleted.

### Exercises

In each of the exercises that follow, try them on your own. Full solutions are available should you get stuck.

1. What is the present value of a deferred annuity with a deferral period of 17 years at 6.7% compounded semi-annually followed by a 10-year annuity due paying \$1,250 at the beginning of every month at 4.78% compounded semi-annually?
2. Your objective is an annuity due paying \$5,000 semi-annually for 5.5 years at 4% compounded quarterly. How far in advance of this would you need to invest \$20,000 at 6.82% compounded monthly? Express answer in years and months.
3. Jeff and Sarah want an ordinary annuity to pay their daughter \$1,000 monthly for three years and nine months for the duration of her educational studies starting August 1, 2024. What lump-sum amount do they need to invest on August 1, 2014, if the deferred annuity can earn 6.6% compounded monthly during the accumulation stage and 4% compounded quarterly during the income payments stage?
4. At the age of 44, Parker just finished all the arrangements on his parents' estate. He is going to invest his \$80,000 inheritance at 5.5% compounded quarterly until he retires at age 55, and then wants to receive month-end payments for the next 25 years. The income annuity is expected to earn 3.85% compounded annually. What are his monthly annuity payments during his retirement?
5. Once Jason graduated college at age 22, he invested \$350 into his RRSP at the beginning of every month until age 40. He then stopped his contributions and let the amount earn interest until today, when at age 62 he decided to retire. He wants his retirement money to last until age 85. If his account can earn 10.4% compounded quarterly before age 62 and 4.8% compounded annually after that, how much money can he expect to receive at the end of every quarter?



An interactive H5P element has been excluded from this version of the text. You can view it online here:  
<https://pressbooks.nsc.ca/businessmath2500/?p=170#h5p-47>

Timelines for the exercises are included in Solutions to Exercises

## 8.2 Review, Symbols, and Formulas

### KEY CONCEPTS

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#### Deferred Annuities

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- The stages of deferred annuities
- The four common unknown variables and how to solve for them

### THE FORMULAS YOU NEED TO KNOW

---

#### Symbols Used

---

$C/Y$  = compounding frequency

$i$  = periodic interest rate

$N$  = number of annuity payments

$PMT$  = annuity payment amount

$PV_{Ord}$  = present value of an ordinary annuity

$P/Y$  = payment frequency

#### Formulas Introduced

---

No new formulas are introduced

ORDER OF OPERATIONS EXERCISE ANSWERS

---

1. 15
2. 10
3. 40
4. 100
5. 55
6. 10
7. \$1,021.70
8. \$7,645.52
9. \$19,215.37
10. 400
11. \$25,967.38
12. \$1,345.87
13. \$5,978.08
14. \$1,056,767.41
15. \$2,156,764.06
16. \$41,271.46
17. \$83,228.60
18. \$2,339.07
19. \$26,450.25
20. \$14,561.43

*FIGURING OUT THE COST: DISCOUNTS*

---

1. D\$ = \$411.60; N = \$568.40
2. L = \$800; D\$ = \$200
3. d = 42.625%; N = \$1,133.16; D\$ = \$841.84
4. d = 26.7904%; L = \$500; D\$ = \$133.95
5. \$259.97
6. \$24.32
7. 31.985%
8. \$19.99
9. d = 43.972%; L = \$51.76
10. L = \$24; N = \$13.92
11. d = 62.8%; L = \$700; D\$ = \$439.60
12. d = 26%; N = \$303.99; D\$ = \$96
13. \$1,026
14. d = 63.14%; N = \$29.49
15. \$1,194,553.06
16. 2 years 216 days
17. Reduced by 17.4154%
- 18a. 3%
- 18b. \$2.50
- 19a. 52.2328%
- 19b. \$3,073.82
- 20a. \$74.99

20b. \$60.91

20c. L = \$78.74; N = \$63.96; D\$ = \$14.78

### *MARKUP: SETTING THE REGULAR PRICE*

---

1. S=\$321.95; S<sub>BE</sub>=\$236.95; M\$=\$133.53; MoC%=70.8683%; MoS%=41.4754%

2. C=\$653.59; S<sub>BE</sub>=\$849.67; M\$=\$346.41; MoC%=53.0011%; MoS%=34.6413%

3. C=\$118.25; S=\$301.53; E=\$153.13; MoS%=60.7833%; S<sub>BE</sub>=\$271.38

4. M\$=\$96.25; P=\$41.25; C=\$178.74; S<sub>BE</sub>=\$223.74

5. C=\$744.83; S=\$1,284.83; P=\$204.83; MoC%=72.5%; MoS%=42.0289%

6. M\$=81.60; S=\$201.60; E=\$51.36; MoS%=40.4762%; S<sub>BE</sub>=\$171.36

7. P=\$175; S=\$1,447.37; C=\$1,172.37; MoC%=23.4568%; S<sub>BE</sub>=\$1,272.37

8. C=\$220; S=\$287.50; M\$=\$67.50; MoS%=23.4783%; MoC%=30.6818%

9. \$112.50

10. 125%

11. \$27,996.79

12. \$57.05

13a. \$733.03

13b. \$219.91

13c. 53.3151%

13d. \$698.03

14a. \$119.95

14b. \$40.85

14c. 51.6435%

14d. 34.0559%

14e. 28.8887%

15. \$20.82

- 16. \$102.40
- 17. 42.8571%
- 18. \$488.09
- 19a. \$40.47
- 19b. \$21.79
- 19c. \$62.26
- 19d. \$14.51
- 19e. \$12.20; \$2.31 reduction
- 20a. \$12.65
- 20b. \$17.71
- 20c. \$0.63
- 20d. MoS% = 28.5714%
- 20e. C = \$15.18; S = \$21.25; P = \$0.76; MoS% = 28.5647%

#### *MARKDOWN: SETTING THE SALE PRICE*

---

- 1. D\$ = \$153.95; S<sub>onsale</sub> = \$285.90
- 2. S = \$299.95; d = 33.3389%
- 3. D\$ = \$275; d = 26.1905%
- 4. S<sub>onsale</sub> = \$25,525; d = 11.2945%
- 5. S = \$19,701.42; D\$ = \$6,501.47
- 6. S = \$319.42; S<sub>onsale</sub> = \$281.09
- 7. S<sub>onsale</sub> = \$53.99
- 8. D\$ = \$45; d = 52.9474%
- 9. d = 14.9%
- 10. S = \$133.30; D\$ = \$93.31
- 11. S<sub>onsale</sub> = \$384.99; S = \$699.98

$$12. d = 36.7893$$

$$13a. P_{\text{onsale}} = \$2.59$$

$$13b. P = \$74.55$$

$$14. d = 10.9512\%$$

$$15a. S_{\text{onsale}} = \$1,519.05$$

$$15b. P_{\text{onsale}} = \$303.81$$

$$15c. S = \$2,025.40$$

$$15d. P = \$810.16$$

$$16. S_{\text{onsale}} = \$367.20$$

$$17a. S_{\text{onsale}} = \$282.91$$

$$17b. d = 29.2637\%$$

$$17c. MoS\% = 23.8026\%$$

$$18. D = 32.4325\%$$

$$19a. S_{\text{onsale}} = \$1,680$$

$$19b. MoS\% \text{ when on sale} = 32.8571\%$$

$$19c. P_{\text{onsale}} = \$192.00$$

$$20a. S_{\text{onsale}} = \$1,614.06$$

$$20b. S = \$2,152.08$$

$$20c. M\$ = \$956.48; MoC\% = 80\%$$

### SALES TAXES

---

$$1. S_{\text{tax}} = \$682.49$$

$$2. S_{\text{tax}} = \$750.74$$

$$3. S_{\text{tax}} = \$747.49$$

$$4. S_{\text{tax}} = \$727.99$$

$$5. S = \$796.45; \text{HST Tax Amount} = \$103.54$$

6.  $S = \$818.17$ ; GST Tax Amount = \$40.91; PST Tax Amount = \$40.91
7. January Remit \$5,471.40; February Refund \$2,202.96
8. Winter Remit \$7,171.20; Spring Refund \$1,025.25; Summer Remit \$9,803.30; Fall Refund \$1,191.65
- 9a. Alberta is best = \$1,396.45
- 9b. Savings = \$17.48
10.  $S = \$529.99$ ; GST Tax Amount = \$26.50
11. Can't purchase at  $S_{\text{tax}} = \$3,050.94$ ; Down Payment = \$45.08
12.  $S_{\text{tax}} = \$4,477.30$ ; GST Tax Amount = \$199.88; PST Tax Amount = \$279.83
13. Q1 Remit \$17,865; Q2 Remit \$35,370; Q3 Refund \$8,105; Q4 Refund \$24,210
14. March Remit \$951.15; April Refund; \$1,257.15; May Remit \$2,953.65
15. Higher in Ontario by 1.5274%.
16. Total PST = \$13.02; Total GST = \$52.46; Total HST = \$13
17. Remit \$378,227.85
18. PST Rate = 9.505%
- 19a.  $S = \$352.72$
- 19b.  $S = \$1,682.81$
- 19c.  $S = \$19,465.09$
- 19d.  $S = \$4,736.57$
20. Remit \$31,331.89

#### *INVOICING: TERMS OF PAYMENT AND CASH DISCOUNTS*

---

1. \$133,568.68
2. \$96,513.10
3. \$46,744.67
4. Balance Remaining = \$21,928.01
5. Balance Remaining = \$434,964.99

6. Balance Remaining = \$7,065.01

7. \$13,692.78

8. \$232,444.92

9. \$204,664.43

10. \$35,901.77

11. \$12,872.93

12. \$23,372.94

13. \$2,654.57

14. \$5,479.14

15. \$4,241.10

16. \$24,772.64

17. \$40,608.80

18. N = \$13,511.24; Late penalty amount = \$519.66

19a. Choose Plan #2

19b. Savings = \$724.92

20. \$250,214.41

### 3.1 TIME VALUE OF MONEY

---

*"A bird in the hand is worth two in the bush"* relates to the concept of Time Value of Money as it implies that having something tangible today is better than having the uncertain potential of something in the future. It speaks to the earning potential that the money in hand can generate over time.

A "delay to liquidity" is created by time. This delay affects the value of your wealth by creating risk, transaction costs, and opportunity costs. The farther you are from your liquidity, the more you are taking away from the present value of your non-liquid wealth.

### 3.2 VALUING A SERIES OF CASH FLOWS

---

Analyzing the Discount Rate applied to a lump-sum payment will help you determine which will have a greater value for you. The lump-sum payment offers you more liquidity sooner.

The Present Value and Future Value of an annuity will be greater as the cash flow or as the time increases. The Present Value of an annuity will be greater as the rate decreases and the Future Value of an annuity will be greater as the rate increases.

### 3.3 CALCULATING THE RELATIONSHIP OF TIME AND VALUE

---

In the Slide Show, the value of 'i' (the interest rate per period) is used as the discount rate and 'N' (number of compounding periods) is used in place of time.

You should increase positive cash flows at an earlier date and decrease negative cash flows at further down the road, as it is more valuable for you to have liquidity sooner rather than later.

### 4.1 PRINCIPAL, RATE, TIME

---

1. If you want to earn \$1,000 of simple interest at a rate of 7% in a span of five months, how much money must you invest?

**Solution:**

**Step 1:** Given information:

$$I = \$1,000; r = 7\% \text{ annually}; t = 5 \text{ months}$$

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{5}{12}$

**Step 3:** Solve for  $P$ .

$$\begin{aligned} P &= \frac{I}{rt} \\ &= \frac{\$1,000}{0.07 \left( \frac{5}{12} \right)} \\ &= \$34,285.71 \end{aligned}$$

I must invest \$34,285.71.

2. If you placed \$2,000 into an investment account earning 3% simple interest, how many months does it take for you to have \$2,025 in your account?

**Solution:**

**Step 1:** Given information:

$$I = \$2,025 - \$2,000 = \$25; P = \$2,000; r = 3\% \text{ annually}$$

**Step 2:** Convert annual  $r$  to match monthly  $t$ ;  $r = \frac{3\%}{12}$

**Step 3:** Solve for  $t$ .

$$\begin{aligned}
 t &= \frac{I}{Pr} \\
 &= \frac{\$25}{\$2,000 \left( \frac{0.03}{12} \right)} \\
 &= 5 \text{ months}
 \end{aligned}$$

It takes 5 months to have \$2,025 in the account.

3. A \$3,500 investment earned \$70 of interest over the course of six months. What annual rate of simple interest did the investment earn?

### **Solution:**

**Step 1:** Given information:

$$P = \$3,500; I = \$70; t = 6 \text{ months}$$

**Step 2:** Convert the time period from months to years;  $t = \frac{6}{12}$

**Step 3:** Solve for  $r$ .

$$\begin{aligned}
 r &= \frac{I}{Pt} \\
 &= \frac{\$70}{\$3,500 \left( \frac{6}{12} \right)} \\
 &= 0.04 \text{ or } 4\%
 \end{aligned}$$

The investment earned 4% simple interest.

## **TIME AND DATES**

1. Brynn borrowed \$25,000 at 1% per month from a family friend to start her entrepreneurial venture on December 2, 2011. If she paid back the loan on June 16, 2012, how much simple interest did she pay?

### **Solution:**

**Step 1:** Given information:

$$P = \$25,000; r = 1\% \text{ monthly}; t = \text{December 2, 2011 to June 16, 2012}$$

Use DATE function on calculator to get the number of days. Total days for  $t = 197$

**Step 2:** Convert both the monthly  $r$  and the daily  $t$  to annual numbers;

$$r = 1\% \times 12 = 12\% \text{ annually}; t = \frac{197}{365}$$

**Step 3:** Solve for  $I$ .

$$\begin{aligned} I &= Prt \\ &= \$25,000(0.12) \left( \frac{197}{365} \right) \\ &= \$1,619.18 \end{aligned}$$

She paid \$1,619.18 simple interest.

2. If \$6,000 principal plus \$132.90 of simple interest was withdrawn on August 14, 2011, from an investment earning 5.5% interest, on what day was the money invested?

**Solution:**

Need to calculate  $t$  in days first.

**Step 1:** Given information:

$$I = \$132.90; P = \$6,000; r = 5.5\% \text{ annually}$$

**Step 2:** Convert annual  $r$  to match daily  $t$ ;  $r = \frac{5.5\%}{365}$

**Step 3:** Solve for  $t$ .

$$\begin{aligned} t &= \frac{I}{Pr} \\ &= \frac{\$132.90}{\$6,000 \left( \frac{0.055}{365} \right)} \\ &= 146.995454 \text{ days} \rightarrow 147 \text{ days} \end{aligned}$$

Use the DATE function on the calculator to find the date when the money was invested.

The money was invested on March 20, 2011.

## 4.2 MOVING MONEY INVOLVING SIMPLE INTEREST

1. An accountant needs to allocate the principal and simple interest on a loan payment into the appropriate ledgers. If the amount received was \$10,267.21 for a loan that spanned April 14 to

July 31 at 9.1%, how much was the principal and how much was the interest?

**Solution:**

**Step 1:** Given information:

$S = \$10,267.21$ ;  $r = 9.1\%$  annually;  $t = \text{April 14 to July 31} = 108 \text{ days}$

**Step 2:** Convert daily  $t$  to match annual  $r$ ;  $t = \frac{108}{365}$

**Step 3:** Solve for  $P$ .

$$\begin{aligned} P &= \frac{S}{1 + rt} \\ &= \frac{\$10,267.21}{1 + 0.091 \times \frac{108}{365}} \\ &= \$9,998 \end{aligned}$$

**Step 4:** Solve for  $I$ .

$$\begin{aligned} I &= S - P \\ &= \$10,267.21 - \$9,998 \\ &= \$269.21 \end{aligned}$$

The principal was \$9,998 and the simple interest on the loan was \$269.21.

- Suppose Robin borrowed \$3,600 on October 21 and repaid the loan on February 21 of the following year. What simple interest rate was charged if Robin repaid \$3,694.63?

**Solution:**

**Step 1:** Given information:

$P = \$3,600$ ;  $S = \$3,694.63$ ;  $t = \text{October 21 to February 21} = 123 \text{ days}$

**Step 2:** Compute  $I$ .

$$\begin{aligned} I &= S - P \\ &= \$3,694.63 - \$3,600 \\ &= \$94.63 \end{aligned}$$

**Step 3:** Convert daily  $t$  to match annual  $r$ ;  $t = \frac{123}{365}$

**Step 4:** Solve for  $r$ .

$$\begin{aligned} r &= \frac{I}{Pt} \\ &= \frac{\$94.63}{\$3,600 \left( \frac{123}{365} \right)} \\ &= 0.078004 \text{ or } 7.8004\% \end{aligned}$$

The simple interest charged was 7.80%.

3. Jayne needs to make three payments to Jade requiring \$2,000 each 5 months, 10 months, and 15 months from today. She proposes instead making a single payment eight months from today. If Jade agrees to a simple interest rate of 9.5%, what amount should Jayne pay?

**Solution:**

**Step 1:** Given information:

$$r = 9.5\% \text{ annually}$$

**Payment #1:**  $P = \$2,000$ ;  $t = 8\text{ months} - 5\text{ months} = 3\text{ months}$

**Payment #2:**  $S = \$2,000$ ;  $t = 10\text{ months} - 8\text{ months} = 2\text{ months}$

**Payment #3:**  $S = \$2,000$ ;  $t = 15\text{ months} - 8\text{ months} = 7\text{ months}$

Replacement payment is eight months from today.

**$r = 9.5\%$  annually throughout**

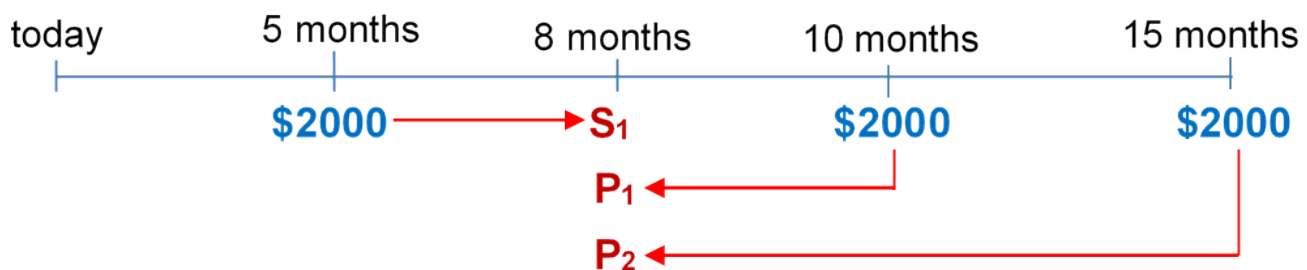


Figure 8.2.2: Timeline [Image Description]

**Payment #1:**

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{3}{12}$

**Step 3:** Solve for  $S_1$ .

$$\begin{aligned}
 S_1 &= P(1 + rt) \\
 &= \$2,000 \times \left(1 + 0.095 \times \frac{3}{12}\right) \\
 &= \$2,000 \times 1.02375 \\
 &= \$2,047.50
 \end{aligned}$$

**Payment #2:**

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{2}{12}$

**Step 3:** Solve for  $P_1$ .

$$\begin{aligned}
 P_1 &= \frac{S}{1 + rt} \\
 &= \frac{\$2,000}{1 + 0.095 \times \frac{2}{12}} \\
 &= \frac{\$2,000}{1.0158\bar{3}} \\
 &= \$1,968.83
 \end{aligned}$$

**Payment #3:**

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{7}{12}$

**Step 3:** Solve for  $P_2$ .

$$\begin{aligned}
 P_2 &= \frac{S}{1 + rt} \\
 &= \frac{\$2,000}{1 + 0.095 \times \frac{7}{12}} \\
 &= \frac{\$2,000}{1.05541\bar{6}} \\
 &= \$1,894.99
 \end{aligned}$$

$$\begin{aligned}
 \text{Replacement payment eight months from today} \&= \$2,047.50 + \\
 \$1,968.83 + \$1,894.99 \&= \$5,911.32 \end{aligned}$$

Jayne should pay \$5,911.32.

4. Merina is scheduled to make two loan payments to Bradford in the amount of \$1,000 each, two months and nine months from now. Merina doesn't think she can make those payments and offers Bradford an alternative plan where she will pay \$775 seven months from now and another payment seven months later. Bradford determines that 8.5% is a fair interest rate. What is the amount of the second payment?

### Solution:

The unknown payment is 14 months from today.

On the original loan payments, both payments are late and should be charged interest.

Under the proposed payments, need to calculate the equivalent payment for the seven month payment on the 14 month day so that the amount remaining for the second payment can be determined. These payments should be equivalent to the original loan payments.

**Step 1:** Given information:

$$r = 8.5\% \text{ annually}$$

### Original Agreement:

**Payment #1:**  $P = \$1,000$ ;  $t = 14 \text{ months} - 2 \text{ months} = 12 \text{ months late}$

**Payment #2:**  $P = \$1,000$ ;  $t = 14 \text{ months} - 9 \text{ months} = 5 \text{ months late}$

### Proposed Agreement:

**Payment #1:**  $P = \$775$ ;  $t = 14 \text{ months} - 7 \text{ months} = 7 \text{ months}$

**Payment #2:** Unknown (x), but 14 months from today.

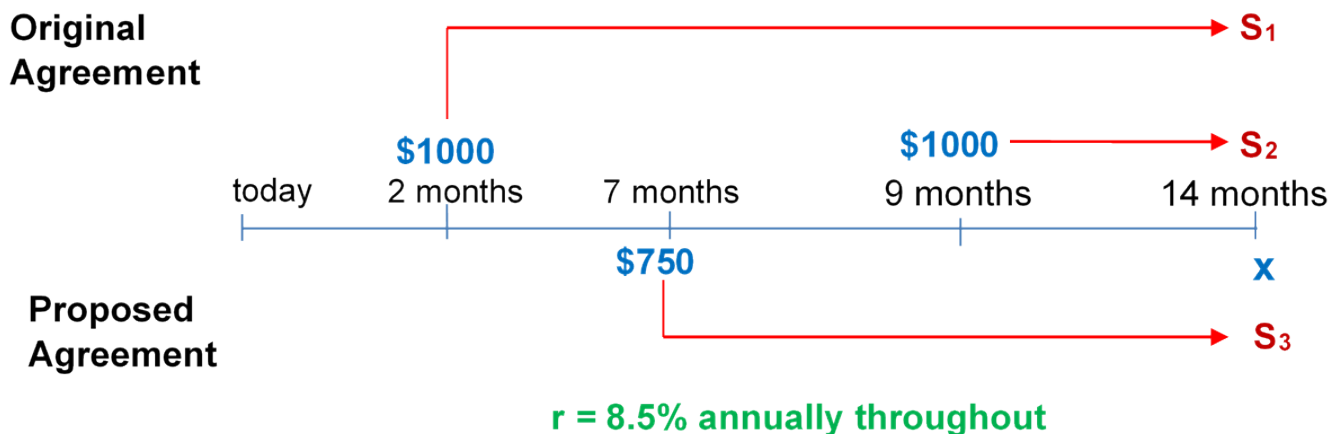


Figure 8.2.4: Timeline [Image Description]

**Original Agreement Payment #1:**

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{12}{12} = 1$  year

**Step 3:** Solve for  $S_1$ .

$$\begin{aligned} S_1 &= P(1 + rt) \\ &= \$1,000 \times (1 + 0.085 \times 1) \\ &= \$1,085 \end{aligned}$$

### Original Agreement Payment #2:

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{5}{12}$

**Step 3:** Solve for  $S_2$ .

$$\begin{aligned} S_2 &= P(1 + rt) \\ &= \$1,000 \times \left(1 + 0.085 \times \frac{5}{12}\right) \\ &= \$1,035.42 \end{aligned}$$

### Proposed Agreement Payment #1:

**Step 2:** Convert monthly  $t$  to match annual  $r$ ;  $t = \frac{7}{12}$

**Step 3:** Solve for  $S_3$ .

$$\begin{aligned} S_3 &= P(1 + rt) \\ &= \$775 \times \left(1 + 0.085 \times \frac{7}{12}\right) \\ &= \$813.43 \end{aligned}$$

Now make the two agreements equivalent to each other.

$$\begin{aligned} \text{Total owing under original agreement} &= \text{Total owing under proposed agreement} \\ \$1,085 + \$1,035.42 &= \$813.43 + x \\ x &= \$1,306.99 \end{aligned}$$

To make the alternate loan payments equivalent to the original payments, Merina must pay \$1,306.99 14 months from today.

### 4.3 SAVINGS ACCOUNTS AND SHORT-TERM GICS

1. Canadian Western Bank offers a Summit Savings Account with posted interest rates as indicated in the table below. Only each tier is subject to the posted rate, and interest is calculated daily based on the closing balance.

Summit Savings Account Interest Rates Based on Balance for Solution 8.3

Balance	Interest Rate
\$0 – \$5,000.00	0%
\$5,000.01 – \$1,000,000.00	1.05%
\$1,000,000.01 and up	0.80%

December's opening balance was \$550,000. Two deposits in the amount of \$600,000 each were made on December 3 and December 21. Two withdrawals in the amount of \$400,000 and \$300,000 were made on December 13 and December 24, respectively. What interest for the month of December will be deposited to the account on January 1?

#### Solution:

**Step 1:** Interest rates as per table in question.

December opening balance = \$550,000

December 3 Deposit = \$600,000

December 13 Withdrawal = \$400,000

December 21 Deposit = \$600,000

December 24 Withdrawal = \$300,000

**Step 2:** Set up Table.

**Dec 1 to Dec 3:**

**Closing Balance In Account:** \$550,000

**# of Days:**  $3 - 1 = 2$

**0% \$0 to \$5,000 (This portion only):**

$$P = \$5,000$$

$$I = \$0$$

**1.05% \$5,000.01 to \$1,000,000 (This portion only):**

$$P = \$545,000$$

$$I = \$545,000(0.0105) \left( \frac{2}{365} \right)$$
$$= \$31.356164$$

---

**Dec 3 to Dec 13:**

**Closing Balance In Account:**  $\$550,000 + \$600,000 = \$1,150,000$

**# of Days:**  $13 - 3 = 10$

**0% \$0 to \$5,000 (This portion only):**

$$P = \$5,000$$

$$I = \$0$$

**1.05% \$5,000.01 to \$1,000,000 (This portion only):**

$$P = \$995,000$$

$$\begin{aligned} I &= \$995,000(0.0105) \left( \frac{10}{365} \right) \\ &= \$286.232876 \end{aligned}$$

**0.8% \$1,000,000.01 and up (This portion only):**

$$P = \$150,000$$

$$\begin{aligned} I &= \$150,000(0.008) \left( \frac{10}{365} \right) \\ &= \$32.876712 \end{aligned}$$

---

**Dec 13 to Dec 21**

**Closing Balance In Account:**  $\$1,150,000 - \$400,000 = \$750,000$

**# of Days:**  $21 - 13 = 8$

**0% \$0 to \$5,000 (This portion only):**

$$P = \$5,000$$

$$I = \$0$$

**1.05% \$5,000.01 to \$1,000,000 (This portion only):**

$$P = \$745,000$$

$$I = \$745,000(0.0105) \left( \frac{8}{365} \right)$$
$$= \$171.452054$$

---

**Dec 21 to Dec 24**

**Closing Balance In Account:**  $\$750,000 + \$600,000 = \$1,350,000$

**# of Days:**  $24 - 21 = 3$

**0% \$0 to \$5,000 (This portion only):**

$$P = \$5,000$$

$$I = \$0$$

**1.05% \$5,000.01 to \$1,000,000 (This portion only):**

$$P = \$995,000$$

$$\begin{aligned} I &= \$995,000(0.0105) \left( \frac{3}{365} \right) \\ &= \$85.869863 \end{aligned}$$

**0.8% \$1,000,000.01 and up (This portion only):**

$$P = \$350,000$$

$$\begin{aligned} I &= \$350,000(0.008) \left( \frac{3}{365} \right) \\ &= \$23.013698 \end{aligned}$$

---

**Dec 24 to Jan 1**

**Closing Balance In Account:**  $\$1,350,000 - \$300,000 = \$1,050,000$

**# of Days:**  $31 + 1 - 24 = 8$

**0% \$0 to \$5,000 (This portion only):**

$$P = \$5,000$$

$$I = \$0$$

**1.05% \$5,000.01 to \$1,000,000 (This portion only):**

$$P = \$995,000$$

$$I = \$995,000(0.0105) \left( \frac{8}{365} \right)$$
$$= \$228.986301$$

**0.8% \$1,000,000.01 and up (This portion only):**

$$\begin{aligned}
 P &= \$50,000 \\
 I &= \$50,000(0.008) \left( \frac{8}{365} \right) \\
 &= \$8.767123
 \end{aligned}$$

**Step 3:** Total Monthly Interest Earned,  $I$ .

$$\begin{aligned}
 I &= \$31.356164 + \$286.232876 + \$32.876712 + \$171.452054 + \$85.869863 + \\
 &+ \$23.013698 + \$228.986301 + \$8.767123 \\
 I &= \$868.55
 \end{aligned}$$

The total monthly interest earned is \$868.55.

- If you place \$25,500 into an 80-day short-term GIC at TD Canada Trust earning 0.55% simple interest, how much will you receive when the investment matures?

**Solution:**

**Step 1:** Given information:

$$P = \$25,500; t = 80 \text{ days}; r = 0.55\%; \text{annually}$$

**Step 2:** Convert daily  $t$  to match annual  $r$ ;  $t = \frac{80}{365}$

**Step 3:** Solve for  $S$ .

$$\begin{aligned}
 S &= P(1 + rt) \\
 &= \$25,500 \times \left( 1 + 0.0055 \times \frac{80}{365} \right) \\
 &= \$25,530.74
 \end{aligned}$$

When the investment matures I will receive \$25,530.74.

3. Interest rates in the GIC markets are always fluctuating because of changes in the short-term financial markets. If you have \$50,000 to invest today, you could place the money into a 180-day GIC at Canada Life earning a fixed rate of 0.4%, or you could take two consecutive 90-day GICs. The current posted fixed rate on 90-day GICs at Canada Life is 0.3%. Trends in the short-term financial markets suggest that within the next 90 days short-term GIC rates will be rising. What does the short-term 90-day rate need to be 90 days from now to arrive at the same maturity value as the 180-day GIC? Assume that the entire maturity value of the first 90-day GIC would be reinvested.

### **Solution:**

**Step 1:** Given information:

**For the first GIC investment option:**

$$P = \$50,000; r = 0.4\% \text{ per year}; t = 180 \text{ days}$$

**For the second GIC investment option:**

$$\text{Initial } P = \$50,000; r = 0.3\% \text{ per year}; t = 90 \text{ days}$$

then,

$$P = S \text{ of first 90-day GIC}; S = \text{maturity value of 180-day GIC}; \\ t = 90 \text{ days}$$

**Step 2:** Transforming both time variables;  $t = \frac{180}{365}$  and  $t = \frac{90}{365}$

**Step 3: (1st GIC option):**

$$S_1 = \$50,000 \left( 1 + 0.004 \times \frac{180}{365} \right) \\ = \$50,098.63$$

**Step 3: (2nd GIC option, 1st GIC):**

$$S_2 = \$50,000 \left( 1 + 0.003 \times \frac{90}{365} \right) \\ = \$50,036.99$$

**Step 3: (2nd GIC option, 2nd GIC):**

$$I = (S \text{ of 180-day GIC}) - (S \text{ of 90-day GIC})$$

$$\begin{aligned}
 I &= \$50,098.63 - \$50,036.99 \\
 &= \$61.64 \text{ (what the second 90-day GIC must earn in interest)}
 \end{aligned}$$

$$\begin{aligned}
 r &= \frac{I}{Pt} \\
 &= \frac{\$61.64}{\$50,036.99 \left( \frac{90}{365} \right)} \\
 &= 0.004995 \text{ or } 0.4995\%
 \end{aligned}$$

The short-term 90-day rate needs to be 0.50% 90 days from now to arrive at the same maturity value as the 180-day GIC.

#### 4.6: APPLICATION: TREASURY BILLS & COMMERCIAL PAPERS

1. A 60-day, \$90,000 face value commercial paper was issued when yields were 2.09%. What was its purchase price?

##### **Solution:**

**Step 1:** Given information:

$$t = \frac{60}{365}; r = 2.09\%; S = \$90,000$$

**Step 2:** Solve for  $P$ .

$$\begin{aligned}
 P &= \frac{S}{1 + rt} \\
 &= \frac{\$90,000}{1 + 0.0209 \times \frac{60}{365}} \\
 &= \$89,691.85
 \end{aligned}$$

Its purchase price was \$89,691.85.

2. A 90-day Province of Ontario T-bill with a \$35,000 face value matures on December 11. Farrah works for Hearthplace Industries and notices that the company temporarily has some extra cash available. If she invests the money on October 28, when the yield is 4.94%, and sells the T-bill on November 25, when the yield is 4.83%, calculate how much money Farrah earned and the rate of return she realized.

##### **Solution:**

**Calculate the purchase price for the T-bill:**

**Step 1:** Given information:

$$r = 4.94\% \text{ (only the rate on the day of purchase matters); } S = \$35,000$$

$$t = \text{October 28 to December 11}$$

$$= 3 + 30 + 11$$

$$= 44 \text{ days left on T-bill or } 44/365$$

(Only the time remaining on the T-bill is important.)

**Step 2:** Solve for  $P$ .

$$\begin{aligned} P &= \frac{S}{1 + rt} \\ &= \frac{\$35,000}{1 + 0.0494 \times \frac{44}{365}} \\ &= \$34,792.81 \end{aligned}$$

**Calculate the price when sold for the T-bill:**

**Step 1:** Given information:

$$r = 4.83\% \text{ (only the rate on the day of sale matters); } S = \$35,000$$

$$t = \text{November 25 to December 11}$$

$$= 5 + 11$$

$$= 16 \text{ days left on T-bill or } 16/365$$

(Only the time remaining on the T-bill is important.)

**Step 2:** Solve for  $P$ .

$$\begin{aligned} P &= \frac{S}{1 + rt} \\ &= \frac{\$35,000}{1 + 0.0483 \times \frac{16}{365}} \\ &= \$34,926.05 \end{aligned}$$

**Calculate the amount of interest:**

$$\begin{aligned} \text{Amount earned} &= \text{Sold price} - \text{Purchase price} \\ &= \$34,926.05 - \$34,792.81 \\ &= \$133.24 \end{aligned}$$

**Calculate Farrah's rate of return:****Step 1:** Given information:

$$I = \$133.24; P = \$34,792.81$$

$$\begin{aligned} t &= \text{October 28 to November 25 (the time held)} \\ &= 3 + 25 \\ &= 28 \text{ days or } 28/365 \end{aligned}$$

**Step 2:** Solve for  $r$ .

$$\begin{aligned} r &= \frac{I}{Pt} \\ &= \frac{\$133.24}{\$34,792.81 \times \frac{28}{365}} \\ &= 0.049920 \text{ or } 4.99\% \end{aligned}$$

Farrah earned \$133.24 and the rate of return she realized was 4.99%.

3. Philippe purchased a \$100,000 Citicorp Financial 220-day commercial paper for \$96,453.93. He sold it 110 days later to Damien for \$98,414.58, who then held onto the commercial paper until its maturity date.
  - a) What is Philippe's actual rate of return?
  - b) What is Damien's actual rate of return?
  - c) What is the rate of return Philippe would have realized if he had held onto the note instead of selling it to Damien?

**Solution:****a)****Step 1:** Given information:

$$P = \$96,453.93; t = \frac{110}{365}; S = \$98,414.58$$

**Step 2:** Calculate  $I$ .

$$\begin{aligned} I &= S - P \\ &= \$98,414.58 - \$96,453.93 \\ &= \$1,960.65 \end{aligned}$$

**Step 3:** Solve for  $r$ .

$$\begin{aligned}
 r &= \frac{I}{Pt} \\
 &= \frac{\$1,960.65}{\$96,453.93 \times \frac{110}{365}} \\
 &= 0.067449 \text{ or } 6.74\%
 \end{aligned}$$

Philippe's actual rate of return is 6.74%.

**b)**

**Step 1:** Given information:

$$S = \$100,000; t = \frac{110}{365}; P = \$98,414.58$$

**Step 2:** Calculate  $I$ .

$$\begin{aligned}
 I &= S - P \\
 &= \$100,000 - \$98,414.58 \\
 &= \$1,585.42
 \end{aligned}$$

**Step 3:** Solve for  $r$ .

$$\begin{aligned}
 r &= \frac{I}{Pt} \\
 &= \frac{\$1,585.42}{\$98,414.58 \times \frac{110}{365}} \\
 &= 0.053454 \text{ or } 5.35\%
 \end{aligned}$$

Damien's actual rate of return is 5.35%.

**c)**

**Step 1:** Given information:

$$S = \$100,000; t = \frac{220}{365}; P = \$96,453.93$$

**Step 2:** Solve for  $I$ .

$$\begin{aligned}
 I &= S - P \\
 &= \$100,000 - \$96,453.93 \\
 &= \$3,546.07
 \end{aligned}$$

**Step 3:** Solve for  $r$ .

$$\begin{aligned}
 r &= \frac{I}{Pt} \\
 &= \frac{\$3,546.07}{\$96,453.93 \times \frac{220}{365}} \\
 &= 0.060995 \text{ or } 6.10\%
 \end{aligned}$$

Philippe would have realized 6.10% rate of return if he had held onto the note instead of selling it to Damien.

#### IMAGE DESCRIPTIONS

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**Figure 8.2.2:** Timeline showing \$2000 at 5 months from today moving to 8 months from today as  $S_1$ . \$2000 at 10 months from today moving back to 8 months from today as  $P_1$ . \$2000 at 15 months from today moving back to 8 months from today as  $P_2$ . Interest rate  $r = 9.5\%$  annually throughout. [Back to Figure 8.2.2]

**Figure 8.2.4:** Timeline showing the Original Agreement of \$1000 at 2 months from today moving to 14 months from today as  $S_1$  and \$1000 at 9 months from today moving to 14 months from today as  $S_2$ . The Proposed Agreement shows \$750 at 7 months from today moving to 14 months from today as  $S_3$  and  $X$  at 14 months from today. [Back to Figure 8.2.4]

## 5.1: COMPOUND INTEREST FUNDAMENTALS

---

1. Calculate the periodic interest rate if the nominal interest rate is 7.75% compounded monthly.

### Solution:

$$\begin{aligned}\text{Periodic Rate, } i &= \frac{\text{Nominal Rate}}{\text{Compounds per Year}} \\ &= \frac{7.75\%}{12} \\ &= 0.6458\% \text{ per month}\end{aligned}$$

The periodic interest rate is 0.65%.

2. Calculate the compounding frequency for a nominal interest rate of 9.6% if the periodic interest rate is 0.8%.

### Solution:

$$\begin{aligned}\text{Compounds Per Year, } C/Y &= \frac{\text{Nominal Rate}}{\text{Periodic Rate}} \\ &= \frac{9.6\%}{0.8\%} \\ &= 12 \text{ (monthly)}\end{aligned}$$

The compounding frequency is 12 (monthly).

3. Calculate the nominal interest rate if the periodic interest rate is 2.0875% per quarter.

### Solution:

$$\begin{aligned}\text{Nominal Rate, } I/Y &= (\text{Periodic Rate}) \times (\text{Compounds Per Year}) \\ &= 2.0875\% \times 4 \\ &= 8.35\% \text{ compounded quarterly}\end{aligned}$$

The nominal interest rate is 8.35% compounded quarterly.

4. After a period of three months, Alese saw one interest deposit of \$176.40 for a principal of \$9,800. What nominal rate of interest is Alese earning?

### Solution:

**Step 1:** First convert the interest amount into a periodic interest rate per quarter.

$$\text{Portion} = \text{Rate} \times \text{Base}$$

$$I = i \times PV$$

$$\$176.40 = i \times \$9,800$$

$$i = \frac{\$176.40}{\$9,800}$$

$$i = 0.018 \text{ or } 1.8\% \text{ per quarter}$$

**Step 2:** Now convert the result in Step 1 to a nominal rate.

$$\begin{aligned} \text{Nominal Rate, } I/Y &= (\text{Periodic Rate}) \times (\text{Compounds Per Year}) \\ &= 1.8\% \times 4 \\ &= 7.2\% \text{ compounded quarterly} \end{aligned}$$

Alese is earning 7.2% compounded quarterly.

## 5.2: DETERMINING THE FUTURE VALUE

---

1. Find the future value if \$53,000 is invested at 6% compounded monthly for 4 years and 3 months.

**Solution:**

**Step 1:** Given information:

$$PV = \$53,000; C/Y = \text{monthly} = 12; t = 4\frac{3}{12} \text{ years}; I/Y = 6\%$$

**Step 2:** Find  $i$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compound per year (C/Y)}} = \frac{6\%}{12} = 0.5\%$$

**Step 3:** Find  $n$ .

$$n = (\text{Number of Years}) \times C/Y = \left(4\frac{3}{12}\right) \times 12 = 4.25 \times 12 = 51$$

**Step 4:** Solve for  $FV$ .

$$\begin{aligned}
 FV &= PV(1 + i)^{51} \\
 &= \$53,000(1 + 0.005)^{51} \\
 &= \$53,000(1.005)^{51} \\
 &= \$68,351.02
 \end{aligned}$$

The future value is \$68,351.02.

Calculator Instructions for Solution 9.2 Question 1

N	I/Y	PV	PMT	FV	P/Y	C/Y
51	6	-53,000	0	?	12	12

2. Find the future value if \$24,500 is invested at 4.1% compounded annually for 4 years; then 5.15% compounded quarterly for 1 year, 9 months; then 5.35% compounded monthly for 1 year, 3 months.

**Solution:**

**Step 1:** Find  $FV_1$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{4.1\%}{1} = 4.1\%$$

$$n = (\text{Number of Years}) \times C/Y = 4 \times 1 = 4$$

$$\begin{aligned}
 FV_1 &= PV_1(1 + i)^n \\
 &= \$24,500(1 + 0.041)^4 \\
 &= \$24,500(1.041)^4 \\
 &= \$28,771.93049 \text{ (This becomes PV for the next calculation in Step 2.)}
 \end{aligned}$$

**Step 2:** Find  $FV_2$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{5.15\%}{4} = 1.2875\%$$

$$n = (\text{Number of Years}) \times C/Y = \left(1\frac{9}{12}\right) \times 4 = 1.75 \times 4 = 7$$

$$\begin{aligned}
 FV_2 &= PV_2(1+i)^n \\
 &= \$28,771.93049(1.012875)^7 \\
 &= \$31,467.33516 \text{ (This becomes PV for the next calculation in Step 3.)}
 \end{aligned}$$

**Step 3:** Find  $FV_3$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{5.35\%}{12} = 0.4458\bar{3}\%$$

$$n = (\text{Number of Years}) \times C/Y = \left(1\frac{3}{12}\right) \times 12 = 1.25 \times 12 = 15$$

$$\begin{aligned}
 FV_3 &= PV_3(1+i)^n \\
 &= \$31,467.33516(1.004458\bar{3})^{15} \\
 &= \$33,638.67
 \end{aligned}$$

The future value is \$33,638.67.

Calculator Instructions for Solution 9.2 Question 2

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	4	4.1	-24,500	0	?	1	1
2	7	5.15	±(FV from Step 1)	0	?	4	4
3	15	5.35	±(FV from Step 2)	0	?	12	12

3. Nirdosh borrowed \$9,300  $4\frac{1}{4}$  years ago at 6.35% compounded semi-annually. The interest rate changed to 6.5% compounded quarterly  $1\frac{3}{4}$  years ago. What amount of money today is required to pay off this loan?

**Solution:**

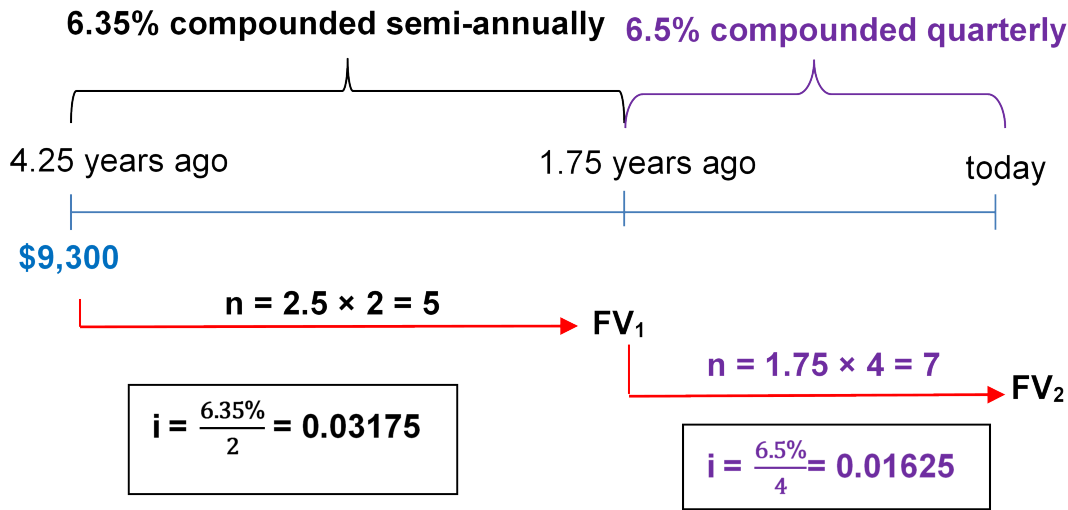


Figure 9.2.3: Timeline [Image Description]

**Step 1:** Find  $FV_1$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{6.35\%}{2} = 3.175\%$$

$$n = (\text{Number of Years}) \times C/Y = 2.5 \times 2 = 5$$

$$\begin{aligned} FV_1 &= PV(1 + i)^n \\ &= \$9,300(1.03175)^5 \\ &= \$10,873.14892 \text{ (This becomes PV for the next calculation in Step 2.)} \end{aligned}$$

**Step 2:** Find  $FV_2$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{6.5\%}{4} = 1.625\%$$

$$n = (\text{Number of Years}) \times C/Y = 1.75 \times 4 = 7$$

$$\begin{aligned} FV_2 &= PV(1 + i)^n \\ &= \$10,873.14892(1.01625)^7 \\ &= \$12,171.92 \text{ (Round at this step.)} \end{aligned}$$

It is required today \$12,171.92 to pay off the loan.

Calculator Instructions for Solution 9.2 Question 3

Step	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	5	6.35	+9,300	0	?	2	2
2	7	6.5	± (FV from Step 1)	0	?	4	4

### 5.3: Determining the Present Value

1. A debt of \$37,000 is owed 21 months from today. If prevailing interest rates are 6.55% compounded quarterly, what amount should the creditor be willing to accept today?

#### Solution:

**Step 1:** Given information:

$$FV = \$37,000; I/Y = 6.55\%; t = \frac{21}{12} = 1.75 \text{ years};$$

$$C/Y = \text{quarterly} = 4.$$

**Step 2:** Find  $i$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{6.55\%}{4} = 1.6375\%$$

**Step 3:** Find  $n$ .

$$n = (\text{Number of Years}) \times C/Y = \frac{21}{12} \times 4 = 7$$

**Step 4:** Solve for  $PV$ .

$$\begin{aligned} PV &= \frac{FV}{(1 + i)^n} \\ &= \frac{\$37,000}{(1.016375)^7} \\ &= \$33,023.56 \end{aligned}$$

The creditor should be willing to accept \$33,023.56 today?

## Calculator Instructions for Solution 9.3 Question 1

N	I/Y	PV	PMT	FV	P/Y	C/Y
7	6.55	?	0	37,000	4	4

2. For the first 4½ years, a loan was charged interest at 4.5% compounded semi-annually. For the next 4 years, the rate was 3.25% compounded annually. If the maturity value was \$45,839.05 at the end of the 8½ years, what was the principal of the loan?

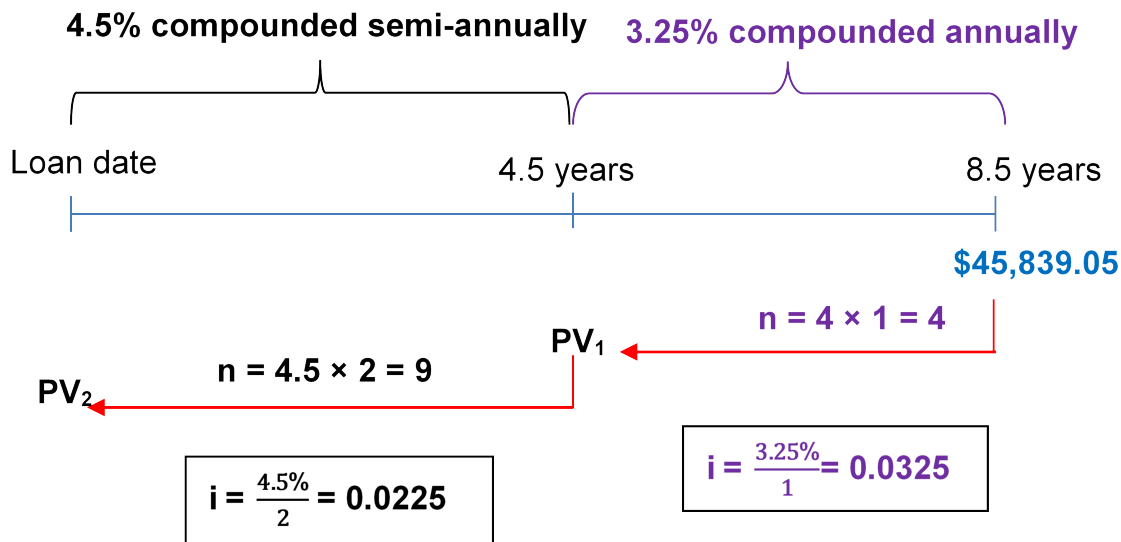
**Solution:**

Figure 9.3.2: Timeline [Image Description]

**Step 1:** Find  $PV_1$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{3.25\%}{1} = 3.25\%$$

$$n = (\text{Number of Years}) \times C/Y = 4 \times 1 = 4$$

$$\begin{aligned} PV_1 &= \frac{FV}{(1 + i)^n} \\ &= \frac{\$45,839.05}{(1.0325)^4} \\ &= \$40,334.37829 \text{ (This becomes FV for the next calculation in Step 2.)} \end{aligned}$$

**Step 2:** Find  $PV_2$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{4.5\%}{2} = 2.25\%$$

$$n = (\text{Number of Years}) \times C/Y = 4.5 \times 2 = 9$$

$$\begin{aligned} PV_2 &= \frac{FV}{(1+i)^n} \\ &= \frac{\$40,334.37829}{(1.0225)^9} \\ &= \$33,014.56 \text{ (Round at this step.)} \end{aligned}$$

The principal of the loan is \$33,014.56.

Calculator Instructions for Solution 9.3 Question 2

Steps	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	4	3.25	?	0	-45,839.05	1	1
2	9	4.5	?	0	±(PV from Step 1)	2	2

#### 5.4: Equivalent Payments

- A winning lottery ticket offers the following two options:
  - A single payment of \$1,000,000 today or
  - \$250,000 today followed by annual payments of \$300,000 for the next three years.

If money can earn 9% compounded annually, which option should the winner select? How much better is that option in current dollars?

#### **Solution:**

- The \$1,000,000 is already today.
- To fairly compare the payment plan, move all money to today as well.

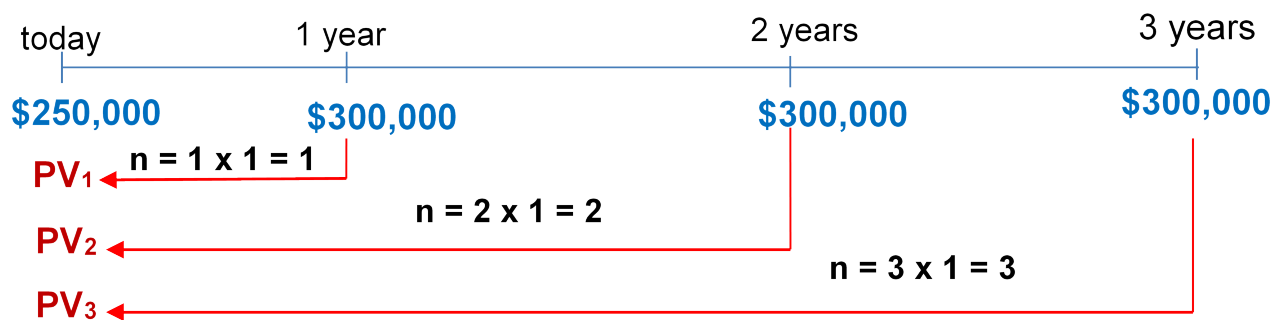


Figure 9.4.1: Timeline [Image Description]

Focal Date = Today

**Step 1:** Find  $i$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{9\%}{1} = 9\%$$

**Step 2:** Find  $n$  of the payments.

$$n = (\text{Number of Years}) \times C/Y$$

$$\text{Payment \#1: } n = 1 \times 1 = 1$$

$$\text{Payment \#2: } n = 2 \times 1 = 2$$

$$\text{Payment \#3: } n = 3 \times 1 = 3$$

**Step 3:** Find the present value of the payments.

$$\begin{aligned} PV_1 &= \frac{FV}{(1+i)^n} \\ &= \frac{\$300,000}{(1.09)^1} \\ &= \$275,229.3578 \end{aligned}$$

$$\begin{aligned} PV_2 &= \frac{FV}{(1+i)^n} \\ &= \frac{\$300,000}{(1.09)^2} \\ &= \$252,503.998 \end{aligned}$$

$$\begin{aligned}
 PV_3 &= \frac{FV}{(1+i)^n} \\
 &= \frac{\$300,000}{(1.09)^3} \\
 &= \$231,655.044
 \end{aligned}$$

$$\begin{aligned}
 \text{Total Present Value Today} &= \$250,000 + \$275,229.3578 + \$252,503.998 + \$231,655.044 \\
 &= \$1,009,388.40
 \end{aligned}$$

Payment plan is better by  $\$1,009,388.40 - \$1,000,000 = \$9,388.40$ .

Calculator Instructions for Solution 9.4 Question 1

Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	1	9	?	0	300,000	1	1
2	2	9	?	0	300,000	1	1
3	3	9	?	0	300,000	1	1

2. James is a debt collector. One of his clients has asked him to collect an outstanding debt from one of its customers. The customer has failed to pay three amounts: \$1,600 eighteen months ago, \$2,300 nine months ago, and \$5,100 three months ago. In discussions with the customer, James finds she desires to clear up this situation and proposes a payment of \$1,000 today, \$4,000 nine months from now, and a final payment two years from now. The client normally charges 16.5% compounded quarterly on all outstanding debts. What is the amount of the third payment?

**Solution:**

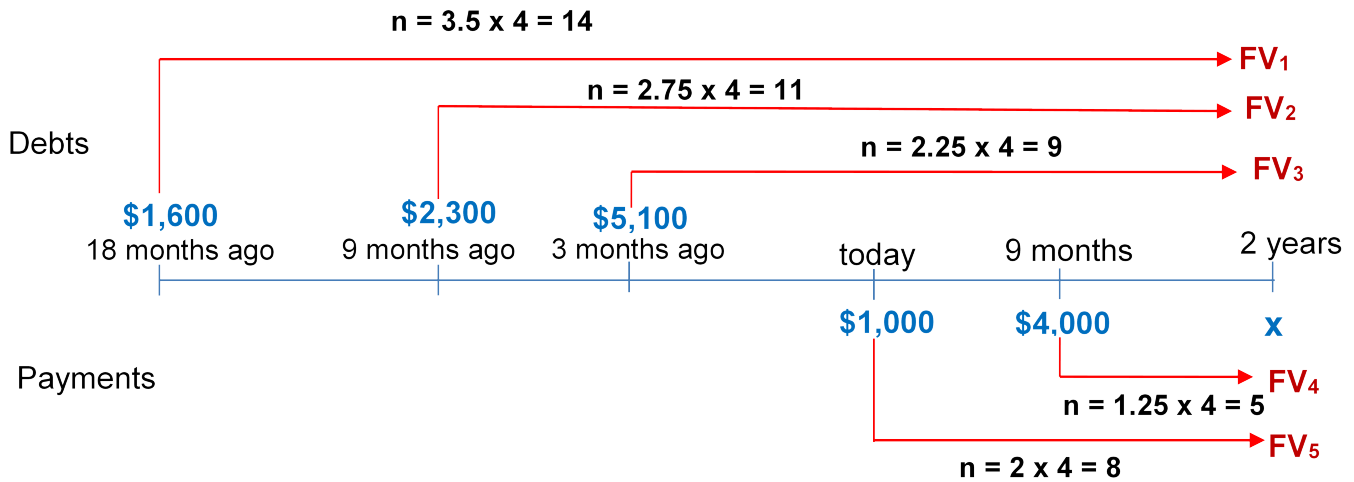


Figure 9.4.2: Timeline [Image Description]

Focal Date = 2 years from today

**Step 1:** Find  $i$ .

$$i = \frac{\text{Nominal Rate (I/Y)}}{\text{Compounds per Year (C/Y)}} = \frac{16.5\%}{4} = 4.125\%$$

**Step 2:** Find  $n$  of the payments.

$$n = (\text{Number of Years}) \times C/Y$$

$$\text{Payment \#1: } n = 3.5 \times 4 = 14$$

$$\text{Payment \#2: } n = 2.75 \times 4 = 11$$

$$\text{Payment \#3: } n = 2.25 \times 4 = 9$$

$$\text{Payment \#4: } n = 1.25 \times 4 = 5$$

$$\text{Payment \#5: } n = 2 \times 4 = 8$$

**Step 3:** Find the future value of the payments.

$$FV_1 = \$1,600(1 + 0.04125)^{14} = \$2,817.670366$$

$$FV_2 = \$2,300(1 + 0.04125)^{11} = \$3,587.839398$$

$$FV_3 = \$5,100(1 + 0.04125)^9 = \$7,337.790461$$

$$FV_4 = \$4,000(1 + 0.04125)^5 = \$4,895.928462$$

$$FV_5 = \$1,000(1 + 0.04125)^8 = \$1,381.783859$$

Total Dated Debts = Total Dated Payments

$$FV_1 + FV_2 + FV_3 = x + FV_4 + FV_5$$

$$\$2,817.670366 + \$3,587.839398 + \$7,337.790461 = x + \$4,895.928462 + \$1,381.783859$$

$$\$13,743.30023 = x + \$6,277.712321$$

$$x = \$7,465.59$$

The amount of the third payment is \$7,465.59.

Calculator Instructions for Solution 9.4 Question 2

Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
Original 1	14	16.5	1,600	0	?	4	4
Original 2	11	16.5	2,300	0	?	4	4
Original 3	9	16.5	5,100	0	?	4	4
Proposed 1	8	16.5	1,000	0	?	4	4
Proposed 2	5	16.5	4,000	0	?	4	4

3. Four years ago, Aminata borrowed \$5,000 from Randal with interest at 8% compounded quarterly to be repaid one year from today. Two years ago, Aminata borrowed another \$2,500 from Randal at 6% compounded monthly to be repaid two years from today. Aminata would like to restructure the payments so that she can pay 15 months from today and 2½ years from today. The first payment is to be twice the size of the second payment. Randal accepts an interest rate of 6.27% compounded monthly on the proposed agreement. Calculate the amounts of each payment assuming the focal date is 15 months from today.

### Solution:

First, calculate the amounts owing under Aminata's original loans.

#### Original Loan 1:

$$i = \frac{I/Y}{C/Y} = \frac{8\%}{4} = 2\%$$

$$n = (\text{Number of Years}) \times C/Y = 5 \times 4 = 20$$

$$\begin{aligned} FV_1 &= PV(1 + i)^n \\ &= \$5,000(1.02)^{20} \\ &= \$7,429.74 \text{ (Due in 1 year from today.)} \end{aligned}$$

#### Original Loan 2:

$$i = \frac{I/Y}{C/Y} = \frac{6\%}{12} = 0.5\%$$

$$n = (\text{Number of Years}) \times C/Y = 4 \times 12 = 48$$

$$\begin{aligned} FV_2 &= PV(1 + i)^n \\ &= \$2,500(1.005)^{48} \\ &= \$3,176.22 \text{ (Due in 2 year from today.)} \end{aligned}$$

Calculator Instructions for Solution 9.4 Question 3

Loan	N	I/Y	PV	PMT	FV	P/Y	C/Y
1	20	8	5,000	0	?	4	4
2	48	6	2,500	0	?	12	12

Now calculate the equivalent payments under the proposed arrangement:

1 year = 12 months

2 years = 24 months

2.5 years = 30 months

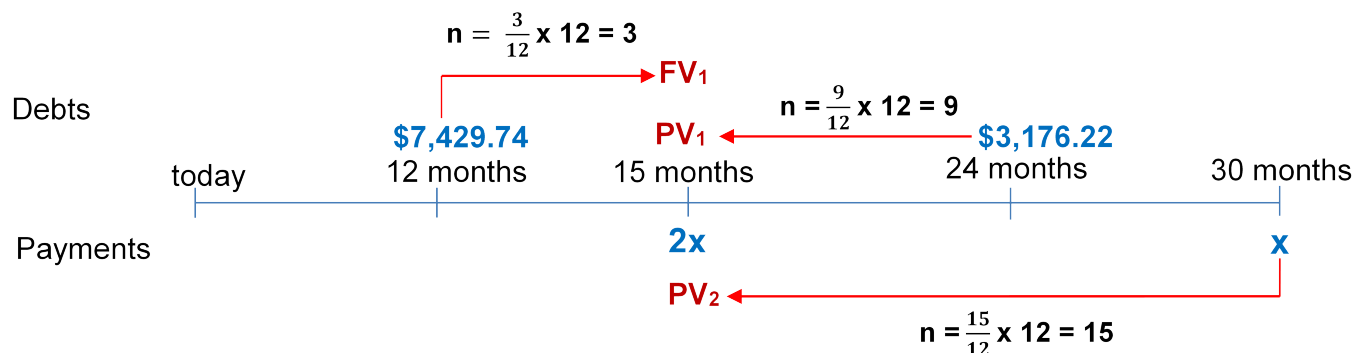


Figure 9.4.3: Timeline [Image Description]

$$i = \frac{I/Y}{C/Y} = \frac{6.27\%}{12} = 0.5225\%$$

Total Dated Debts = Total Dated Payments

$$FV_1 + PV_1 = 2x + PV_2$$

$$7,429.74(1.005225)^3 + \frac{3,176.22}{(1.005225)^9} = 2x + \frac{x}{(1.005225)^{15}}$$

$$7,546.810744 + 3,030.686729 = 2x + 0.924806x$$

$$\$10,577.49747 = 2.924806x$$

$$x = \$3,616.48 \text{ (second payment)}$$

$$2x = 2(\$3,616.48) = \$7,232.96 \text{ (first payment)}$$

The amount of each payment is \$7,232.96.

Calculator Instructions for Solution 9.4 Question 3 Calculating Payments

Payment	N	I/Y	PV	PMT	FV	P/Y	C/Y
Original 1	3	6.27	7,429.74	0	?	12	12
Original 2	9	6.27	?	0	3,176.22	12	12
Proposed 1	15	6.27	?	0	1	12	12

## 5.5 DETERMINING THE INTEREST RATE

1. Your company paid an invoice five months late. If the original invoice was for \$6,450 and the amount paid was \$6,948.48, what monthly compounded interest rate is your supplier charging on late payments?

### Solution:

**Step 1:** Given information:

$$PV = \$6,450; FV = \$6,948.48; C/Y = \text{monthly} = 12$$

**Step 2:** Find  $n$ .

$$n = (\text{Number of Years}) \times C/Y = \frac{5}{12} \times 12 = 5$$

**Step 3:** Using the formula for  $FV$ , rearrange and solve for  $i$ .

$$FV = PV(1 + i)^n$$

$$\$6,948.48 = \$6,450(1 + i)^5$$

$$1.077283 = (1 + i)^5$$

$$1.077283^{\frac{1}{5}} = (1 + i)$$

$$1.014999 = 1 + i$$

$$i = 0.014999$$

**Step 4:** Solve for the nominal rate,  $I/Y$ .

$$I/Y = i \times 12$$

$$= 0.179999$$

$$= 18\% \text{ (compounded monthly)}$$

The supplier is charging 18% compounded monthly on late payments?

Calculator Instructions for Solution 9.5 Question 1

N	I/Y	PV	PMT	FV	P/Y	C/Y
5	?	-6,450	0	6,948.48	12	12

2. At what monthly compounded interest rate does it take five years for an investment to double?

**Solution:**

**Step 1:** Pick any two values for PV and FV where FV is double the PV.

$$PV = \$10,000; FV = \$20,000$$

**Step 2:** Find  $n$ .

$$n = (\text{Number of Years}) \times C/Y = 5 \times 12 = 60$$

**Step 3:** Using the formula for  $FV$  solve for  $i$ .

$$FV = PV(1 + i)^n$$

$$\$20,000 = \$10,000(1 + i)^{60}$$

$$2 = (1 + i)^{60}$$

$$2^{\frac{1}{60}} = (1 + i)$$

$$1.011619 = 1 + i$$

$$i = 0.011619$$

**Step 4:** Solve for the nominal rate,  $I/Y$ .

$$\begin{aligned}\text{Nominal Rate} &= i \times 12 \\ &= 0.139428 \\ &= 13.94\% \text{ compounded monthly}\end{aligned}$$

At monthly compounded interest rate does it take five years for an investment to double.

The investment will double in five years at 13.94% compounded monthly.

Calculator Instructions for Solution 9.5 Question 2

N	I/Y	PV	PMT	FV	P/Y	C/Y
60	?	-10,000	0	20,000	12	12

3. Indiana just received a maturity value of \$30,320.12 from a semi-annually compounded investment that paid 4%, 4.1%, 4.35%, 4.75%, and 5.5% in consecutive years. What amount of money did Indiana invest? What fixed quarterly compounded nominal interest rate is equivalent to the variable rate his investment earned?

**Solution:**

**Step 1:** Given information:

Year 1:  $I/Y=4\%$ ;  $C/Y=2$

Year 2:  $I/Y=4.1\%$ ;  $C/Y=2$

Year 3:  $I/Y=4.35\%$ ;  $C/Y=2$

Year 4:  $I/Y=4.75\%$ ;  $C/Y=2$

Year 5:  $I/Y=5.5\%$ ;  $C/Y=2$

**Step 2:** Calculate  $n$  and  $i$  for all years:

$$n = (\text{Number of Years}) \times C/Y = 1 \times 2 = 2$$

$$\text{Year 1: } i = \frac{I/Y}{C/Y} = \frac{4\%}{2} = 2\%$$

$$\text{Year 2: } i = \frac{I/Y}{C/Y} = \frac{4.1\%}{2} = 2.05\%$$

$$\text{Year 3: } i = \frac{I/Y}{C/Y} = \frac{4.35\%}{2} = 2.175\%$$

$$\text{Year 4: } i = \frac{I/Y}{C/Y} = \frac{4.75\%}{2} = 2.375\%$$

$$\text{Year 5: } i = \frac{I/Y}{C/Y} = \frac{5.5\%}{2} = 2.75\%$$

**Step 3:** Solve for  $PV$ .

$$\text{Year 5: } PV = \frac{\$30,320.12}{(1 + 0.0275)^2} = \$28,718.86385$$

$$\text{Year 4: } PV = \frac{\$28,718.86385}{(1 + 0.02375)^2} = \$27,401.82101$$

$$\text{Year 3: } PV = \frac{\$27,401.82101}{(1 + 0.02175)^2} = \$26,247.63224$$

$$\text{Year 2: } PV = \frac{\$26,247.63224}{(1 + 0.0205)^2} = \$25,203.68913$$

$$\text{Year 1: } PV = \frac{\$25,203.68913}{(1 + 0.02)^2} = \$24,225$$

**Step 4:** Solve for  $n$ .

$$n = (\text{Number of Years}) \times C/Y = 5 \times 4 = 20$$

**Step 5:** Use the formula for  $FV$  and rearrange for  $i$ .

$$FV = PV(1 + i)^n$$

$$\$30,320.12 = \$24,225(1 + i)^{20}$$

$$1.251604 = (1 + i)^{20}$$

$$1.251604^{\frac{1}{20}} = 1 + i$$

$$1.011284 = 1 + i$$

$$i = 0.011284$$

**Step 6:** Find the nominal rate,  $I/Y$ .

$$\begin{aligned} I/Y &= i \times C/Y \\ &= 0.011284 \times 4 \\ &= 0.045138 \\ &= 4.51\% \end{aligned}$$

\text{compounded quarterly}

\$24,225 investment earned 4.51% compounded quarterly.

Calculator Instructions for Solution 9.5 Question 3

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
Year 5	2	5.5	?	0	30,320.12	2	2
Year 4	2	4.75	?	0	±PV from above	2	2
Year 3	2	4.35	?	0	±PV from above	2	2
Year 2	2	4.1	?	0	±PV from above	2	2
Year 1	2	4	?	0	±PV from above	2	2
Nominal rate	20	?	-24,225	0	30,320.12	4	4

## 5.6: EQUIVALENT AND EFFECTIVE INTEREST RATES

1. The HBC credit card has a nominal interest rate of 26.44669% compounded monthly. What effective rate is being charged?

### Solution:

**Step 1:** Given information:

$$I/Y = 26.44669\%; C/Y_{\text{Old}} = 12; C/Y_{\text{New}} = 1$$

**Step 2:**

$$\begin{aligned}
 i_{\text{Old}} &= \frac{I/Y}{C/Y_{\text{Old}}} \\
 &= \frac{26.44669\%}{12} \\
 &= 2.203890\%
 \end{aligned}$$

**Step 3:**

$$\begin{aligned}
 i_{\text{New}} &= (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1 \\
 &= (1 + 0.02203890)^{\frac{12}{1}} - 1 \\
 &= (1.02203890)^{12} - 1 \\
 &= 1.299 - 1 \\
 &= 0.299
 \end{aligned}$$

29.9% effectively

## Calculator Instructions (using ICONV) for Solution 9.6 Question 1

NOM	C/Y	EFF
26.44669	12	?

2. Louisa is shopping around for a loan. TD Canada Trust has offered her 8.3% compounded monthly, Conexus Credit Union has offered 8.34% compounded quarterly, and ING Direct has offered 8.45% compounded semi-annually. Rank the three offers and show calculations to support your answer.

**Solution:**

Convert all to effective rates to facilitate a fair comparison.

**TD Canada Trust:**

**Step 1:** Given information:

$$I/Y = 8.3\%; C/Y_{\text{Old}} = 12; C/Y_{\text{New}} = 1$$

**Step 2:**

$$\begin{aligned} i_{\text{Old}} &= \frac{I/Y}{C/Y_{\text{Old}}} \\ &= \frac{8.3\%}{12} \\ &= 0.691\bar{6}\% \end{aligned}$$

**Step 3:**

$$\begin{aligned} i_{\text{New}} &= (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1 \\ &= (1 + 0.00691\bar{6})^{\frac{12}{1}} - 1 \\ &= (1.00691\bar{6})^{12} - 1 \\ &= 1.086231 - 1 \\ &= 0.086231 \end{aligned}$$

8.6231% effectively

**CONEXUS Credit Union:**

**Step 1:** Given information:

$$I/Y = 8.34\%; C/Y_{\text{Old}} = 4; C/Y_{\text{New}} = 1$$

**Step 2:**

$$\begin{aligned}
 i_{\text{Old}} &= \frac{I/Y}{C/Y_{\text{Old}}} \\
 &= \frac{8.34\%}{4} \\
 &= 2.085\%
 \end{aligned}$$

**Step 3:**

$$\begin{aligned}
 i_{\text{New}} &= (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1 \\
 &= (1 + 0.02085)^{\frac{4}{1}} - 1 \\
 &= (1.02085)^4 - 1 \\
 &= 1.086044 - 1 \\
 &= 0.086045
 \end{aligned}$$

8.6045% effectively

**ING Direct:**

**Step 1:** Given information:

$$I/Y = 8.45\%; C/Y_{\text{Old}} = 2; C/Y_{\text{New}} = 1$$

**Step 2:**

$$\begin{aligned}
 i_{\text{Old}} &= \frac{I/Y}{C/Y_{\text{Old}}} \\
 &= \frac{8.45\%}{2} \\
 &= 4.225\%
 \end{aligned}$$

**Step 3:**

$$\begin{aligned}
 i_{\text{New}} &= (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1 \\
 &= (1 + 0.04225)^{\frac{2}{1}} - 1 \\
 &= (1.04225)^2 - 1 \\
 &= 1.086285 - 1 \\
 &= 0.086285
 \end{aligned}$$

8.6285% effectively

### Ranking:

Rankings of Companies Based on Effective Rate for Solution 9.6 Question 2

Rank	Company	Effective Rate
1	ING Direct	8.6285%
2	TD Canada Trust	8.6231%
3	CONEXUS Credit Union	8.6045%

Calculator Instructions (using ICONV) for Solution 9.6 Question 2

Company	NOM	C/Y	EFF
TD	8.3	12	?
CONEXUS	8.34	4	?
ING	8.45	2	?

3. The TD Emerald Visa card wants to increase its effective rate by 1%. If its current interest rate is 19.067014% compounded daily, what new daily compounded rate should it advertise?

### Solution:

First calculate the effective rate.

**Step 1:** Given information:

$$I/Y = 19.067014\%; C/Y_{\text{Old}} = 365; C/Y_{\text{New}} = 1$$

**Step 2:**

$$\begin{aligned}
 i_{\text{Old}} &= \frac{I/Y}{C/Y_{\text{Old}}} \\
 &= \frac{19.067014\%}{365} \\
 &= 0.052238\%
 \end{aligned}$$

**Step 3:**

$$\begin{aligned}
 i_{\text{New}} &= (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1 \\
 &= (1 + 0.00052238)^{\frac{365}{1}} - 1 \\
 &= (1.00052238)^{365} - 1 \\
 &= 1.209999 - 1 \\
 &= 0.21
 \end{aligned}$$

21 % effectively

Now convert it back to a daily rate after making the adjustment (reverse steps 2 & 3):

#### Step 1:

$$i_{\text{New}} = 21\% + 1\% = 22\%; C/Y_{\text{Old}} = 365; C/Y_{\text{New}} = 1$$

#### Step 3:

$$\begin{aligned}
 i_{\text{New}} &= (1 + i_{\text{Old}})^{\frac{C/Y_{\text{Old}}}{C/Y_{\text{New}}}} - 1 \\
 0.22 &= (1 + i_{\text{Old}})^{\frac{365}{1}} - 1 \\
 1.22 &= (1 + i_{\text{Old}})^{365} \\
 1.22^{\frac{1}{365}} &= 1 + i_{\text{Old}} \\
 1.000544 &= 1 + i_{\text{Old}} \\
 i_{\text{Old}} &= 0.000544
 \end{aligned}$$

#### Step 2:

$$\begin{aligned}
 i_{\text{Old}} &= \frac{I/Y}{C/Y_{\text{Old}}} \\
 0.000544 &= \frac{I/Y}{365} \\
 I/Y &= 0.198905
 \end{aligned}$$

19.89% compounded daily

### 5.7: Determining the Number of Compounds

1. You just took over another financial adviser's account. The client invested \$15,500 at 6.92% compounded monthly and now has \$24,980.58. How long (in years and months) has this client had the money invested?

**Solution:****Step 1:** Given information:

$$PV = \$15,500; I/Y = 6.92\%; FV = \$24,980.58$$

**Step 2:** Calculate  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{6.92\%}{1} = 0.57\bar{6}\%$$

**Step 3:** Use the formula for  $FV$ , rearrange and solve for  $n$ .

$$FV = PV(1 + i)^n$$

$$\$24,980.58 = \$15,500(1 + 0.0057\bar{6})^n$$

$$1.611650 = (1.0057\bar{6})^n$$

$$\ln(1.611650) = n \times \ln(1.0057\bar{6})$$

$$0.477258 = n \times 0.005750$$

$$n = 83 \text{ monthly compounds}$$

$$\text{Years} = \frac{83}{12} = 6.91\bar{6} \text{ which is 6 years plus } 0.91\bar{6} \times 12 = 11 \text{ months}$$

6 years, 11 months

Calculator Instructions for Solution 9.7 Question 1

N	I/Y	PV	PMT	FV	P/Y	C/Y
?	6.92	15,500	0	24,980.58	12	12

2. Your organization has a debt of \$30,000 due in 13 months and \$40,000 due in 27 months. If a single payment of \$67,993.20 was made instead using an interest rate of 5.95% compounded monthly, when was the payment made? Use today as the focal date.

**Solution:****Step 1:** First figure out what the money is worth today.**Original Agreement:**

Payment #1 = \$30,000 due in 13 months

Payment #2 = \$40,000 due in 27 months

$$I/Y = 5.95\%; C/Y = 12$$

**Proposed Agreement:**

\$67,993.20 due in  $x$  months

**Step 2:** Focal date = today

**Step 3:** Calculate  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{5.95\%}{12} = 0.4958\bar{3}\%$$

**Step 4:** Calculate  $n$  of the payments.

**Payment #1:**

$$\begin{aligned} n &= (\text{Number of Years}) \times (\text{Compounds Per Year}) \\ &= 1\frac{1}{12} \times 12 \\ &= 1.08\bar{3} \times 12 \\ &= 13 \end{aligned}$$

**Payment #2:**

$$\begin{aligned} n &= (\text{Number of Years}) \times (\text{Compounds Per Year}) \\ &= 2\frac{3}{12} \times 12 \\ &= 2.25 \times 12 \\ &= 27 \end{aligned}$$

**Step 5:** Calculate  $PV$  of the payments.

**Payment #1:**

$$PV = \frac{\$30,000}{(1.004958)^{13}} = \$28,131.73574$$

**Payment #2:**

$$PV = \frac{\$40,000}{(1.004958)^{27}} = \$34,999.55193$$

**Step 6:** Find the total  $PV$  of the payments.

$$\text{Total today} = \$28,131.73574 + \$34,999.55193 = \$63,131.28768$$

**Now figure out where the payment occurs:**

**Step 1:**

$$PV = \$63,131.28768; FV = \$67,993.20; I/Y = 5.95\%; C/Y = 12$$

**Step 2:** Find  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{5.95\%}{12} = 0.4958\bar{3}\%$$

**Step 3:** Use the formula for  $FV$ , rearrange and solve for  $n$ .

$$FV = PV(1 + i)^n$$

$$\$67,993.20 = \$63,131.28768(1 + 0.004958)^n$$

$$1.121112 = (1.004958)^n$$

$$\ln(1.077012) = n \times \ln(1.004958)$$

$$0.074191 = n \times 0.004946$$

$$n = 15 \text{ monthly compounds}$$

**Step 4:** Convert the time to years and months.

$$\begin{aligned} \text{Number of years} &= \frac{15}{12} \\ &= 1.25 \text{ which is 1 year plus } 0.25 \times 12 = 3 \text{ months} \end{aligned}$$

Payment is made 15 months from today.

Calculator Instructions for Solution 9.7 Question 2

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
Payment 1	13	5.95	?	0	30,000	12	12
Payment 2	27	5.95	?	0	40,000	12	12
Timing of Payment	?	5.95	6,3131.28768	0	67,993.2	12	12

3. A \$9,500 loan requires a payment of \$5,000 after 1½ years and a final payment of \$6,000. If the interest rate on the loan is 6.25% compounded monthly, when should the final payment be made? Use today as the focal date. Express your answer in years and months.

**Solution:****Step 1:** Given information:

$$P = \$9,500; I/Y = 6.25\%; C/Y = 12$$

Payment #1 = \$5,000 due in  $1\frac{1}{2}$  years

Payment #2 = \$6,000 due in  $x$  years

**Step 2:** Focal date = today

**Step 3:** Find  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{6.25\%}{12} = 0.5208\bar{3}\%$$

**Step 4:** Calculate  $n$  for the first payment.

**Payment #1:**

$$\begin{aligned} n &= (\text{Number of Years}) \times (\text{Compounds Per Year}) \\ &= 1\frac{1}{2} \times 12 \\ &= 1.5 \times 12 \\ &= 18 \end{aligned}$$

**Payment #2:**

$$n = ?$$

**Step 5:** Calculate  $PV$  of the payments.

**Payment #1:**

$$\begin{aligned} \$5,000 &= PV(1 + 0.005208\bar{3})^{18} \\ PV &= \frac{\$5,000}{(1.005208\bar{3})^{18}} \\ &= \$4,553.65956 \end{aligned}$$

**Payment #2:**

$$\begin{aligned} \$6,000 &= PV(1 + 0.005208\bar{3})^n \\ PV &= \frac{\$6,000}{(1.005208\bar{3})^n} \end{aligned}$$

**Step 6:** Solve for  $n$  of the final payment.

$$\begin{aligned} \$9,500 &= \$4,553.65956 + \frac{\$6,000}{(1.005208)^n} \\ \$4,946.34044 &= \frac{\$6,000}{(1.005208)^n} \quad (1.005208)^n \\ &= \frac{\$6,000}{\$4,946.34044} \quad (1.005208)^n &= 1.213018 \\ &= \ln(1.213018) \quad n \times 0.005194 &= 0.193111 \quad n &= 37.173874 \\ &\text{monthly compounds (round up to)} \quad 38 \quad \text{months} \end{aligned}$$

**Step 7:** Convert the time to years and months.

$$\begin{aligned} \text{Number of years} &= \frac{38}{12} \\ &= 3.1\bar{6} \text{ which is 3 years plus } 0.1\bar{6} \times 12 = 2 \text{ months} \end{aligned}$$

3 years, 2 months

#### Image Descriptions

**Figure 9.2.3:** This timeline indicates \$9300 at 4.25 years ago. The interest rate of 6.35% compounded semi-annually goes from 4.25 years ago to 1.75 years ago, giving  $i = 0.03175$ . The interest rate of 6.5% compounded quarterly goes from 1.75 years ago to today, giving  $i = 0.01625$ . \$9300 moves from 4.25 years ago to 1.75 years ago as  $FV_1$ , with  $n = 2.5 \times 2 = 5$ .  $FV_1$  at 1.75 years ago moves to today as  $FV_2$  with  $n = 1.75 \times 4 = 7$ . [Back to Figure 9.2.3]

**Figure 9.3.2:** This timeline indicates \$45,839.05 at 8.5 years. The interest rate of 4.5% compounded semi-annually goes from Loan date to 4.5 years, giving  $i = 4.5\%/2 = 0.0225$ . The interest rate of 3.25% compounded annually goes from 4.5 years to 8.5 years, giving  $i = 3.25\%/1 = 0.0325$ . \$9300 moves from 8.5 years to 4.5 years as  $PV_1$ , with  $n = 4 \times 1 = 4$ .  $FV_1$  at 4.5 years moves to loan date as  $FV_2$  with  $n = 4.5 \times 2 = 9$ . [Back to Figure 9.3.2]

**Figure 9.4.1:** This timeline shows \$250,000 at today, \$300,000 at 1 year, \$300,000 at 2 years, \$300,000 at 3 years. The \$300,000 at 1 year moves back to today as  $PV_1$ , with  $n = 1 \times 1 = 1$ . The \$300,000 at 2 years moves back to today as  $PV_2$ , with  $n = 2 \times 1 = 2$ . The \$300,000 at 3 years moves back to today as  $PV_3$ , with  $n = 3 \times 1 = 3$ . [Back to Figure 9.4.1]

**Figure 9.4.2:** This is a timeline with debts above the line and payments below the line. The debt of \$1600 at 18 months ago is brought to 2 years as  $FV_1$  with  $n = 3.5 \times 4 = 14$ . The debt of \$2300 at 9 months ago is brought to 2 years as  $FV_2$  with  $n = 2.75 \times 4 = 11$ . The debt of \$5100 at 3 months ago is brought to 2 years as  $FV_3$  with  $n = 2.25 \times 4 = 9$ . The payment of \$1000 at today is brought to 2 years as  $FV_4$  with  $n = 2 \times 4 = 8$ . The payment of \$4000 at 9 months is brought to 2 years as  $FV_5$  with  $n = 1.25 \times 4 = 5$ . There is a payment of  $x$  at 2 years. [Back to Figure 9.4.2]

**Figure 9.4.3:** This is a timeline with debts above the line and payments below the line. The debt of \$7,429.74 at 12 months is brought to 15 months as  $FV_1$  with  $n = (3/12) \times 12 = 3$ . The debt of \$3,176.22 at 24 months is brought to 15 months as  $PV_1$  with  $n = (9/12) \times 12 = 9$ . The payment of  $x$  at 30 months is

brought to 15 months as  $PV_2$  with  $n = (15/12) \times 12 = 15$ . There is a payment of  $2x$  at 15 months. [Back to Figure 9.4.3]

## 6.1: APPLICATION: LONG-TERM GICS

1. Sanchez placed \$11,930 into a five-year interest payout GIC at 4.2% compounded monthly. Calculate the interest payout amount every month.

### Solution:

**Step 1:** Given information:

$$PV = \$11,930;$$

$$\text{Nominal Rate, } I/Y = 4.2\%;$$

$$\text{Compoundings per Year, } C/Y = 12.$$

**Step 2:** Calculate  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{4.2\%}{12} = 0.35\%$$

**Step 3:** Calculate  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds per Year}) = 5 \times 12 = 60$$

**Step 4:** Calculate the future value.

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$11,930(1 + 0.0035)^{60} \\ &= \$14,712.38405 \end{aligned}$$

**Step 5:** Find the interest amount earned,  $I$ .

$$I = FV - PV = \$14,712.38405 - \$11,930 = 2,782.38$$

Calculator Instructions for Solution 10.1 Question 1

N	I/Y	PV	PMT	FV	P/Y	C/Y
60	24.2	-11,930	0	?	12	12

2. TD Canada Trust is offering its five-year Stepper GIC at annually escalating rates of 1.15%, 2%, 2.75%, 3.5%, and 4.5%. All rates are compounded semi-annually. Alternatively, it is offering a five-year fixed rate GIC at 2.7% compounded monthly. What total interest amount does an

\$18,000 investment earn under each option?

**Solution:**

**Calculate the interest on the 5-year Stepper GIC:**

**Step 1:** Given information:

Year 1:  $I/Y = 1.15\%$ ;  $C/Y = 2$

Year 2:  $I/Y = 2\%$ ;  $C/Y = 2$

Year 3:  $I/Y = 2.75\%$ ;  $C/Y = 2$

Year 4:  $I/Y = 3.5\%$ ;  $C/Y = 2$

Year 5:  $I/Y = 4.5\%$ ;  $C/Y = 2$

$$PV = \$18,000$$

**Step 2:** Calculate  $i$ .

$$\text{Year 1: } i = \frac{I/Y}{C/Y} = \frac{1.15\%}{2} = 0.575\%$$

$$\text{Year 2: } i = \frac{I/Y}{C/Y} = \frac{2\%}{2} = 1\%$$

$$\text{Year 3: } i = \frac{I/Y}{C/Y} = \frac{2.75\%}{2} = 1.375\%$$

$$\text{Year 4: } i = \frac{I/Y}{C/Y} = \frac{3.5\%}{2} = 1.75\%$$

$$\text{Year 5: } i = \frac{I/Y}{C/Y} = \frac{4.5\%}{2} = 2.25\%$$

**Step 3:** Calculate  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$\text{For all years: } n = 1 \times 2 = 2$$

**Step 4:** Calculate the future value for each time segment.

$$\text{Year 1: } FV = \$18,000(1 + 0.00575)^2 = \$18,207.59513$$

$$\text{Year 2: } FV = \$18,207.59513(1 + 0.01)^2 = \$18,573.56779$$

$$\text{Year 3: } FV = \$18,573.56779(1 + 0.01375)^2 = \$19,087.85247$$

$$\text{Year 4: } FV = \$19,087.85247(1 + 0.0175)^2 = \$19,761.77296$$

$$\text{Year 5: } FV = \$19,761.77296(1 + 0.0225)^2 = \$20,661.06$$

**Step 5:** Calculate the amount of interest,  $I$ .

$$\begin{aligned} I &= FV - PV \\ &= \$20,661.06 - \$18,000 \\ &= \$2,661.06 \end{aligned}$$

Calculator Instructions for Solution 10.1 Question 2

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
Year 1	2	1.15	-18,000	0	?	2	2
Year 2	2	2	±FV from above	0	?	2	2
Year 3	2	2.75	±FV from above	0	?	2	2
Year 4	2	3.5	±FV from above	0	?	2	2
Year 5	2	4.5	±FV from above	0	?	2	2

### Calculate the Interest on the Fixed Rate GIC:

**Step 1:** Given information.

$$PV = \$18,000; I/Y = 2.7\%; C/Y = 12; t = 5 \text{ years.}$$

**Step 2:** Calculate  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{2.7\%}{12} = 0.225\%$$

**Step 3:** Calculate  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 5 \times 12 = 60$$

**Step 4:** Calculate the future value.

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$18,000(1 + 0.00225)^{60} \\ &= \$20,598.54 \end{aligned}$$

**Step 5:** Calculate the amount of interest,  $I$ .

$$\begin{aligned} I &= FV - PV \\ &= \$20,598.54 - \$18,000 \\ &= \$2,598.54 \end{aligned}$$

Calculator Instructions for Solution 10.1 Question 2 Calculating the Interest

N	I/Y	PV	PMT	FV	P/Y	C/Y
60	2.7	-18,000	0	?	12	12

3. Calculate the interest earned on each of the following five-year GICs. Rank the GICs from best to worst based on the amount of interest earned on a \$15,000 investment.
- An interest payout GIC earning 4.5% compounded quarterly.
  - A fixed rate compound interest GIC earning 4.2% compounded monthly.
  - A variable rate quarterly compound interest GIC earning consecutively 3.9% for 1.5 years, 4.25% for 1.75 years, 4.15% for 0.75 years, and 4.7% for 1 year.
  - An escalator rate GIC earning semi-annually compounded rates of 1.25%, 2%, 3.5%, 5.1%, and 7.75% in successive years.

**Solution:****a)****Step 1:** Given information.

$$PV = \$15,000; I/Y = 4.5\%; C/Y = 4; t = 5 \text{ years}$$

**Step 2:** Find  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{4.5\%}{4} = 1.125\%$$

**Step 3:** Find the total interest amount.

$$\begin{aligned} I &= PV \times i \\ &= \$15,000 \times 0.01125 \\ &= \$168.75 \end{aligned}$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 5 \times 4 = 20$$

$$\text{Total interest} = \$168.75 \times 20 = \$3,375$$

**b)****Step 1:** Given information.

$$PV = \$15,000; I/Y = 4.2\%; C/Y = 12; t = 5 \text{ years}$$

**Step 2:** Find  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{4.2\%}{12} = 0.35\%$$

**Step 3:** Find  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 5 \times 12 = 60$$

**Step 4:** Calculate the future value.

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$15,000(1 + 0.0035)^{60} \\ &= \$18,498.39 \end{aligned}$$

**Step 5:** Calculate the amount of interest,  $I$ .

$$\begin{aligned} I &= FV - PV \\ &= \$18,498.39 - \$15,000 \\ &= \$3,498.39 \end{aligned}$$

Calculator Instructions for Solution 10.1 Question 3 Part B

N	I/Y	PV	PMT	FV	P/Y	C/Y
60	4.2	-15,000	0	?	12	12

**c)**

**First Time Segment (today to 1.5 years):**

$$i = \frac{I/Y}{C/Y} = \frac{3.9\%}{4} = 0.975\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 1\frac{1}{2} \times 4 = 6$$

$$FV_1 = \$15,000 (1 + 0.00975)^6 = \$15,169.16$$

**Second Time Segment (1.5 years to 3.25 years):**

$$i = \frac{I/Y}{C/Y} = \frac{4.25\%}{4} = 1.0625\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 1\frac{3}{4} \times 4 = 7$$

$$FV_2 = \$15,1899.16916 (1 + 0.010625)^7 = \$17,1120.03668$$

**Third Time Segment (3.25 years to 4 years):**

$$i = \frac{I/Y}{C/Y} = \frac{4.15\%}{4} = 1.0375\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 0.75 \times 4 = 3$$

$$FV_3 = \$17,1120.03668 (1 + 0.010375)^3 = \$17,1658.44538$$

**Fourth Time Segment (last year):**

$$i = \frac{I/Y}{C/Y} = \frac{4.7\%}{4} = 1.175\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 1 \times 4 = 4$$

$$FV_4 = \$17,1658.44538 (1 + 0.01175)^4 = \$18,1503.13505$$

$$\begin{aligned} I &= FV - PV \\ &= \$18,503.14 - \$15,000 \\ &= \$3,503.14 \end{aligned}$$

Calculator Instructions for Solution 10.1 Question 3 Part C

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
1st segment	6	3.9	-15,000	0	?	4	4
2nd segment	7	4.25	±FV from above	0	?	4	4
3rd segment	3	4.15	±FV from above	0	?	4	4
4th segment	4	4.7	±FV from above	0	?	4	4

**d)**

**Step 1:** Given information:

Year 1: I/Y=1.25%; C/Y=2

Year 2: I/Y=2%; C/Y=2

Year 3: I/Y=3.5%; C/Y=2

Year 4: I/Y=5.1%; C/Y=2

Year 5: I/Y=7.75%; C/Y=2

**Step 2:** For all years:

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 1 \times 2 = 2$$

$$\text{Year 1: } i = \frac{I/Y}{C/Y} = \frac{1.25\%}{2} = 0.625\%$$

$$\text{Year 2: } i = \frac{I/Y}{C/Y} = \frac{2\%}{2} = 1\%$$

$$\text{Year 3: } i = \frac{I/Y}{C/Y} = \frac{3.5\%}{2} = 1.75\%$$

$$\text{Year 4: } i = \frac{I/Y}{C/Y} = \frac{5.1\%}{2} = 2.55\%$$

$$\text{Year 5: } i = \frac{I/Y}{C/Y} = \frac{7.75\%}{2} = 3.875\%$$

**Step 3:** Calculate the future value for each time segment.

$$\text{Year 1: } FV = \$15,000(1 + 0.00625)^2 = \$15,188.08594$$

$$\text{Year 2: } FV = \$15,188.08594(1 + 0.01)^2 = \$15,493.36646$$

$$\text{Year 3: } FV = \$15,493.36646(1 + 0.0175)^2 = \$16,040.37913$$

$$\text{Year 4: } FV = \$16,040.37913(1 + 0.0255)^2 = \$16,868.86873$$

$$\text{Year 5: } FV = \$16,868.86873(1 + 0.03875)^2 = \$18,201.54$$

$$\begin{aligned} I &= FV - PV \\ &= \$18,201.54 - \$15,000 \\ &= \$3,201.54 \end{aligned}$$

### Calculator Instructions:

Calculator Instructions for Solution 10.1 Question 3 Part D

Calculation	N	I/Y	PV	PMT	FV	P/Y	C/Y
Year 1	2	1.25	-15,000	0	?	2	2
Year 2	2	2	±FV from above	0	?	2	2
Year 3	2	3.5	±FV from above	0	?	2	2
Year 4	2	5.1	±FV from above	0	?	2	2
Year 5	2	7.75	±FV from above	0	?	2	2

Options Ranked by Interest Earned for Solution 10.1 Question 3 Part D

Rank	Option	Interest Earned
1	C	\$3,503.14
2	B	\$3,498.39
3	A	\$3,375.00
4	D	\$3,201.54

## 6.2: Application: Long-Term Promissory Notes

- Determine the proceeds of the sale on a six-year interest-bearing promissory note for \$5,750 at 6.9% compounded monthly, discounted two years and three months before its due date at a discount rate of 9.9% compounded quarterly.

### Solution:

**Step 1:** The timeline below represents the situation.

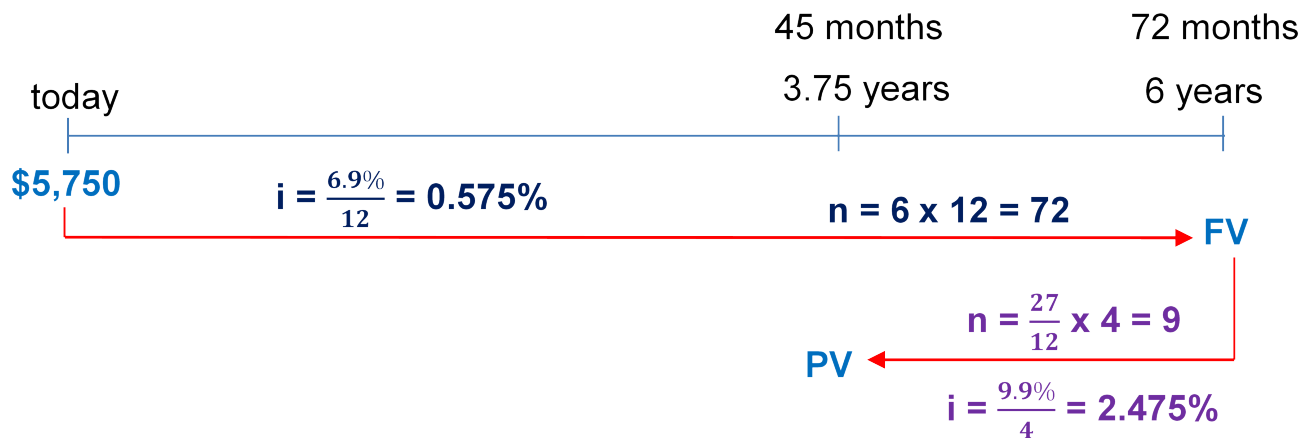


Figure 10.2.1: Timeline [Image Description]

**Step 2:** Calculate the maturity value (FV) of the original note.

$$i = \frac{I/Y}{C/Y} = \frac{6.9\%}{12} = 0.575\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 6 \times 12 = 72$$

$$\begin{aligned}
 FV &= PV(1 + i)^n \\
 &= \$5,750(1 + 0.00575)^{72} \\
 &= \$8,688.62
 \end{aligned}$$

**Step 3:** Find the proceeds by discounting the original note back 2.25 years to 3.75 years. Proceeds = PV.

$$i = \frac{I/Y}{C/Y} = \frac{9.9\%}{4} = 2.475\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = \frac{27}{12} \times 4 = 9$$

$$\begin{aligned} PV &= \frac{FV}{(1 + i)^n} \\ &= \frac{\$8,688.62}{(1.02475)^9} \\ &= \$6,972.52 \text{ (Proceeds)} \end{aligned}$$

The proceeds are \$6,972.52.

Calculator Instructions for Solution 10.2 Question 1

Action	N	I/Y	PV	PMT	FV	P/Y	C/Y
Original Note	72	6.9	-5,750	0	?	12	12
Proceeds	9	9.9	?	0	8,688.62	4	4

2. A \$36,555 interest-bearing note at 5% compounded monthly is issued on October 15, 2011, for a term of 87 months. Fifty-seven months later, the note is sold to yield a discount amount of \$11,733.41. What quarterly compounded discount rate is being used?

### Solution:

**Step 1:** The timeline below represents the situation.

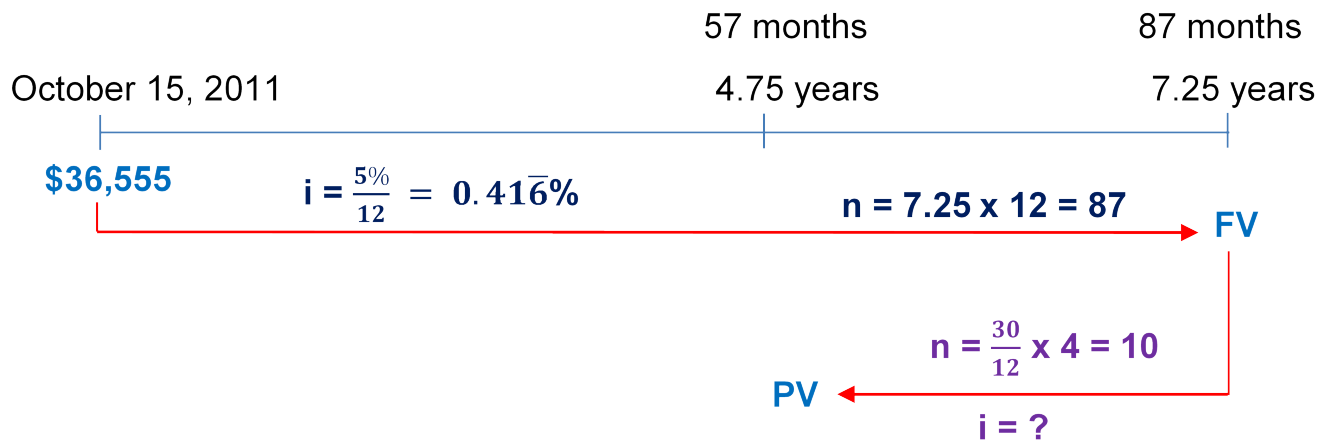


Figure 10.2.2: Timeline [Image Description]

**Step 2:** Calculate the maturity value (FV) of the original note.

$$i = \frac{I/Y}{C/Y} = \frac{5\%}{12} = 0.41\overline{6}\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 7.25 \times 12 = 87$$

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$36,555(1 + 0.0041\overline{6})^{87} \\ &= \$52,486.97 \end{aligned}$$

**Step 3:** Find the proceeds by discounting the original note back 2.5 years to 4.75 years. Proceeds = PV

$$\begin{aligned} PV &= FV - \text{Discount amount} \\ &= \$52,486.97 - \$11,733.41 \\ &= \$40,753.56 \text{ (Proceeds)} \end{aligned}$$

**Step 4:** Find the interest rate compounded quarterly when the original note discounted back from 87 months to 57 months.

$$\begin{aligned} FV &= PV(1 + i)^n \\ \$52,486.97 &= \$40,753.56(1 + i)^{10} \\ (1 + i)^{10} &= \frac{\$52,486.97}{\$40,753.56} \\ (1 + i)^{10} &= 1.287911 \\ 1 + i &= 1.287911^{\frac{1}{10}} \\ i &= 0.025624 \end{aligned}$$

(rate per quarter)

$$\begin{aligned} \text{Nominal rate} &= i \times (\text{compounds per year}) \\ &= 0.025624 \times 4 \\ &= 0.1025 \end{aligned}$$

Interest Per Year = 10.25% compounded quarterly

Calculator Instructions for Solution 10.2 Question 2

Action	N	I/Y	PV	PMT	FV	P/Y	C/Y
Original Note	87	5	-36,555	0	?	12	12
Discount Rate	10	?	-40,753.56	0	52,486.97	4	4

3. A seven-year interest-bearing note for \$19,950 at 8.1% compounded quarterly is issued on January 19, 2006. Four years and 11 months later, the note is discounted at 14.55% compounded monthly. Determine the proceeds on the note and how much interest the original owner of the note realized.

**Solution:**

**Step 1:** The timeline below represents the situation.

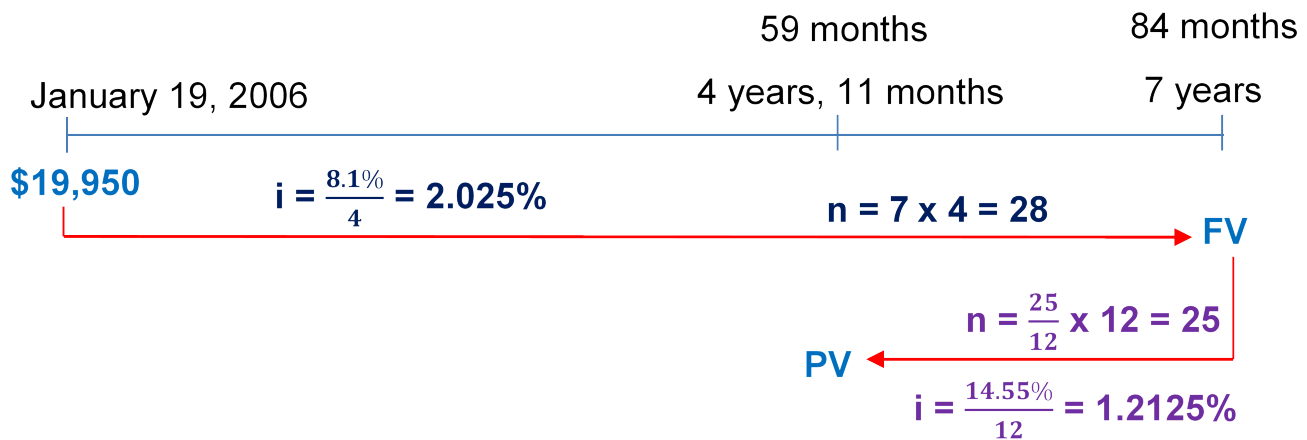


Figure 10.2.3: Timeline [Image Description]

**Step 2:** Calculate the maturity value (FV) of the original note.

$$i = \frac{I/Y}{C/Y} = \frac{8.1\%}{4} = 2.025\%$$

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 7 \times 4 = 28$$

$$\begin{aligned} FV &= PV(1 + i)^n \\ &= \$19,950(1 + 0.02025)^{28} \\ &= \$34,972.59 \end{aligned}$$

**Step 3:** Find the proceeds by discounting the original note back 25 months (84 – 59) to 4 years 11 months.

$$\begin{aligned}
 PV &= \frac{FV}{(1 + i)^n} \\
 &= \frac{\$34,972.59}{(1.012125)^{25}} \\
 &= \$25,874.62 \text{ (Proceeds)}
 \end{aligned}$$

Calculate the interest amount,  $I$ .

$$\begin{aligned}
 I &= FV - PV \\
 &= \$25,874.62 - \$19,950 \\
 &= \$5,924.62
 \end{aligned}$$

Proceeds are \$25,874.62 and the interest realized is \$5,924.62.

Calculator Instructions for Solution 10.2 Question 3

Action	N	I/Y	PV	PMT	FV	P/Y	C/Y
Original Note	28	8.1	-19,950	0	?	4	4
Proceeds	25	14.55	?	0	25,874.62	12	12

### 10.3: Application: Strip Bonds

1. A \$15,000 face value Government of Manitoba strip bond has 19.5 years left until maturity. If the current market rate is posted at 6.7322% compounded semi-annually, what is the purchase price for the bond?

#### Solution:

**Note:** A **Strip Bond** is an investment entitling the owner to receive only the face value of the bond at maturity.

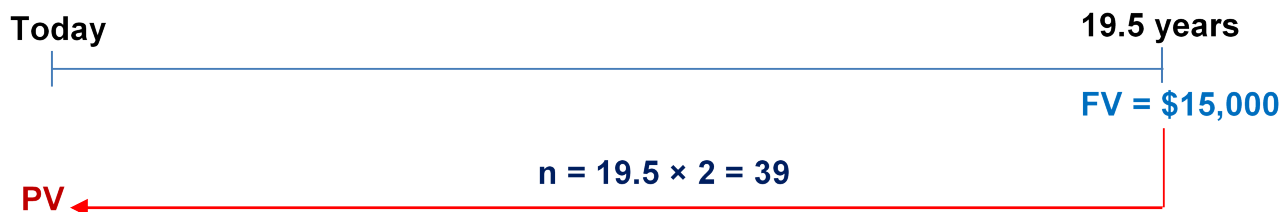


Figure 10.3.1: Timeline [Image Description]

**Step 1:** Given information:

$$FV = \$15,000; t = 19.5 \text{ years}; I/Y = 6.7322\%; C/Y = 2$$

**Step 2:** Calculate  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year}) = 19.5 \times 2 = 39$$

**Step 3:** Calculate  $i$ .

$$i = \frac{I/Y}{C/Y} = \frac{6.7322\%}{2} = 3.3661\%$$

**Step 4:** Solve for  $PV$ .

$$\begin{aligned} PV &= \frac{FV}{(1 + i)^n} \\ &= \frac{\$15,000}{(1.033661)^{39}} \\ &= \$4,124.24 \end{aligned}$$

The purchase price of the bond is \$4,124.24.

Calculator Instructions for Solution 10.3 Question 1

N	I/Y	PV	PMT	FV	P/Y	C/Y
39	6.7322	?	0	15,000	2	2

2. An investor purchased a \$7,500 face value strip bond for \$2,686.01 on May 29, 2006. The strip bond had been issued on May 29, 2002, with a 25-year maturity. The investor sold the strip bond on November 29, 2012, for \$3,925.28.
  - a) What was the market yield when the investor purchased the strip bond?
  - b) What was the market yield when the investor sold the strip bond?
  - c) What actual yield did the investor realize on the strip bond?

**Solution:**

**a)**

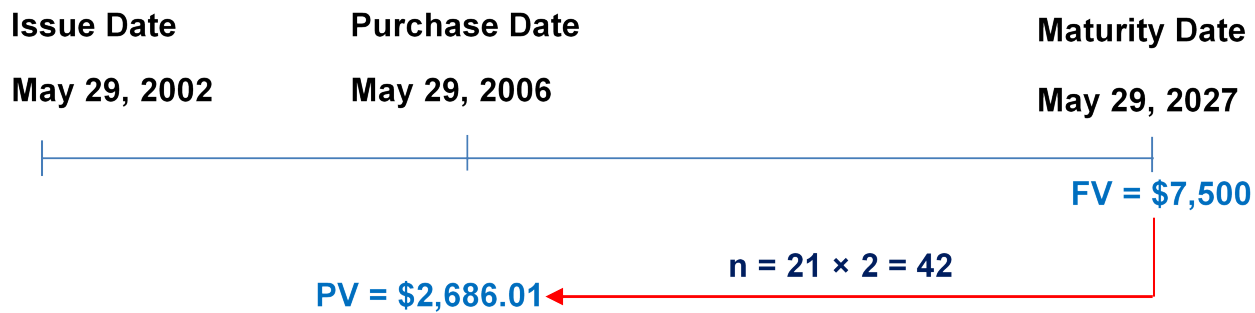


Figure 10.3.2a: Timeline [Image Description]

**Step 1:** Given information:

$$PV = \$2,686.01; FV = \$7,500$$

**Step 2:** Calculate the number of years.

$$\text{Number of Years} = (\text{May 29, 2027}) - (\text{May 29, 2006}) = 21$$

**Step 3:** Calculate  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 21 \times 2 = 42$$

**Step 4:** Calculate  $i$ .

$$\begin{aligned} FV &= PV(1+i)^n \\ \$7,500 &= \$2,686.01 (1+i)^{42} \\ 2.792246 &= (1+i)^{42} \\ 2.792246^{\frac{1}{42}} &= 1+i \\ 1.024750 &= 1+i \\ i &= 0.024750 \\ \text{\textit{(rate per half year)}} \end{aligned}$$

**Step 5:** Calculate the nominal interest rate,  $I/Y$ .

$$\begin{aligned} I/Y &= i \times (\text{compounds per year}) \\ &= 0.024750 \times 2 \\ &= 0.0495 \end{aligned}$$

Market yield = 4.95% compounded semi-annually

Calculator Instructions for Solution 10.3 Question 2 Part A

N	I/Y	PV	PMT	FV	P/Y	C/Y
42	?	-2,686.01	0	7,500	2	2

**b)**

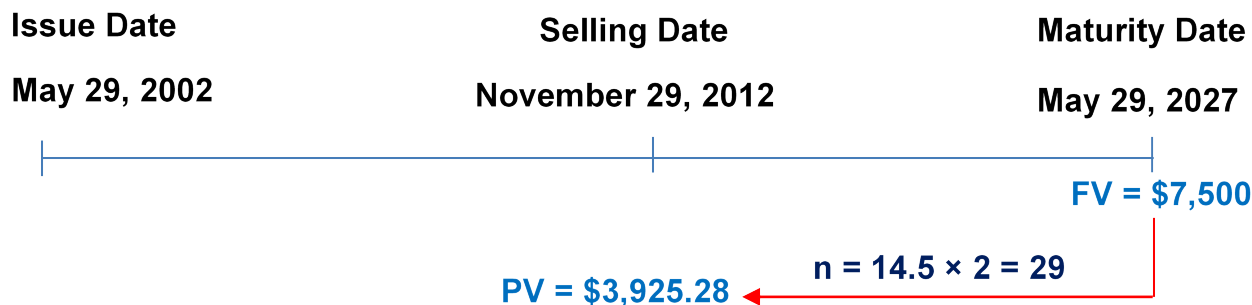


Figure 10.3.2b: Timeline [Image Description]

**Step 1:** Given information:

$$PV = \$3,925.28; FV = \$7,500$$

**Step 2:** Find the number of years.

$$\text{Number of Years} = (\text{May 29, 2027}) - (\text{November 29, 2012}) = 14.5$$

**Step 3:** Find  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 14.5 \times 2 = 29$$

**Step 4:** Find  $i$ .

$$\begin{aligned} FV &= PV(1+i)^n \\ \$7,500 &= \$3,925.28 (1+i)^{29} \\ \left(\frac{\$7,500}{\$3,925.28}\right)^{\frac{1}{29}} - 1 &= 0.022577 \end{aligned}$$

(rate per half year)

**Step 5:** Calculate the nominal interest rate,  $I/Y$ .

$$\begin{aligned} \text{Nominal rate} &= i \times (\text{compounds per year}) \\ &= 0.022577 \times 2 \\ &= 0.045155 \end{aligned}$$

Market yield = 4.5155% compounded semi-annually

Calculator Instructions for Solution 10.3 Question 2 Part B

N	I/Y	PV	PMT	FV	P/Y	C/Y
29	?	-3,925.28	0	7,500	2	2

c)

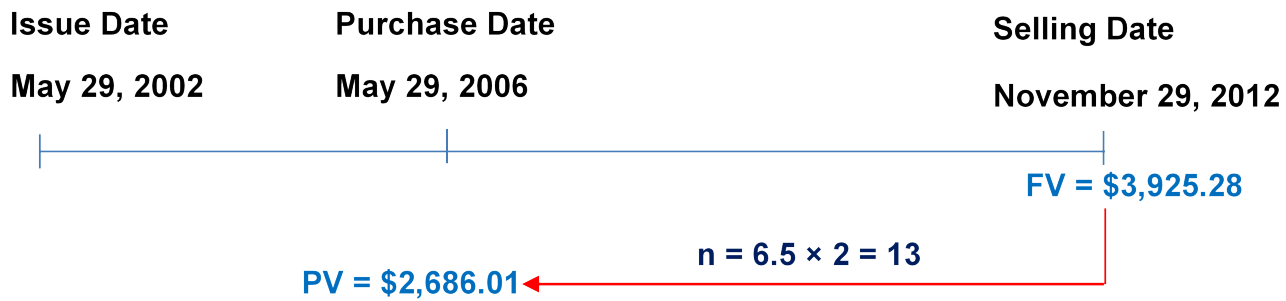


Figure 10.3.2c: Timeline [Image Description]

**Step 1:** Given information:

$$PV = \$2,686.01; FV = \$3,925.28$$

**Step 2:** Find the number of years.

$$\text{Number of Years} = (\text{November 29, 2012}) - (\text{May 29, 2006}) = 6.5$$

**Step 3:** Find  $n$ .

$$n = (\text{Number of Years}) \times (\text{Compounds Per Year})$$

$$n = 6.5 \times 2 = 13$$

**Step 4:** Find  $i$ .

$$\begin{aligned} FV &= PV(1+i)^n \\ \$3,925.28 &= \$2,686.01 (1+i)^{13} \\ i &= \left(\frac{\$3,925.28}{\$2,686.01}\right)^{\frac{1}{13}} - 1 \\ i &= 0.029613 \end{aligned}$$

$i = 0.029613$  (rate per half year)

**Step 5:** Calculate the nominal interest rate,  $I/Y$ .

$$\begin{aligned} \text{Nominal rate} &= i \times (\text{compounds per year}) \\ &= 0.029613 \times 2 \\ &= 0.059226 \end{aligned}$$

Market yield = 5.9226% compounded semi-annually

Calculator Instructions for Solution 10.3 Question 2 Part C

N	I/Y	PV	PMT	FV	P/Y	C/Y
13	?	-2,686.01	0	3,925.28	2	2

## 7.1: FUNDAMENTALS OF ANNUITIES

- Each year, Buhler Industries saves up \$1 million to distribute in Christmas bonuses to its employees. To do so, at the end of every month the company invests \$81,253.45 into an account earning 5.5% compounded monthly. Calculate the number of the payments (investments).

### Solution:

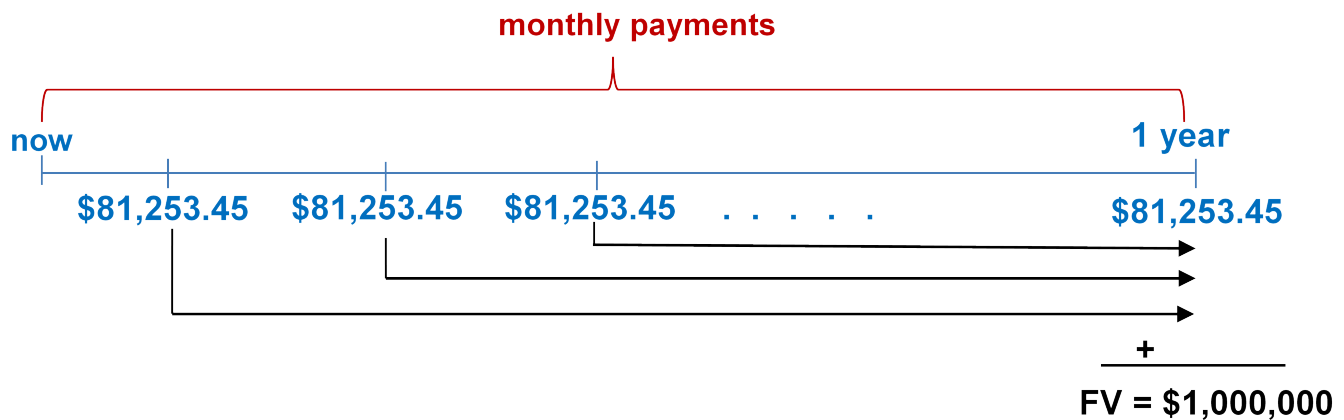


Figure 7.1.1: Timeline [Image Description]

Annuity type is **Ordinary Simple Annuity** (payment and compounding frequencies coincide and payments at end of the interval).

**Step 1:** Given information:

Years = 1;  $P/Y = \text{monthly} = 12$

**Step 2:** Calculate  $N$ .

$N = (\text{Number of Years}) \times (\text{Payments Per Year})$

$N = 1 \times 12 = 12 \text{ payments}$

- Marie has decided to start saving for a down payment on her home. If she puts \$1,000 every quarter for five years into a GIC earning 6% compounded monthly she will have \$20,979.12. She will make her first deposit three months from now. Draw an annuity timeline and determine the annuity type. Calculate the number of the payments (deposits).

### Solution:

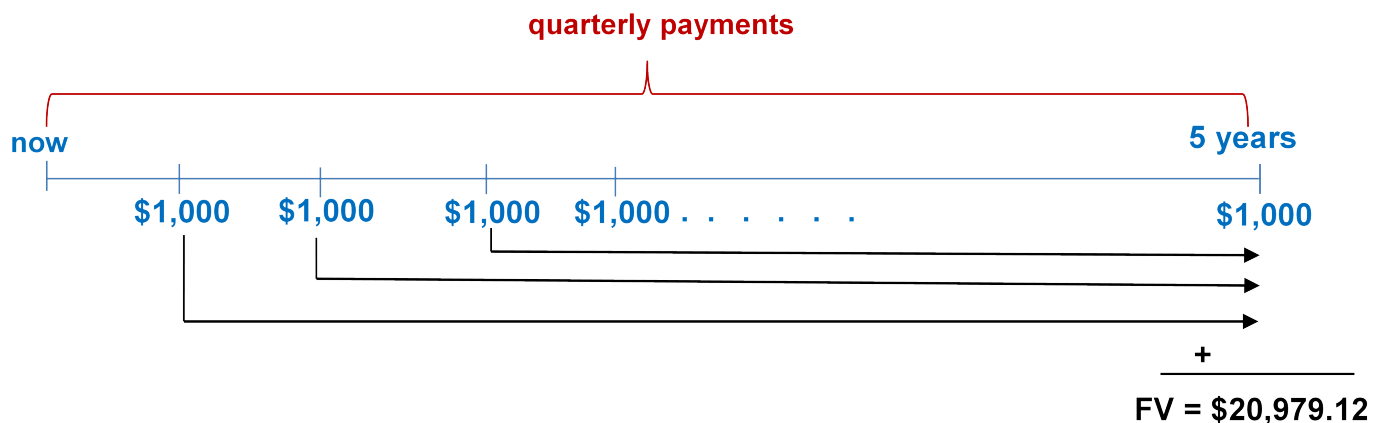


Figure 7.1.2: Timeline [Image Description]

Annuity type is **Ordinary General Annuity** (payment and compounding frequencies do not coincide and payments at end of the interval).

**Step 1:** Given information.

Years = 5;  $P/Y$  = quarterly = 4

**Step 2:** Calculate  $N$ .

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 5 \times 4 = 20 \text{ payments}$$

3. Steve takes out a two-year gym membership worth \$500. The first of his monthly \$22.41 payments is due at signing and includes interest at 8% compounded annually. Calculate the number of the payments.

**Solution:**

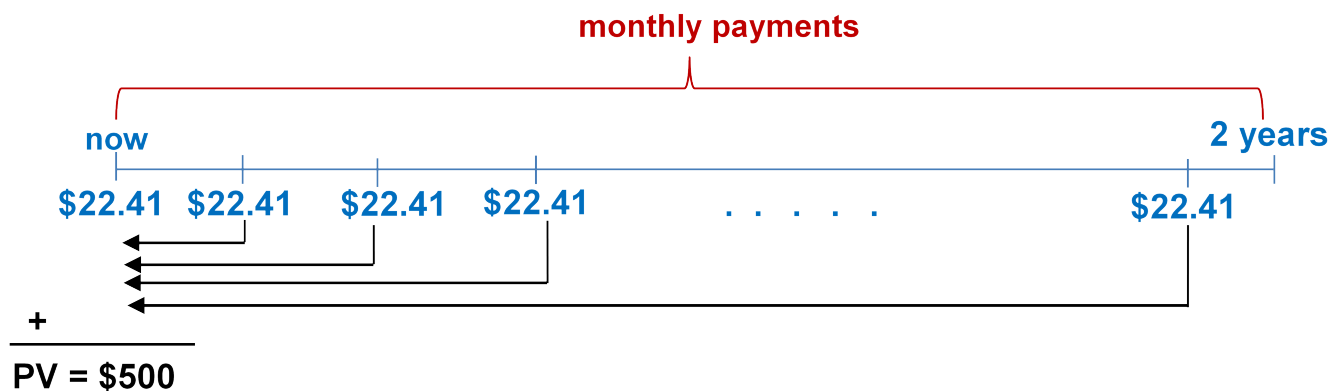


Figure 7.1.3: Timeline [Image Description]

Annuity type is **General Annuity Due** (payment and compounding frequencies do not coincide and payments at beginning of the interval).

**Step 1:** Given information.

$$\text{Years} = 2; P/Y = \text{monthly} = 12$$

**Step 2:** Calculate  $N$

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 2 \times 12 = 24 \text{ payments}$$

## 7.2: FUTURE VALUE OF ANNUITIES

---

1. You are a financial adviser. Your client is thinking of investing \$600 at the end of every six months for the next six years with the invested funds earning 6.4% compounded semi-annually. Your client wants to know how much money she will have after six years. What do you tell your client?

**Solution:**

**Using BA2+ calculator:**

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 6 \times 2 = 12 \text{ payments}$$

$$I/Y = 6.4$$

$$P/Y = 2$$

$$PV = 0$$

$$C/Y = 2$$

$$PMT = -600$$

$$CPT FV = \$8,612.62$$

The client will have \$8,612.62 after six years.

2. McDonald's major distribution partner, The Martin-Brower Company, needs at least \$1 million to build a new warehouse in Medicine Hat two years from today. To date, it has invested \$500,000. If it continues to invest \$50,000 at the end of every quarter into a fund earning 6% quarterly, will it have enough money to build the warehouse two years from now? Show calculations to support your answer.

**Solution:**

**Using BA2+ calculator:**

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 2 \times 4 = 8 \text{ payments}$$

$$I/Y = 6$$

$$P/Y = 4$$

$$C/Y = 4$$

$$PV = -500,000$$

$$PMT = -50,000$$

$$CPT FV = \$984,188.25$$

No, the fund is \$15,111.75 (\$1,000,000 - \$984,888.25) short of the money required.

3. The Saskatchewan Roughriders started a rainy-day savings fund three-and-a-half years ago to help pay for stadium improvements. At the beginning of every quarter the team has deposited \$20,000 into the fund, which has been earning 4.85% compounded semi-annually. How much money is in the fund today?

**Solution:****Using BA2+ calculator:****Mode = BGN**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 3.5 \times 4 = 14 \text{ payments}$$

$$I/Y = 4.85$$

$$P/Y = 4$$

$$C/Y = 2$$

$$PV = 0$$

$$PMT = -20,000$$

$$CPT FV_{\text{due}} = \$306,680.93$$

\$306,680.93 is in the fund today.

4. Carlyle plans to make month-end contributions of \$400 to his RRSP from age 20 to age 40. From age 40 to age 65, he plans to make no further contributions to his RRSP. The RRSP can earn 9% compounded annually from age 20 to age 60, and then 5% compounded annually from age 60 to age 65. Under this plan, what is the maturity value of his RRSP when he turns 65?

**Solution:**

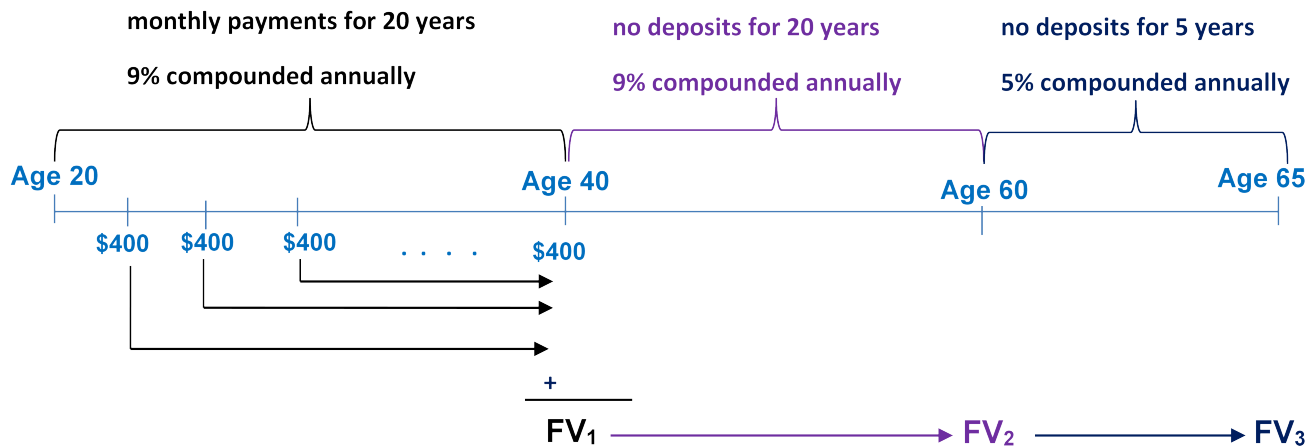


Figure 7.2.4: Timeline [Image Description]

### Using BA2+ calculator:

Mode = END

**Step 1:** Find  $FV_1$ .

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 20 \times 12 = 240 \text{ payments}$$

$$I/Y = 9$$

$$P/Y = 12$$

$$C/Y = 1$$

$$PV = 0$$

$$PMT = -400$$

$$CPT FV_1 = \$255,540.6808 \text{ (This becomes PV in Step 2)}$$

**Step 2:** Find  $FV_2$ .

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 20 \times 1 = 20 \text{ payments}$$

$$I/Y = 9$$

$$P/Y = 1$$

$$C/Y = 1$$

$$PMT = 0$$

$$PV = \$255,540.6808$$

$$CPT FV_2 = \$1,432,154.943 \text{ (This becomes PV in Step 3)}$$

**Step 3:** Find  $FV_3$

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 5 \times 1 = 5 \text{ payments}$$

$$I/Y = 5$$

$$P/Y = 1$$

$$C/Y = 1$$

$$PMT = 0$$

$$PV = \$1,432,154.943$$

$$CPT FV_3 = \$1,827,832.95$$

The maturity value of his RRSP is \$1,827,832.95 when he turns 65.

### 7.3: PRESENT VALUE OF ANNUITIES

1. When Sinbad retires, he expects his RRSP to pay him \$2,000 at the end of every month for 25 years. If his retirement annuity earns 3.8% compounded quarterly, how much money does he need to have in his RRSP when he retires?

#### Solution:



Figure 7.3.1: Timeline [Image Description]

#### Using BA2+ calculator:

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 25 \times 12 = 300 \text{ payments}$$

$$I/Y = 3.8$$

$$P/Y = 12$$

$$C/Y = 4$$

$$FV = 0$$

$$PMT = -2,000$$

$$CPT PV = \$387,444.19$$

He needs to have \$387,444.19 in the RRSP when he retires.

2. Sandy's parents would like to have an annuity pay her \$500 at the beginning of every month

from September 1, 2012, to April 1, 2017, to help with her university tuition and living expenses. On May 1, 2017, they would like to give her a graduation gift of \$5,000. If the annuity can earn 6.15% compounded quarterly, how much money must be in the account on September 1, 2012? (Use years and months in the calculations).

### Solution:

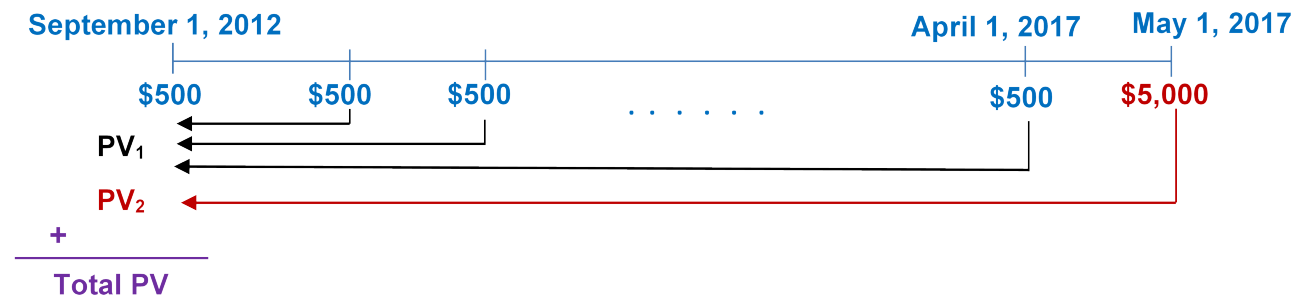


Figure 7.3.2: Timeline [Image Description]

$$\text{Number of Years} = (\text{May 1, 2017}) - (\text{September 1, 2012}) = 4\frac{8}{12}$$

### Compute Total PV:

Using BA2+ calculator:

Mode = BGN

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 4\frac{8}{12} \times 12 = 56 \text{ payments}$$

$$I/Y = 6.5$$

$$P/Y = 12$$

$$C/Y = 4$$

$$PMT = -500$$

$$FV = -5,000$$

$$CPT PV = \$28,188.43$$

\$28,188.43 must be in the account on September 1, 2012.

## 7.4: ANNUITY PAYMENT AMOUNTS

1. To save approximately \$30,000 for a down payment on a home four years from today, what amount needs to be invested at the end of every month at 4.5% compounded semi-annually?

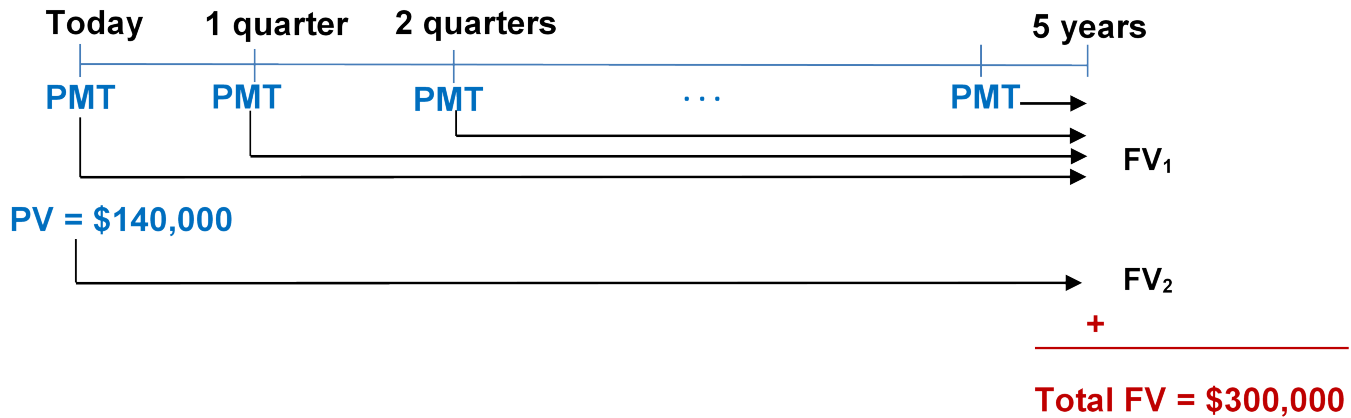
**Solution:**

Figure 7.4.2: Timeline [Image Description]

**Using BA2+ calculator:****Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 4 \times 12 = 48 \text{ payments}$$

$$I/Y = 4.5$$

$$P/Y = 12$$

$$C/Y = 2$$

$$PV = 0$$

$$FV = 30,000$$

$$CPT \text{ PMT} = \$572.08$$

The required end of month deposit is \$572.08

2. Sinclair does not believe in debt and will only pay cash for all purchases. He has already saved up \$140,000 toward the purchase of a new home with an estimated cost of \$300,000. Suppose his investments earn 7.5% compounded monthly. How much does he need to contribute at the beginning of each quarter if he wants to purchase his home in five years?

**Solution:**

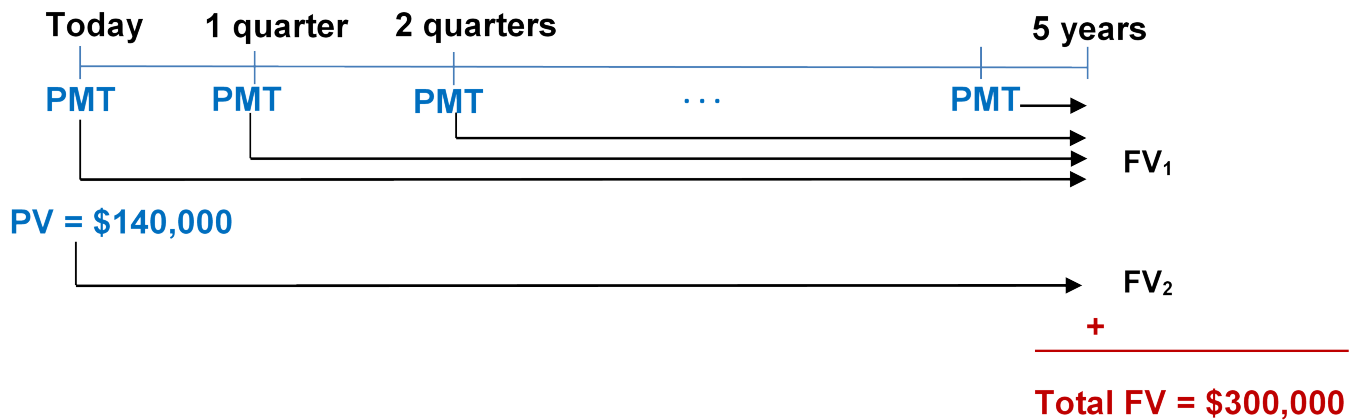


Figure 7.4.2: Timeline [Image Description]

Using BA2+ calculator:

Mode = BGN

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 5 \times 4 = 20 \text{ payments}$$

$$I/Y = 7.5$$

$$P/Y = 4$$

$$C/Y = 12$$

$$PV = -140,000$$

$$FV = 300,000 \text{ (Opposite sign to PV)}$$

$$CPT \text{ PMT} = -\$3,943.82$$

The required beginning of quarter deposit is \$3,943.82.

- The Kowalskis' only child is eight years old. They want to start saving into an RESP such that their son will be able to receive \$5,000 at the end of every quarter for four years once he turns 18 and starts attending postsecondary school. When the annuity is paying out, it is forecast to earn 4% compounded monthly. While they make contributions at the end of every month to the RESP, it will earn 8% compounded semi-annually. Additionally, at the end of every year of contributions the government places a \$500 grant into the RESP. What is the monthly contribution payment by the Kowalskis?

### Solution:

**Step 1:** Calculate the present value of the payments to the child to determine the money needed in the RESP when he turns 18.

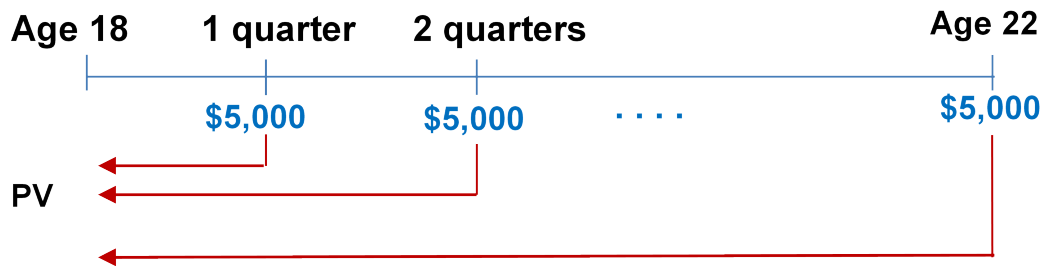


Figure 7.4.3-1: Timeline [Image Description]

**Using BA2+ calculator:**

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 4 \times 4 = 16 \text{ payments}$$

$$I/Y = 4$$

$$P/Y = 4$$

$$C/Y = 12$$

$$PMT = -5,000$$

$$FV = 0$$

$$CPT PV = \$73,569.21991$$

**Step 2:** Calculate the future value of the RESP grant at age 18.

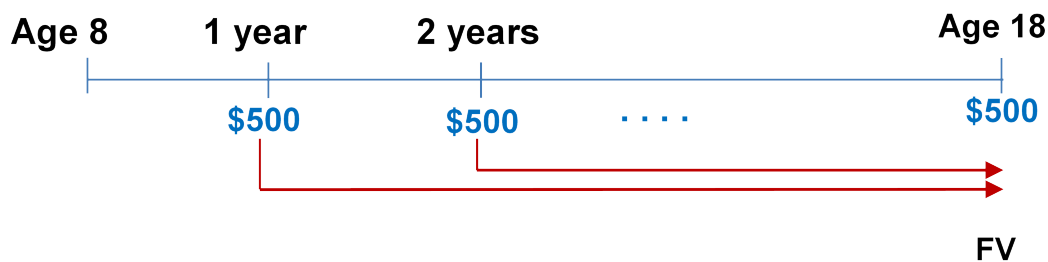


Figure 7.4.3-2: Timeline [Image Description]

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 10 \times 1 = 10 \text{ payments}$$

$$I/Y = 8$$

$$P/Y = 1$$

$$C/Y = 2$$

$$PV = 0$$

$$PMT = -500$$

$$CPT FV = \$7,298.548671$$

**Step 3:** Calculate the payments required from 8 years old to 18 years old.

$$\text{Required FV} = \$73,569.21991 - \$7,298.548671 = \$66,270.67124$$

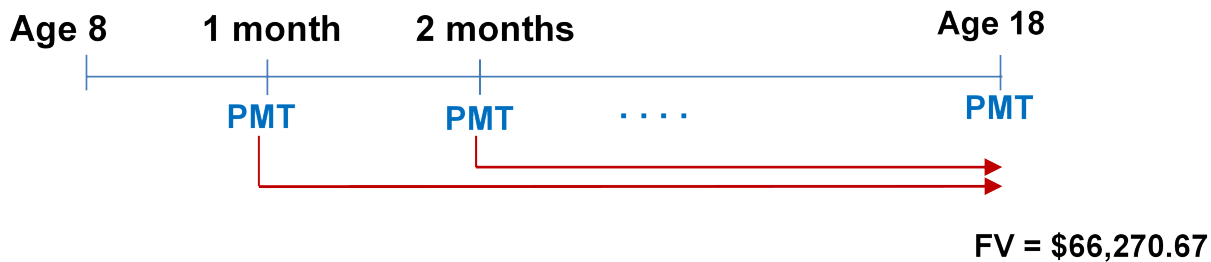


Figure 7.4.3-3: Timeline [Image Description]

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 10 \times 12 = 120 \text{ payments}$$

$$I/Y = 8$$

$$P/Y = 12$$

$$C/Y = 2$$

$$PV = 0$$

$$FV = -66,270.67$$

$$CPT PMT = -\$364.88$$

Therefore, a monthly deposit of \$364.88 is required from age 8 to 18.

## 7.5: NUMBER OF ANNUITY PAYMENTS

1. Amarjit wants to save up for a down payment on his first home. A typical starter home in his area sells for \$250,000 and the bank requires a 10% down payment. If he starts making \$300 month-end contributions to an investment earning 4.75% compounded monthly, how long will it take for Amarjit to have the necessary down payment?

**Solution:**

$$\text{Required future down payment} = 10\% \times \$250,000 = \$25,000.$$

**Using BA2+ calculator:**

**Mode = END**

$$I/Y = 4.75$$

$$P/Y = 12$$

$$C/Y = 12$$

$$PMT = -300$$

$$PV = 0$$

$$FV = 25,000$$

$$CPT N = 72.161266 \text{ rounded up to } 73 \text{ monthly payments}$$

$$\begin{aligned} \text{Number of years} &= \frac{73}{12} \\ &= 6.08\overline{3} \\ &= 6 \text{ years and } 0.08\overline{3} \times 12 = 1 \text{ month} \end{aligned}$$

6 years, 1 month

2. Hi-Tec Electronics is selling a 52" LG HDTV during a special “no sales tax” event for \$1,995 with end of month payments of \$100 including interest at 15% compounded semi-annually. How long will it take a consumer to pay off her new television?

### **Solution:**

**Mode = END**

$$I/Y = 15$$

$$P/Y = 12$$

$$C/Y = 2$$

$$PV = 1,995$$

$$PMT = -100$$

$$FV = 0$$

$$CPT N = 22.9783316 \text{ rounded up to } 23 \text{ monthly payments}$$

$$\begin{aligned} \text{Number of years} &= \frac{23}{12} \\ &= 1.91\overline{6} \\ &= 1 \text{ year and } 0.91\overline{6} \times 12 = 11 \text{ month} \end{aligned}$$

1 year, 11 months

3. Most financial institutions tout the benefits of “topping up” your mortgage payments—that is, increasing from the required amount to any higher amount. Assume a 25-year mortgage for \$200,000 at a fixed rate of 5% compounded semi-annually.
  - a) How many fewer payments does it take to pay off your mortgage if you increased your

monthly payments by 10%?

b) How much money is saved by “topping up” the payments? Assume that all payments are equal amounts in your calculations.

### Solution:

a)

**Step 1:** Find the original PMT.

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 25 \times 12 = 300 \text{ payments}$$

$$I/Y = 5$$

$$P/Y = 12$$

$$C/Y = 2$$

$$PV = 200,000$$

$$FV = 0$$

$$CPT \text{ PMT} = -\$1,163.21$$

Therefore, the original PMT is \$1,163.21.

**Step 2:** Find the “Topped up” PMT.

$$\text{“Topped up” PMT} = \$1,163.21(1.10) = \$1,279.53$$

**Step 3:** Find the new term with the “Topped up” PMT.

**Mode = END**

$$I/Y = 5$$

$$P/Y = 12$$

$$C/Y = 2$$

$$PV = 200,000$$

$$PMT = -1,279.53$$

$$CPT \text{ } N = 251.3724 \text{ rounded up to } 252 \text{ monthly payments}$$

**Step 4:** How many fewer payments =  $300 - 252 = 48$

There are 48 fewer monthly payments. The mortgage is paid off four years earlier (after 21 years rather than 25 years).

b)

Regular payments =  $300 \times \$1,163.21 = \$348,963$

Topped up payments =  $252 \times \$1,279.53 = \$322,441.56$

Savings =  $\$348,963 - \$322,441.56$

Savings =  $\$26,521.44$

## 7.6: ANNUITY INTEREST RATES

---

1. Francisco just changed occupations. Unfortunately, he is not able to transfer his company pension with him to his new company. The administrators of the pension plan offer him the choice of a lump-sum payout of \$103,075 today or beginning-of-month payments of \$535 for the next 25 years. What semi-annually compounded rate of return are the pension administrators using in their calculations?

### Solution:

**Mode = BGN**

$N = (\text{Number of Years}) \times (\text{Payments Per Year})$

$N = 25 \times 12 = 300 \text{ payments}$

$P/Y = 12$

$C/Y = 2$

$PV = 103,075$

$FV = 0$

$PMT = -535$

$CPT I/Y = 3.9019$

3.9019% compounded semi-annually.

2. An investment today requires \$1,125.51 to purchase. In return, the investment pays out \$30 after every six months for the next 20 years, along with an additional final lump-sum payout of \$1,000. What semi-annually compounded interest rate is being earned on the investment?

### Solution:

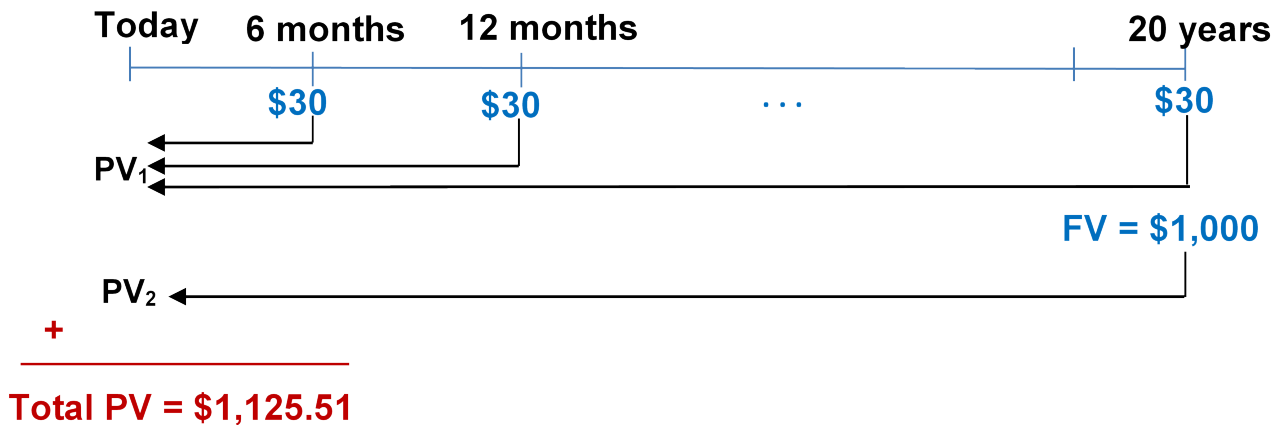


Figure 7.6.2: Timeline [Image Description]

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 20 \times 2 = 40 \text{ payments}$$

$$P/Y = 2$$

$$C/Y = 2$$

$$PV = 1,125.51$$

$$PMT = -30 \text{ (Opposite sign to PV)}$$

$$FV = -1,000 \text{ (Opposite sign to PV)}$$

$$CPT I/Y = 5.0000$$

5% compounded semi-annually.

- When you buy a car, a cash rebate is usually available if you finance the vehicle through your bank instead of the dealership; if you finance the vehicle through the dealership, you are not eligible for the cash rebate. Assume you can purchase a vehicle for \$24,960 and finance it for four years with month-end payments at 0% through the dealership. Alternatively, you could get a loan from a bank and pay cash for your vehicle, which would entitle you to receive a \$3,500 cash rebate. What monthly compounded interest rate would the bank have to charge to arrive at the same monthly payment as the dealership alternative? What decision rule can you create from this calculation?

**Solution:**

**Finance Though Dealership:**

Determine dealership payment:

$$\text{If } I/Y = 0\%, \text{ then } PMT = PV \div N = \$24,960 \div (12 \times 4) = \$520$$

**Finance Though Bank:**

The PV is reduced by the rebate:

$$PV = \$24,960 - \$3,500 = \$21,460$$

Find the monthly compounded interest rate that allows the same end of month payments of \$520.

**Mode = END**

$$N = (\text{Number of Years}) \times (\text{Payments Per Year})$$

$$N = 4 \times 12 = 48 \text{ payments}$$

$$P/Y = 12$$

$$C/Y = 12$$

$$PV = 21,460$$

$$PMT = -520 \text{ (Opposite sign to PV)}$$

$$FV = 0$$

$$CPT I/Y = 7.6118$$

7.6118% compounded monthly

**Decision rule:**

If you can obtain a loan from the bank for less than 7.6118% compounded monthly, your payments would be lower, and you should borrow from the bank. If unable to obtain a rate lower than 7.6118% from the bank, forego the cash rebate and use the dealership financing.

## VIDEOS

## Videos: Definitions, Ratios and Proportions

1. To run a lemonade stand, Charlie and Sam spent \$20 building a booth, and \$100 on lemons and sugar. If they brought in \$300 in Sales, find their Costs, Net Profit and Gross Profit.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1558#oembed-1>

2. Detailed information about Minute Maid Lemonade company is supplied below:

### Income Statement

Sales \$20,000,000

– COGS 12,000,000

= Gross Profit 8,000,000

– Operating expenses 5,000,000

= Net Profit 3,000,000

Create a common size income statement (express each line as a percent of sales.)

Find the Percent Net Margin for both Charlie and Sam's lemonade stand, and for Minute Maid. Which company had higher Profit Margins?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1558#oembed-2>

3. Sarah likes to keep the investments in her RRSP in a ratio of 1:3:2 in Cash, Bonds, and Stocks. Last year she invested \$12,000 in total.

	%	Initial	%change	Interest	Today	New Total	Difference
CASH			+3%				
BONDS			+8%				
STOCKS			-2%				
<b>TOTAL</b>		\$12,000	-				

- (a) How did she invest her initial \$12,000?
- (b) Over the year, her cash investments rose by 3%, her Bonds by 8%, and her stocks lowered by 2%. What was her profit or loss over the year?
- (c) This year, Sarah has another \$2540 to invest. How should she invest her money so that the ratio of total investments returns to 1:3:2?



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## CHAPTER ATTRIBUTION

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## Videos: Markups and Markdowns



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1. Arizona computers marks up merchandise by 25%.
  - a. If a laptop costs \$500 how much would it sell for?
  - b. A desktop sells for \$750, how much did it cost?
  - c. The company had total sales of \$150,000 for the month. Calculate the Cost of Goods Sold and Gross Profit and Percent Margin.



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- d. The company is having a sale where all goods are marked down by 10% of their selling price. If a tablet is on sale for \$200 how much did it cost?



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Q2 ArgCo Fine Handbags prices its product to give a 40% margin.

- a. What rate of markup do they use?
- b. If the company had profits of \$600,000 what was their cost of goods sold?
- c. If operating expenses are 5% of sales what is the net profit for part (b)?
- d. The most recent shipment of bags had a cost of \$270,000. What was the selling price?



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## CHAPTER ATTRIBUTION

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## Videos: Currency Conversions

exchange rates to answer these problems:

Currency		In CAD (1 unit of foreign currency buys__CAD)	
Australia (AUD)	0.8423	USA (USD)	1.1218
Russian Ruble (RUB)	0.1783	Euro (EUR)	1.4245
New Zealand (NZD)	0.7168	UK pounds (GBP)	1.6523

For example, 1 GBP = 1.6523 CAD

How much Canadian dollars will it take to buy:

(a) 500 Rubles?

(b) 500 Euros?



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- You have decided to visit the Thrifty Market in downtown Bellingham. You purchase a quart bottle of Black Cherry Pop for \$1.25 per bottle. Is this a better deal than Canadian Pop, which sells for \$5.10 per six pack? Solve the problem by changing the US price into a Canadian price per Canadian six pack.

4 quarts = 1 US gallon

6 cans = 1 six pack

1 can of pop = 355 ml

1 gallon = 3.7843 liters.

1 liter = 1000 ml



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3. You are planning a winter surfing Vacation. You will begin on the North Shore in Hawaii, try to catch the fifty year storm at Bell's Beach, Australia and finish your trip with a little longboarding on New Zealand's North Island. You estimate you will need \$500 in each currency.

(a) How much Canadian will you need to spend?

(b) When you land in Auckland NZ you discover that you still have \$65 USD and \$90 AUD in your wallet. How much NZD currency can you get?



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## Videos: Trade and Cash Discounts



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1. A company makes snowboards that retail at \$1250. The company offers chained discounts of 30% and 5% to its retailers.
  - a. Calculate the price a retailer would pay for the snowboards.
  - b. What single discount (single equivalent discount) is equivalent to these two discounts?
  - c. What is the percent markup of the retailers?
  - d. Due to a slowdown in sales the manufacturer would like to increase the total discount given to its retailers to 40%. What additional chained discount must they offer?



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2. On November 3, the Springfield power plant receives 10 fuel rods. The rods have a list price of \$20,000 each with discounts of 5% and 10% and payment terms of 2/10, 1/15 net 30.



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- a. What is the last day the invoice can be paid? How much must be paid?
- b. What is the last day the 2% discount may be taken? How much must be paid?
- c. If the supplier offers cash discounts on partial payments and \$100,000 is paid after 5 days and another \$50,000 15 days after the invoice date. How much will Springfield owe at the end of 30 days?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1564#oembed-4>

3. An invoice for \$10,000 has payment terms 3/10, 2/20, n/45. Discounts are allowed for partial payments. The company made a payment of \$3,000 9 days after the date of invoice, and a second payment 18 days after the date of the invoice that reduced the balance owing to \$2,000. What is size of the second payment?



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## CHAPTER ATTRIBUTION

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## Videos: Linear Functions and Costs

### WHAT IS A LINEAR EQUATION?



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1. You're concerned about your cell phone bill for this month and you can't remember how your plan actually works. You have been careful to keep track of how much data used each month and this month you have used 160 Mb. Using the payment information from your on-line banking you compile the following:

Month	Mb of Data used	Amount paid
June	120	\$40
July	80	\$30
August	200	\$60

- a. What is the independent variable and what is the dependent variable?
- b. Use a graph to determine the amount of your cell phone bill for this month.



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- c. Determine the slope of the line in the graph and describe the slope in the words of the problem.



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- d. If you don't use your cell phone at all for the month of October, how much will you have to pay?
- e. Determine the equation of the line in the graph. Verify your answers from (b) and (d)



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- f. Another company offers a cell phone plan that costs \$20 per month and then charges \$0.15 per Mb. Determine the cost equation and graph it using the same grid.
- g. If you use 175 Mb, which plan is cheaper and by how much?
- h. Use the equations to determine the number of Mb that result in the same cost for both plans.



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2. Baby Sitters R Us charges a flat booking fee plus an hourly rate. On a normal workday, I hire a Sitter for 9 hours, and I pay \$123.50. On Saturday night, I usually hire a sitter for 4 hours while I go to the movies, and I pay \$66.

- a. Find a linear equation giving cost in terms of the number of hours.
- b. Describe the slope in the words of the problem.
- c. If I hire a sitter for 6 hours, how much will it cost me?



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## CHAPTER ATTRIBUTION

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## Videos: System of Equations

### SOLVING A SYSTEM

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$P = \$20 - \$2Q$  (demand function)

$P = \$2 + \$4Q$  (supply function)

Where  $P$  = price and  $Q$  = quantity.



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### CHAPTER ATTRIBUTION

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## Videos: Breakeven Analysis

Question 1: The Hungry Days Ice Cream Company has fixed costs of \$30,000 per month. Each case of ice cream costs \$4.50 to make.

- (a) Write an equation giving Total Cost as a function of the number of cases of ice cream made. Graph the function in the space below.
- (b) The company sells ice cream for \$12 per case. Give an equation for the Revenue function and graph the Revenue function below.
- (c) Label your graph. Show Profit, Loss and Breakeven.



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(d) Calculate the Breakeven point. Express in cases of ice cream, sales dollars and as a percent of capacity if the maximum monthly capacity is 16,000 cases. NOTE: THERE IS A TYPO IN THIS PART THAT I NEED TO FIX IN THE VIDEO:

$$\$12x = \$30x + \$4.50x$$



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(e) Calculate the profit or loss if 4,001 cases are sold



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(f) Calculate the profit or loss if 3,250 cases are sold.

(g) The company wants to make a profit of \$75,000 this month. How many cases must they sell?



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(h) Hungry Days is considering dropping its prices and would like to make a profit of \$82,000 by selling 16,000 cases. What should the new price be? What is the breakeven in units?



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2. Rain City Umbrellas does a thriving business in selling umbrellas. Each month they spend \$1,057 on rent for their prime umbrella kiosk, and \$2,000 on a salary for their salesperson. They buy umbrellas wholesale for \$7.50 per umbrella. They then sell each umbrella for \$26 to wet tourists. Find their breakeven point in sales.



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## USING THE BAI PLUS FOR BREAKEVEN ANALYSIS

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## Videos: Simple Interest

### WHAT IS SIMPLE INTEREST?

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1. You deposit \$1000 into a bank account that pays 10% simple. How much interest would you earn if:
  - (a) You left the money in for 2 years?
  - (b) You left the money in for 6 months?
  - (c) You left the money in for 9 months?
  - (d) Your deposit was March 3 and you withdrew the money November 7 the same year?
  - (e) Your deposit was on April 6 and you withdrew the money on Dec. 23 the same year?
  - (f) You made a deposit November 12, 2012, and withdrew the money March 15, 2013?
  - (g) You made a deposit October 28, 2012 and withdrew the money September 21, 2013?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1572#oembed-2>

2. On December 1, your one year term deposit comes due and you collect the principal and \$120 in interest. The term deposit paid 6% simple interest. How much money did you deposit?
3. 18 months ago you purchased a bond that paid 7.5% simple interest. You cashed in the bond and received the principal plus \$337.50 in interest. Find the principal.



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4. On May 31 you decided it was time to start a new life in the palm trees. You emptied out your bank account. You received your principal of \$1750 and interest of \$437.50. How long had your money been in the bank if the account paid interest at 5% simple?
5. On your 65th birthday you eagerly cash in one of your RRSP's. You receive a total of \$4295. Your original investment was \$3000 and the RRSP paid interest at 7% simple. How many months was your money in the RRSP?
6. How long will it take you to double your money at 8% simple?



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7. On March 6th you deposit \$3000 into a bank account. You withdraw the principal and interest totaling \$3123.29 on December 31. What interest rate was paid?
8. On December 16, 2011 you withdraw your entire Christmas fund of \$2098.63 from your

account. You deposited \$2000 on February 19 of 2011 (not a leap year). What simple interest rate was paid?



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9. You have decided to start a new car fund. You deposit \$6000 into an account paying 10% simple interest. How much money will you have in 50 months?
10. On December 31 you repaid a loan to the Hammerhead Finance and Loan Company. You repaid \$6000 which was principal and interest on a five year loan at 10% simple. How much money did you borrow?



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## CHAPTER ATTRIBUTION

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## Videos: Payment Plans and Making Choices

1. For your first job after graduation you have decided to accept a contract position. The manager has offered to pay you \$30,000 at the end of the first year and \$35,000 at the end of the second year or to pay you \$22,000 now, \$20,000 at the end of the first year and \$20,000 at the end of the second year. Which option is better and by how much, if interest is 10% simple? Use today as your focal date.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1574#oembed-1>

2. You have two debts coming due. A debt of \$1000 in 6 months and another of \$1500 at the end of the year. Interest is charged at 12% simple. You speak to Mr. Hammerhead, the loans manager, and discuss several different payment options.

How much would it cost to payoff both the debts today? (focal date = today)

You would like to defer your repayment and repay the whole debt in 18 months. How much would you have to pay? (focal date = 18 months)



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## Videos: Equations of Value

### WHAT IS AN EQUATION OF VALUE?

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Today, you take out a \$5000 loan at 10% simple interest, which is to be repaid with two equal payments at the end of the first year and at the end of the second year. Find the size of the payments (using today as your focal date).



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You must repay a debt of \$6000 dollars at 6% simple borrowed today. The finance company has agreed to allow you to make three yearly payments with the first payment in one year. The first two payments will be equal and the third payment will be \$500 more than the first two payments. Find the size of the payments using 3 years from now as your focal date.



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# Videos: Compound Interest

## WHAT IS COMPOUND INTEREST?



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## NO REALLY, WHAT IS COMPOUND INTEREST?



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1. Invest \$100 for one year with a nominal rate of 24%. Fill in the table if you compound...

Compound...	<i>m</i>	Nominal ( <i>j<sub>m</sub></i> )	Periodic ( <i>i</i> )	Periods ( <i>n</i> ) in a year	FV of \$100 in 1 year.
Annually					
Semi-annually					
Quarterly					
Monthly					



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2. You acquire \$5,000 and deposit the money in a mutual fund that has an average return of 9% compounded annually. How much will you have when you retire in 25 years?



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3. You deposit \$10,000 at  $j_4 = 12\%$  for 54 months. How much money will you have?



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4. You would like to save \$20,000 so you can take a year off in 42 months. How much should you deposit today if money earns 4% compounded semi-annually?



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5. How much money do you need to invest to earn \$1000 interest in 18 months if the interest rate is 8% compounded quarterly?



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6. Macleans Magazine estimates that it will cost \$130,000 for a four year university education when children born this year reach university in 18 years.
- (a) You have \$10,000 to invest today. What annual rate is required to reach \$130,000?
- (b) How much do you need to invest if you can only expect a rate of return of  $j_2=12\%$ ?
- (c) You deposit \$12,000 and invest in bonds that pay 6% compounded semi-annually. How old will your child be when they can start university?



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## *Videos: Payment Plans and Making Choices, Compound*

3. Sagheer received two offers for a property that he is selling. Offer 1 consists of \$700,000 immediately and offer 2 consists of \$500,000 now and \$110,000 per year at the end of the next 2 years. If money earns 6% compounded semi-annually, which offer has a better economic advantage and by how much, in today's dollars?



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4. Anna borrowed \$1,000 from a local loan shark on January 1, 2010, at  $j_{12}=24\%$ . On January 1 2012, Anna paid the loan shark \$500, and the loan shark agreed to lower the rate to  $j_{12}=18\%$ .

(a) How much money will Anna owe as of January 1, 2014?

(b) If Anna pays the balance, how much interest will she have paid?



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## Videos: Equations of Value and Compound Interest

### WHAT IS AN EQUATION OF VALUE?

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A recap of what we did with simple Interest:



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Today, you take out a \$5000 loan at 10% compounded quarterly, which is to be repaid with two equal payments at the end of the first year and at the end of the second year.

- a. Find the size of the payments.
- b. How much interest will be paid?



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You borrowed \$3,000 from your line of credit 8 months ago and a further \$5,000 three months ago. You arrange with the bank to pay off the line of credit with 2 payments. The first payment, 6 months from today, will be twice as large as the second payment made one year from today. The bank charges you 9% interest, compounded monthly. Suggested focal Date: **6 months from today**.

- (a) Find the size of each payment.
- (b) How much interest did you pay?



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## CHAPTER ATTRIBUTION

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## Videos: Average and Effective Rates

*Equivalent: Invest \$100 in 2 accounts, the following year they have the same balance*

*Effective = Average Annual =  $j_1$*

The humongous bank believes in offering its customers choices. A banker offers you two different options for your investments. The premium growth account offers interest at  $j_2 = 10\%$ . The accelerator account offers an effective rate of 10.25%. Which account is better?



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### USING THE BAI PLUS TO CONVERT INTEREST RATES:



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Complete the following table, with each row being equivalent rates:

Effective	Semi-annual	Quarterly	Monthly
$j_1 = 12\%$			
	$j_2 = 8\%$		
		$j_4 = 10\%$	
			$j_{12} = 6\%$



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1584#oembed-3>

3. Mortgages charge interest on a semi-annual basis but payments are usually made monthly. Change a mortgage rate of  $j_2 = 7\%$  to a rate compounded monthly.

4. Your credit card charges interest at 1.5% per month. Find the effective rate.



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5. Premium Savings bonds have the following rates (all annual).

Year 1: 2% Year 2: 5% Year 3: 12% Year 4: 15% Year 5: 24%

(a) How much would a \$1,000.00 investment be worth at the end of the fifth year?

(b) What effective rate was earned?





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6. Your investments earn a 100% return in the first year and lose 50% in the second year. What is your effective rate of return over the two years?



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7. A mutual fund had returns of 12% compounded monthly in the first year; 18% compounded annually in the second year; and 4% compounded quarterly in the third year.

What would a \$1,000 investment 3 years ago be worth today?

What was the average rate (compounded monthly) of return earned?

The fund lost \$100 in the fourth year – what was the average effective rate of return for the four years?



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## Videos: Introduction to Annuities

### WHAT IS AN ANNUITY?

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1. You decide to begin a savings plan. Starting in one year you will deposit \$100 each year into an account that pays  $j_1 = 12\%$ . How much will you have in 3 years? How much of that is interest?



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2. Financial planners estimate you will require an income of \$2,500 per month to retire. You dial 1-800-psyhic and discover you will live another 15 years. How much money will you need to retire if your first withdrawal will be in one month? (interest is 9% compounded monthly)



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1586#oembed-3>

3. You have just purchased a new house for \$250,000. You can afford to pay \$2,600 per month on your mortgage. How long will it take you to pay off your house if interest is calculated at 12% compounded monthly?



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4. Your new car loan has payments of \$499 at the end of each month for 5 years to pay for a \$25,000 Mercury Stable. What rate of interest is being charged? What is the effective rate? What is the cost of financing?



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## CHAPTER ATTRIBUTION

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## Videos: 5 fingers of finance

### *5 Fingers of Finance: PV and FV both not 0!*

1. You decide to lease a new BMW automobile. The vehicle costs \$30,000 new and you are required to pay all taxes at the beginning of the lease. The lease payments are \$600 per month for three years and you know the lease company charges interest at 6% compounded monthly.

- a) What is the residual value of the car after three years?
- b) How much interest do you pay over the life of the lease?



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2. You are saving for your son's postsecondary education in 10 years. You have \$27,000 in an RESP account and plan to add \$1,000 at the end of the year for the next ten years. How much money will your son have when he goes to school in 10 years if you receive a return of 4.8% compounded annually on the investments?



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## CHAPTER ATTRIBUTION

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## Videos: General Annuities

3. You are shopping for a \$100,000 mortgage and the bank has quoted you a rate of  $j_2=6\%$  .If you will make **monthly** payments for 20 years how much will your payments be?



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4. You want to have accumulated \$4,000 for your European trip four years from now. If interest is 6.4% compounded monthly, find the size of the **quarterly** deposits required for 4 years if deposits are made at the beginning of each quarter.



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### CHAPTER ATTRIBUTION

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## Videos: Annuities Due

### ANNUITIES DUE (START AT THE BEGINNING, BGN ON)

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5. You deposit \$1000 at the end of each quarter into an account that pays  $j_4 = 10\%$ .
- Find the future value in one year.
  - You make the deposits at the beginning of the quarter. How much money will you have at the end of the year?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1592#oembed-1>

6. You begin a savings plan. At the beginning of each month you deposit \$250 into an account that pays  $j_{12} = 9\%$ . How much money will you have after 5 years?



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## SUMMARY OF TYPES OF ANNUITIES

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## Videos: Back-to-Back Annuities

### DEFERRED ANNUITIES

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1. You plan to retire in 21 years. You will need 25 annual payments of \$30,000 with the first payment to be received in 21 years. Interest is calculated at 10% effective.

(a) How much money should you set aside today for your retirement?



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### BACK-TO-BACK ANNUITIES

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(b) Instead of making one large payment you would like to make 21 annual payments starting today. Find the size of these payments.



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(c) How much of your retirement annuity in part (b) is interest?

(d) Re-answer b, assuming you make the payments starting in one year.

(e) How much of your retirement annuity in part (d) is interest?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1594#oembed-3>

2. You are saving money for your retirement. Starting in six months and for five years you invest \$1,500 every 6 months in a mutual fund that pays 6% compounded semi-annually. After five years, the rate drops to 4.8% compounded quarterly and you decide to deposit \$1,000 every quarter for another five years. How much money will you have at the end of ten years?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1594#oembed-4>

3. You would like to save enough to take a 2 year sabbatical in 15 years. Starting today, you make your first of 180 monthly deposits of \$500 into an account that pays  $j_{12}=9\%$ .

- (a) One month after making your last deposit you take your first of 24 monthly payments. Find the size of these payments.
- (b) Two months after making your last deposit you take your first of 24 monthly payments. Find the size of these payments.



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1594#oembed-5>

## SUMMARY OF BACK TO BACK ANNUITIES

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## CHAPTER ATTRIBUTION

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## Videos: Bonds

### BONDS

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You purchase a bond with a face value of \$1,000 and a coupon rate of 6% compounded semi-annually. The bond has a maturity of 10 years.

- a. What is the semi-annual coupon payment?
- b. How much is the bond worth if you sell it in 3 years and the interest rate for similar bonds is 8% compounded semi-annually?
- c. How much is the bond worth if you sell it in 4 years and the interest rate for similar bonds is 5% compounded semi-annually?



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### CHAPTER ATTRIBUTION

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## Videos: Perpetuities and Stocks

### *Perpetuities*

A wealthy BCIT alumnus has decided to make an endowment to the School of Business to fund an annual scholarship for the best business math student. She will donate \$15,000 and the funds will be invested at 6.6% effective.

- (a) How much will the annual scholarship be if the first scholarship is in one year?
- (b) If they decide to give out the first scholarship today, how much will the alumnus have to donate to keep the same scholarship from part (a).
- (c) If she decides to keep her endowment at \$15,000, how much will the annual scholarship be if the first scholarship is today?



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1598#oembed-1>

The Winitall lottery offers you two choices for its grand prize: either a cash prize of \$1,500,000; or \$7,000 per month forever with the first payment today. Which choice should you select if interest is 6% monthly?





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## STOCKS

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You purchase some preferred shares that have a semi-annual dividend of \$1.25. How much should you pay for the shares if the interest rate is 5% compounded semi-annually and the first dividend is in 6 months?

You purchase the shares above for \$50 and the interest rate rises to  $j_2=8\%$ .

- How much would you make or lose? (assume the first dividend is still in 6 months)
- How much money would you make or lose if the interest rate was 6% effective? (assume the first dividend is still in six months)



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## CHAPTER ATTRIBUTION

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## Videos: Mortgages and Amortization, part 1

### INTRO TO MORTGAGES

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You have just purchased a new house and taken a mortgage for \$100,000. The interest rate is 12% compounded monthly and you will make payments for 25 years.

- Find the size of the monthly payment.
- The bank has a policy of rounding the payments up to the next cent. Find the new monthly payment and compute a new  $n$ .
- What was the balance of the loan after three periods?



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- How much of your third payment was Principal? Interest?
- How much did you pay in the first year?
- How much of that was principal?

- g. How much of that was interest?
- h. What was the balance of the loan at the end of the first year?
- i. How much interest did you pay in the third year?



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- j. Assuming the mortgage stays at the same rate until maturity, find the size of your last payment.



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- k. Fill out the Amortization Table for the first year

## MORE VOCABULARY



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## CHAPTER ATTRIBUTION

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## Videos: Mortgages and Amortization, part 2

Banks will apply the 30% rule: no more than 30% of your gross income maybe used for your mortgage payment. Your gross income is \$3,333.33. You are looking for a 5-year term with a 25-year amortization. If banks are charging  $j_2=6\%$ :

- Find the maximum size of the payment and the maximum mortgage.
- How much would you owe at the end of the five-year term?



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- After five years the interest rate increases to  $j_2=8\%$ . Find the size of the new payment. (assume the same amortization period)
- Five years later the interest rate stays the same, but you can afford to pay an extra \$12,000 on your mortgage. Assuming you will amortize the balance over the remaining 15 years, what is your payment?



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- Instead of reducing your payment you would like to keep the payment the same as in part c

and reduce the number payments you have left to make. How much time will you save?



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Your gross income is \$5,000 per month. The rate is  $j_2 = 8\%$  and you are looking for a 3 year term and a 20 year amortization.

- a. If the bank uses the 30% rule, find the size of your maximum mortgage.
- b. After 3 years you renew your mortgage at the same rate but are able to make a \$10,000 payment.
  - i. Find the size of the new monthly payment. Assume you will amortize the remaining balance over 17 years.
  - ii. Instead of reducing your payment you would like to pay your mortgage off faster and keep your payment the same. How much time will you save, assuming the interest rate stays the same?



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You take a 25-year, \$100,000 mortgage with a rate of  $j_2 = 7\%$ .

- a. Find your monthly payment – round up to the next dollar.
- b. Find the size of the last payment if the interest rate stays constant.
- c. How long would it take you to pay off your mortgage using biweekly payments of half the monthly payments, instead of monthly payments?
- d. Find the size of the final payment using biweekly payments.
- e. How much money would you save by making biweekly payments?



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## CHAPTER ATTRIBUTION

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## Videos: Investment Decisions

1. You are considering investing in a business. The business will cost \$60,000 and require another \$10,000 in immediate renovations. The business is expected to return \$20,000 per year for 4 years. The company requires a 10% rate of return.

(a) Calculate the payback for the business



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(b) Calculate the NPV for the Business

(c) Calculate the Internal rate of return



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## USING THE BAI PLUS FOR CASH FLOWS



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2. You are considering investing in a business. The business is expected to cost \$60,000 and require another \$10,000 in immediate renovations. The business is expected to return \$20,000 per year for 4 years and then \$40,000 after the fifth year. The company requires a 10% rate of return.
- (a) Calculate the payback for the business
  - (b) Calculate the NPV for the business
  - (c) Calculate the Internal rate of return



One or more interactive elements has been excluded from this version of the text. You can view them online here: <https://pressbooks.nsc.ca/businessmath2500/?p=1604#oembed-4>

3. A company is considering launching a new product. The initial start-up costs will be \$100,000 and the product will provide returns of \$40,000 in year 1, \$40,000 in year 2 and \$31,757.60 in the third year.
- (a) Calculate the NPV using a MARR of 5%
  - (b) Calculate the NPV using a MARR of 7%
  - (c) Calculate the NPV using a MARR of 6%
  - (d) Calculate the IRR of the project.



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4. You are considering an investment in a gardening business. You estimate your start-up costs to be \$160,301.96. You expect rent and electricity charges to be \$15,000 per year payable at the beginning of the year for 7 years. You expect revenue to be \$50,000 per year for 7 years at which time you will sell your equipment for \$15,000. Your MARR is 12%.



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(a) Calculate the NPV for the business – should you invest?

(b) What is the IRR?



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(c) By switching to solar panels you can save some money on the start-up. How much per year would you need to save to make this a worthwhile investment?





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(d) The makers of your equipment are offering a rebate at the end of the first year. How large would that rebate need to be in order for you to accept this investment?

(e) The salesperson says that you have underestimated the equipment's salvage value. What salvage value would make this an acceptable investment?



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## SUMMARY

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## CHAPTER ATTRIBUTION

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# Glossary

**accrued interest**

Any interest amount that has been calculated but not yet placed (charged or earned) into an account.

**amortization**

A process by which the principal of a loan is extinguished over the course of an agreed-upon time period through a series of regular payments that go toward both the accruing interest and principal reduction. **amortization period** The length of time it will take for the principal of a loan to be reduced to zero.

**amortization schedule**

A table that shows the payment amount, principal component, interest component, and remaining balance for every payment in the annuity. **amortization term** The length of time for which the interest rate and payment agreement between the borrower and the lender will remain unchanged.

**annuity**

A continuous stream of equal periodic payments from one party to another for a specified period of time to fulfill a financial obligation.

**annuity due**

Annuity payments that are each made at the beginning of a payment interval.

**annuity payment**

The dollar amount of the equal periodic payment in an annuity environment.

**average rate of change**

The total change of the function values (output values) divided by the change in the input values

**break even point**

The level of production where revenue equals cost; the minimum level of sales necessary for a company to make a profit

**break-even analysis**

a financial calculation that weighs the costs of a new business, service or product against the unit sell price to determine the point at which you will break even

**closed mortgage**

A type of mortgage that has many strict rules and does not allow the mortgager to pay off the debt in full until the loan matures. Early payment incurs substantial penalties.

**commercial paper**

A short-term financial instrument with maturity no longer than one year that is issued by large corporations.

**compound interest**

A system for calculating interest that primarily applies to long-term financial transactions with a time frame of one year or more; interest is periodically converted to principal throughout a transaction, with the result that the interest itself also accumulates interest.

**Compound interest GIC**

A GIC that uses compound interest rates for which interest is periodically calculated and converted to the principal of the GIC for further compounding.

**Compound interest savings bonds**

Called C-bonds, these bonds annually convert the interest on the savings bond to principal.

**Compounding period**

The amount of time that elapses between the dates of successive conversions of interest to principal.

**consistent system**

A system of equations that has at least one solution

**correlation coefficient**

A value,  $r$ , between -1 and 1;  $r > 0$  suggests a positive (increasing) relationship;  $r < 0$  suggests a negative (decreasing) relationship

**cost**

An outlay of money required to produce, acquire, or maintain a product, which includes both physical goods and services.

**coupon**

A promotion that entitles a consumer to receive certain benefits, usually in the form of a reduction of the selling price for a product.

**coupon handling expense**

A handling charge that is paid to channel members for redeeming a coupon.

**coupon marketing expense**

The expense associated with creating, distributing, and redeeming a coupon.

**coupon redemption expense**

The face value price reduction offered by a coupon.

**current balance**

The balance in an account plus any accrued interest.

**Deferred annuity**

A financial transaction where annuity payments are delayed until a certain period of time has elapsed.

**dependent system**

A consistent system in which the equations represent the same line; has infinite solutions

**discount**

A reduction in the price of a product.

**discount rate**

An interest rate used to remove interest from a future value.

**domain**

The set of possible input values to a function

**effective interest rate**

The true annually compounded interest rate that is equivalent to an interest rate compounded at some other (non-annual) frequency.

**equivalent interest rates**

Interest rates with different compounding that produce the same effective rate and therefore are equal to each other.

**equivalent payment streams**

Equating two or more alternative financial streams such that neither party receives financial gain or harm by choosing either stream.

**equivalent payments**

Two payments that have the same value on the same day factoring in a fair interest rate.

**Escalator interest GIC**

A GIC that uses compound interest rates that usually remain constant during each of a series of time intervals, always rising stepwise throughout the term of the investment with any accrued interest being converted to principal.

**expenses**

A business's financial outlays incurred in the selling of a product.

**extrapolation**

When we predict a value outside the domain and range of the data

**face value of a T-bill**

The maturity value of a T-bill, which is payable at the end of the term. It includes both the principal and interest together.

**feasible region**

The set of possible solutions to a system as shown on a graph

**fixed costs**

Startup costs, such as those for equipment and buildings

**fixed interest rate**

An interest rate that is unchanged for the duration of the transaction.

**focal date**

A point in time to which all monies involved in all payment streams will be moved using time value of money calculations.

**function**

A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say “the output is a function of the input.”

**function notation**

The notation  $\text{output} = f(\text{input})$  defines a function named  $f$ . This would be read “output is  $f$  of input”

**fundamental concept of equivalency**

Two or more payment streams are equal to each other if they have the same economic value on the same focal date.

**future value**

The amount of principal with interest at a future point of time for a financial transaction. If this future point is the same as the end date of the financial transaction, it is also called the maturity value.

**general annuities**

An annuity in which the payment interval does not equal the compounding interval ( $P/Y$  does not equal  $C/Y$ ).

**general annuity due**

An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

**guaranteed investment certificate (GIC)**

An investment that offers a guaranteed rate of interest over a fixed period of time.

**horizontal intercept**

A point at which a line intersects the horizontal axis of a graph. If it exists in a function, it can always be found by solving  $f(x) = 0$ .

**inconsistent system**

A system of equations in which the equations represent two parallel lines; has no solution

**independent system**

A consistent system in which the equations represent lines that intersect only once; has exactly one solution

**interest amount**

The dollar amount of interest that is paid or earned.

**Interest payout GIC**

A GIC where the interest is periodically paid out to the investor, but it is never added to the principal of the GIC. Because the interest does not actually compound, in essence the concepts of simple interest are used.

**interest rate**

The rate of interest that is charged or earned during a specified time period.

**interpolation**

When we predict a value inside the domain and range of the data

**linear function**

A function whose graph produces a line. Linear functions can always be written in the form  $f(x) = b + mx$  or  $f(x) = mx + b$ , where  $b$  is the initial/starting value of the function (when  $x = 0$ ), and  $m$  is the constant rate of change of the function.

**list price**

A price for a product that has been published or advertised in some way.

**literal coefficient**

a variable used to represent a number.

The number the variable represents can be either known or unknown. It can be our usual  $x$  or  $y$ , or it can be other letters, such as  $a$ ,  $b$ , or  $c$ .

**local extrema**

The local maxima and minima of a function (also called 'local extreme values')

**local maximum**

A point where a function changes from increasing to decreasing.

**local minimum**

A point where a function changes from decreasing to increasing.

**loyalty discount**

A discount given from a seller to a purchaser for repeat business.

**mail-in rebate**

A refund that occurs after a product has been purchased.

**maintained markup**

The average level of markup that is maintained across all units sold at various price levels including the selling price and the sale price(s).

**manufacturer's suggested retail price (MRSP)**

A recommended product retail price that a manufacturer sets for a retailer based on market research.

**marginal cost**

The change in total cost if the quantity of items produced increases by one.

**marginal profit**

The change in total profit if the quantity of items produced increases by one.

**marginal revenue**

The change in total revenue if the quantity of items produced increases by one.

**markdown**

A reduction from the regular selling price of a product resulting in a new lower sale price.

**marketing price adjustment**

Any marketing activity executed by a member of the distribution channel for the purposes of altering a product's price.

**markup**

The process of taking a product's cost and increasing it by a certain amount to arrive at a selling price.

**markup amount**

The dollar amount of the expenses and profit combined together into a single number; it represents the difference between the price and cost in dollars.

**markup on cost percentage**

The markup dollars expressed as a rate using cost as the base.

**markup on selling price percentage**

The markup dollars expressed as a rate using the regular selling price as the base.

**maturity date**

The date upon which a transaction, such as a promissory note, comes to an end and needs to be repaid.

**maturity value**

The amount of money at the end of a transaction, which includes both the interest and the principal together.

**mortgagee**

The financial institution that lends the money for a mortgage.

**mortgagor**

The individual or business that borrows the money for a mortgage.

**motgage**

A special type of loan that is collaterally secured by real estate property.

**net price**

The price of the product after a discount is removed from the list price.

**nominal interest rate**

A nominal number for the annual interest rate, which is commonly followed by words that state the compounding frequency.

**objective function**

A function whose value we either want to be as large as possible (want to maximize it) or as small as possible (want to minimize it)

**open mortgage**

A type of mortgage that has very few rules and allows the mortgagor to pay off the debt in full or with additional prepayments at any given point without penalty.

**ordinary general annuities**

An annuity in which the payment interval does not equal the compounding interval, and payments are made at the end of the term.

**ordinary general annuity**

An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are unequal. The first payment occurs one interval after the beginning of the annuity, while the last payment is on the same date as the end of the annuity.

**ordinary simple annuities**

An annuity in which the payment interval equals the compounding interval, and payments are made at the term.

**ordinary simple annuity**

An annuity where payments are made at the end of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs one interval after the beginning of the annuity while the last payment is on the same date as the end of the annuity.

**parallel lines**

Two lines with the same rates of change (same slopes); will never cross

**partial amortization schedule**

Amortization schedules that show only a specified range of payments and not the entire annuity.

**payment frequency**

The number of annuity payments in a complete year.

**payment interval**

The amount of time between each continuous and equal annuity payment.

**period of deferral**

The time segment of a deferred annuity where the single payment earns interest and no contributions are made to the investment.

**periodic interest rate**

The percentage of interest earned or charged at the end of each compounding period.

**perpendicular lines**

Two lines wherein the slope of one line is the negative reciprocal of the other line's slope; form a 90 degree angle at the point where they meet

**piecewise function**

A function in which the formula used depends upon the domain the input lies in.

**present value**

The amount of money at the beginning of a time period in a transaction. If this is in fact the amount at the start of the financial transaction, it is also called the principal. Or it can simply be the amount at some time earlier before the future value was known. In any case, the amount excludes the interest.

**present value of an annuity**

The sum of all the present values for all of the annuity payments when they are moved to the beginning of the first payment interval.

**present value principal for loans**

The present value of all payments on a loan is equal to the principal that was borrowed.

**principal**

The original amount of money that is borrowed or invested in a financial transaction.

**profit**

The amount of money that remains after a business pays all of its costs and expenses.

**promissory note**

A promissory note is a written promise by one party to pay an amount of money to another party on a specific date, or on demand.

**quantity discount**

A discount for purchasing larger quantities of a certain product.

**range**

The set of possible output values of a function

**rate of change**

How the output quantity changes in relation to the input quantity. The units on a rate of change are “output units per input units”

**rebate marketing expense**

The expense associated with creating, distributing, and redeeming a rebate.

**rebate redemption expense**

The face value amount that a consumer will receive as a refund if a submitted rebate fulfills all rebate conditions.

**repayment schedule**

A table that details the financial transactions in an account, including the balance, interest amounts, and payments.

**revenue**

The amount of money a company brings in from sales

**sale discount**

A temporary discount lowering the price from a product's regular selling price.

**sale price**

A price for a product after a markdown that is lower than its regular selling price.

**savings account**

A deposit account that bears interest and has no stated maturity date.

**seasonal discount**

A discount offered to consumers and businesses for purchasing products out of season.

**simple annuities**

An annuity in which the payment interval equals the compounding interval ( $P/Y$  equals to  $C/Y$ ).

**simple annuity**

An annuity in which the payment frequency and compounding frequency are equal.

**simple annuity due**

An annuity where payments are made at the beginning of the payment intervals and the payment and compounding frequencies are equal. The first payment occurs on the same date as the beginning of the annuity, while the end of the annuity is one payment interval after the last payment.

**simple interest**

A system for calculating interest that primarily applies to short-term financial transactions with a time frame of less than one year.

**single equivalent discount**

A single discount rate that is equal to a series of multiple rate discounts.

**slope**

The constant rate of change of a function (the  $m$  in  $f(x) = mx + b$ )

**Strip bond**

A marketable bond that has been stripped of all interest payments.

**system of linear equations**

A set of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously; the solution is a set of numerical values for each variable in the system that will satisfy all the equations in the system at the same time

**time period**

The length of the financial transaction for which interest is charged or earned. It may also be called the term.

**trade discount**

A discount offered to businesses only based on the type of business and its location in the distribution system.

**treasury bills**

Short-term financial instruments with maturities no longer than one year that are issued by both federal and provincial governments.

**variable costs**

Costs that depend on the number of items produced, such as materials and labour

**variable interest rate**

An interest rate that is open to fluctuations over the duration of a transaction.

**vertical intercept**

A point at which a line intersects the vertical axis of a graph (the  $b$  in  $f(x) = mx + b$ )

**vertical line test**

If any vertical line would cross a graph more than once, then the graph does not define only one vertical output for each horizontal input.

**yield**

The percentage increase between the sale price and redemption price on an investment such as a T-bill or commercial paper.

## *Rounding Rules*

How and when to round always is a source of confusion. This Appendix provides you with some rules for rounding your answers. We start with some general guidelines that apply globally throughout the entire textbook, then comment at the rules that apply to some specific sections of text.

### GLOBAL ROUNDING RULES

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1. All interim solutions never get rounded unless there is a logical reason or business process forcing it to be rounded.
2. When writing nonterminating decimals in this textbook, up to six decimals are written. The horizontal line format is used for repeating decimals. If the number is not a final solution, then it is assumed that all decimals or as many as possible are being carried forward.
3. All final numbers are rounded to four decimals in decimal format and two decimals in percent format unless instructions indicate otherwise.
4. Final solutions are rounded according to common business practices, practical limitations, or specific instructions.
5. Zeroes not required at the end of decimals are generally not written unless required to meet a rounding standard or to visually line up a sequence of numbers.

### TOPIC SPECIFIC ROUNDING RULES

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Some specific topics have their own rounding standards. Until you learn about these topics, it does not make any sense to put those standards here. However, the standards are introduced with the relevant topic.

## Version History

Business Mathematics was created by combining chapters from NSCC Math for Bookkeeping and Business Math: A Step-by-Step Handbook Abridged by Sanja Krajisnik, Carol Leppinen, and Jelena Loncar-Vines, Conestoga College.

Both books are Pressbooks hosted condensed adapted versions of *Business Math: A Step-by-Step Handbook* by J. Olivier. All works shared under CC BY-NC-SA licenses.

Additional content added from Financial Empowerment by Bettina Schneider and Saylor Academy CC BY-NC-SA 4.0 and *Mathematics for the Liberal Arts* by Lumen Learning shared under a CC BY license.

NSCC Business Math	Adapted From
<b>1. Basic Math Review</b>	
1.1 Order of Operations	2.1 in Business Math: A Step-by-Step Handbook
1.2 Fractions and Decimals	2.2 in Business Math: A Step-by-Step Handbook
1.3 Rules & Properties of Exponents	Rules for Exponents in <i>Mathematics for the Liberal Arts</i>
1.4 Solving Linear Equations	2.5 in Business Math: A Step-by-Step Handbook
1.5 Solving Word Problems	Apply a Problem-Solving Strategy to Basic Word Problems in <i>Mathematics for the Liberal Arts</i>
1.6 Break-Even Analysis	5.2 in Business Math: A Step-by-Step Handbook
1.7 Formulas	new
<b>2 Mathematics of Merchandising</b>	
2.1 Figuring Out the Cost: Discounts	6.1 in Business Math: A Step-by-Step Handbook
2.2 Markup: Setting the Regular Price	6.2 in Business Math: A Step-by-Step Handbook
2.3 Markdown: Setting the Sale Price	6.3 in Business Math: A Step-by-Step Handbook
2.5 Invoicing: Terms of Payment and Cash Discounts -NSCC	7.4 in Business Math: A Step-by-Step Handbook
<b>3 Evaluating Choices: Time, Risk, and Value</b>	
3.1 The Time Value of Money	Unit 4.1 from Chapter 4 Evaluating Choices: Time, Risk, and Value in Financial Empowerment
3.2 Valuing a Series of Cash Flows	Unit 4.3 from Chapter 4 Evaluating Choices: Time, Risk, and Value in Financial Empowerment
3.3 Calculating the Relationship of Time and Value	Unit 4.2 from Chapter 4 Evaluating Choices: Time, Risk, and Value in Financial Empowerment
<b>4 Simple Interest</b>	
4.1 Principal, Rate, Time	8.1 in Business Math: A Step-by-Step Handbook
4.2 Maturity Value (or Future Value) Moving Money Involving Simple Interest	8.2 in Business Math: A Step-by-Step Handbook
4.3 Application of Future Value: Savings Accounts And Short-Term GICs	8.3 in Business Math: A Step-by-Step Handbook
4.4 Application of Present Value: Treasury Bills and Commercial Paper	8.6 in Business Math: A Step-by-Step Handbook
4.5 Simple Interest Terminology (Interactive Activity)	Chapter 8: Simple Interest Terminology in Business Math: A Step-by-Step Handbook Abridged
4.6 Key Concepts Summary	
4.7 Symbols and Formulas Introduced	Chapter 8: Symbols and Formulas in Business Math: A Step-by-Step Handbook Abridged
4.8 Glossary of Terms	new
<b>5 Compound Interest</b>	
5.1 Compound Interest Fundamentals	9.1 in Business Math: A Step-by-Step Handbook
5.2 Determining the Future (Maturity) Value	9.2 in Business Math: A Step-by-Step Handbook
5.3 Determining the Present Value	9.3 in Business Math: A Step-by-Step Handbook
5.4 Equivalent Payments	9.4 in Business Math: A Step-by-Step Handbook
5.5 Determining the Interest Rate	9.5 in Business Math: A Step-by-Step Handbook
5.6 Compound Interest Terminology (Interactive Activity)	9 in Business Math: A Step-by-Step Handbook Abridged

5.7 Key Concepts Summary	9.7 in Business Math: A Step-by-Step Handbook
5.8 Symbols and Formulas	Chapter 9: Symbols and Formulas in Business Math: A Step-by-Step Handbook Abridged
5.9 Glossary of Terms	New
<b>6 Compound Interest – Applications &amp; Single Payments</b>	
6.1 Application: Long Term GICs	10.1 in Business Math: A Step-by-Step Handbook
6.2 Application: Long Term Promissory Notes	10.2 in Business Math: A Step-by-Step Handbook
6.3 Application: Strip Bonds	10.3
6.4 Key Concepts Summary	10-.5
6.5 Symbols and Formulas	Chapter 10: Symbols and Formulas in Business Math: A Step-by-Step Handbook Abridged
6.6 Glossary of Terms	new
<b>7 Compound Interest – Annuities</b>	
7.1 Fundamentals of Annuities	11.1 in Business Math: A Step-by-Step Handbook
7.2 Future Value of Annuities	11.2
7.3 Present Value of Annuities	11.3
7.4 Annuity Payment Amounts	11.4
7.5 Number of Annuity Payments	11.5
7.6 Annuity Interest Rate	11.6
7.7 Annuities Terminology (Interactive Activity)	Chapter 11: Annuities Terminology (Interactive Activity) in Business Math: A Step-by-Step Handbook Abridged
7.8 Annuities Key Concepts Summary	Chapter 11: Key Concepts Summary in Business Math: A Step-by-Step Handbook Abridged
7.9 Symbols and Formulas	Chapter 11: Symbols and Formulas in Business Math: A Step-by-Step Handbook Abridged
7.10 Glossary of Terms	new
<b>8 Compound Interest – Special Applications</b>	
8.1 Deferred Annuities	12.1 in Business Math: A Step-by-Step Handbook
8.2 Key Concepts	12.5
8.3 Symbols and Formulas	Chapter 12: Symbols and Formulas in Business Math: A Step-by-Step Handbook Abridged
8.4 Glossary of Terms	new